MAT 2784A SOLUTIONS D3.1
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2.16. Résoudre l'équ. diff. d'Euler-Cauchy ;

$$x^2 y'' + x y' + 4y = 0.$$

On pose : $y(x) = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$$x^2 m(m-1) x^{m-2} + x\, m x^{m-1} + 4 x^m = 0$$

$$m(m-1) x^m + m x^m + 4 x^m = 0$$

$$x^m (m^2 - m + m + 4) = 0$$

on a $x^m \neq 0 \Rightarrow m^2 + 4 = 0$

$m^2 = -4$

$m^2 = 4 i^2$

$\Rightarrow m_1 = 2i \ ; \ m_2 = -2i$

deux solutions complexes (distinctes)

on a : $y_1(x) = x^\alpha \cos(\beta \ln x)$

$y_2(x) = x^\alpha \sin(\beta \ln x)$

$\frac{y_1}{y_2} \neq$ const. $\Rightarrow$ ind^{tes}.

Donc : $y_1(x) = \cos(2 \ln x) \ ; \ y_2(x) = \sin(2 \ln x)$

$\Rightarrow$ $\boxed{y(x) = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)}$ c la solution générale.

– Voir Verso (2.17) –

– le graphe est ds la feuille suivante –

2.17. $x^2 y'' + 4xy' + 2y = 0$; $y(1) = 1$; $y'(1) = 2$

On pose : $y(x) = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$x^2 m(m-1) x^{m-2} + 4x\, m x^{m-1} + 2x^m = 0$

$m(m-1) x^m + 4m x^m + 2x^m = 0$

$x^m (m^2 - m + 4m + 2) = 0$

$x^m \neq 0 \Rightarrow m^2 + 3m + 2 = 0$

$\Delta = (3)^2 - 4(1)(2) = 9 - 8 = 1$

$m_1 = \frac{-3-1}{2} = \frac{-4}{2} \Rightarrow m_1 = -2$

$m_2 = \frac{-3+1}{2} = \frac{-2}{2} \Rightarrow m_2 = -1$

$y(x) = C_1 x^{m_1} + C_2 x^{m_2}$

$\Rightarrow$ $\boxed{y(x) = C_1 x^{-2} + C_2 x^{-1}}$ est la solution général

* pour $x = 1$; $y = 1$ $\Rightarrow$ $C_1 + C_2 = 1$.

* $y'(x) = -2C_1 x^{-3} - C_2 x^{-2}$

pour $x = 1$; $y' = 2$ $\Rightarrow$ $-2C_1 - C_2 = 2$

$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + C_2 = -2 \end{cases}$ on peut l'écrire comme suit :

$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

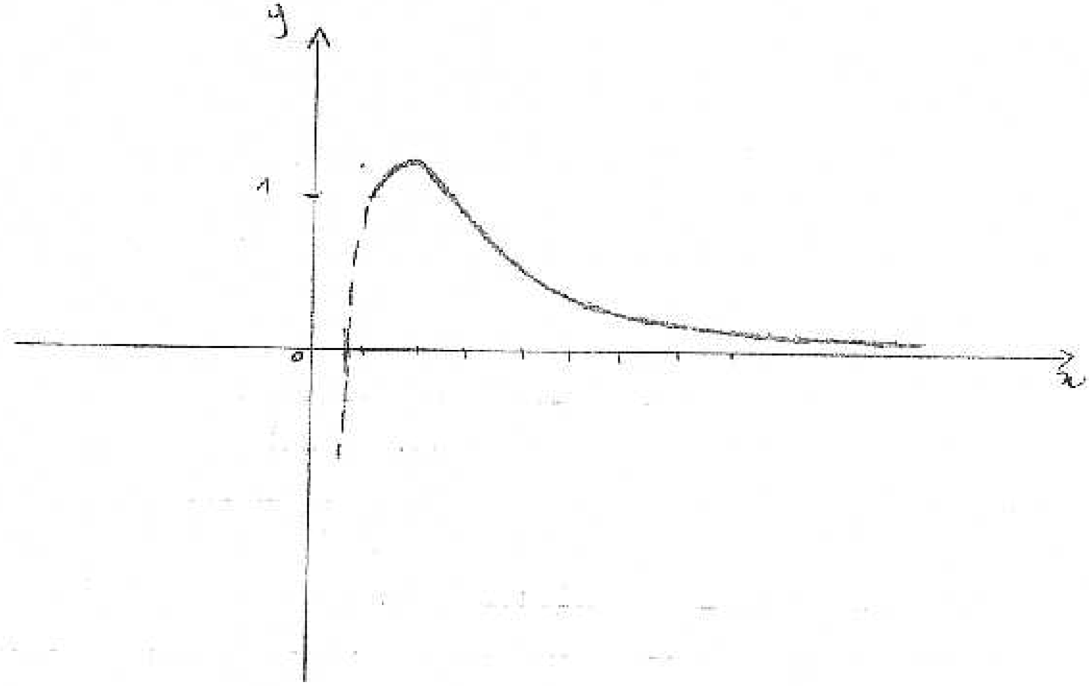
$\det \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 \neq 0$; $\exists$ solution unique (C_1, C_2).

$C_1 = \frac{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}}{-1} = \frac{1+2}{-1} \Rightarrow \boxed{C_1 = -3}$; $C_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}}{-1} = \frac{-2-2}{-1} = \frac{-4}{-1} \Rightarrow \boxed{C_2 = 4}$

$\boxed{y(x) = -3x^{-2} + 4x^{-1}}$ est la solution unique.

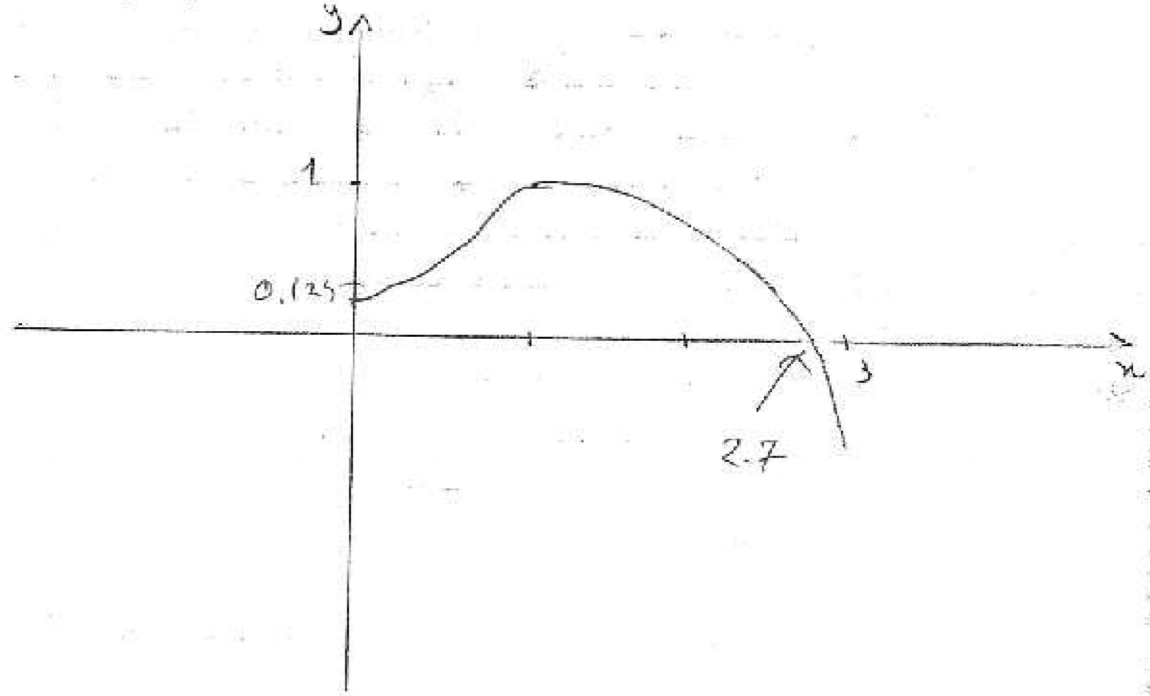
– Suite 2.17 – le graphe. $x \geq x_0$

On a : $y(x) = \frac{-3}{x^2} + \frac{4}{x}$.



– Suite 2.19 – le graphe.

On a : $y(x) = x - x \ln x$.



2.19 : $x^2 y'' - x y' + y = 0$; $y(1) = 1$; $y'(1) = 0$

On pose : $y(x) = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$x^2 m(m-1) x^{m-2} - x m x^{m-1} + x^m = 0$

$x^m m(m-1) - m x^m + x^m = 0$

$x^m (m^2 - m - m + 1) = 0$

$x^m (m^2 - 2m + 1) = 0$

$x^m \neq 0 \Rightarrow m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_1 = m_2 = m = 1.$

On a : $y_1(x) = x$

$y_2(x) = \ln x \cdot y_1(x) = x \ln x.$

On a : $\frac{y_1}{y_2} = \frac{1}{\ln x} \neq$ cte $\Rightarrow y_1, y_2$ indépts.

$\Rightarrow$ $\boxed{y(x) = C_1 x + C_2 x \ln x}$ est la solution générale.

* pour $x = 1$; $y = 1$

~~$C_1 + C_2 = 1$~~ $C_1 + C_2 \cdot 0 = 1 \Rightarrow \boxed{C_1 = 1}$

* $y'(x) = C_1 + C_2 \ln x + C_2 x \cdot \frac{1}{x}$

$y'(x) = C_1 + C_2 \ln x + C_2$

pour $x = 1$; $y' = 0$

$C_1 + C_2 = 0$

$C_2 = -C_1 = -1$

$\boxed{y(x) = x - x \ln x}$ est la solution unique.

– Voir Verso (le graphe) –

3.2 : $y''' + 3y'' - 4y' - 12y = 0$

On pose : $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$y''' = \lambda^3 e^{\lambda x}$.

$e^{\lambda x} (\lambda^3 + 3\lambda^2 - 4\lambda - 12) = 0$

$e^{\lambda x} \neq 0 \Rightarrow \lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$ (équa. caractéristique)

$P(\lambda) = \lambda^3 + 3\lambda^2 - 4\lambda - 12$.

On a pour $\lambda_1 = 2 \Rightarrow P(\lambda_1) = 0$

$$
\begin{array}{rrrr|l}
\lambda^3 & +3\lambda^2 & -4\lambda & -12 & \lambda - 2 \\
-\lambda^3 & \pm 2\lambda^2 & & & \lambda^2 + 5\lambda + 6 \\
\hline
0 & 5\lambda^2 & -4\lambda & & \\
 & -5\lambda^2 & \pm 10\lambda & & \\
\hline
 & & 6\lambda & -12 & \\
 & & -6\lambda & \pm 12 & \\
\hline
 & & 0 & ? &
\end{array}
$$

$\Rightarrow P(\lambda) = (\lambda - 2)(\lambda^2 + 5\lambda + 6)$

$P(\lambda) = 0 \Rightarrow \lambda_1 = 2$; $\lambda^2 + 5\lambda + 6 = 0$

$\Delta = (5)^2 - 4(6) = 25 - 24 = 1$

$\lambda_2 = \frac{-5-1}{2} = \frac{-6}{2} \Rightarrow \lambda_2 = -3$

$\lambda_3 = \frac{-5+1}{2} = \frac{-4}{2} \Rightarrow \lambda_3 = -2$

On a 3 racines réelles distinctes

Donc : $y_1(x) = e^{2x}$

$y_2(x) = e^{-3x}$

$y_3(x) = e^{-2x}$

La solution générale est : $\boxed{y(x) = C_1 e^{2x} + C_2 e^{-3x} + C_3 e^{-2x}}$

– Voir ~~verso~~ p. D3.3

<u>3 _ 6</u> : $y^{(4)} - y = 0$; $y(0) = 0$, $y'(0) = 0$; $y''(0) = 0$

$y'''(0) = 1$

On pose : $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$y''' = \lambda^3 e^{\lambda x}$

$y^{(4)} = \lambda^4 e^{\lambda x}$

$\lambda^4 e^{\lambda x} - e^{\lambda x} = 0$

$e^{\lambda x} (\lambda^4 - 1) = 0$

$e^{\lambda x} \neq 0 \Rightarrow \lambda^4 - 1 = 0$

$\lambda^4 = 1$ <
- $\lambda^2 = 1$ < $\lambda_1 = 1$, $\lambda_2 = -1$ (2 racines réelles distinctes)
- $\lambda^2 = -1 = i^2$ < $\lambda_3 = i$, $\lambda_4 = -i$ (2 racines complexes distinctes)

D'après l'identité d'Euler : $e^{i\varphi} = \cos\varphi + i \sin\varphi$.

$u_3 = e^{(\alpha + i\beta)x} = e^{ix} = \cos x + i \sin x$.

$y_3 = \mathrm{Re}\, u = \cos x$

$y_4 = \mathrm{Im}\, u = \sin x$.

$\Rightarrow$ $\boxed{y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x}$

est la sol. gén.

* pour $x = 0$, $y = 0$ $\Rightarrow$ $C_1 + C_2 + C_3 = 0$

* $y'(x) = C_1 e^{x} - C_2 e^{-x} - C_3 \sin x + C_4 \cos x$

pour $x = 0$, $y' = 0$ $\Rightarrow$ $C_1 - C_2 + C_4 = 0$

* $y''(x) = C_1 e^{x} + C_2 e^{-x} - C_3 \cos x - C_4 \sin x$

pour $x = 0$, $y'' = 0$ $\Rightarrow$ $C_1 + C_2 - C_3 = 0$

* $y'''(x) = C_1 e^{x} - C_2 e^{-x} + C_3 \sin x - C_4 \cos x$

pour $x = 0$, $y''' = 1$ $\Rightarrow$ $C_1 - C_2 - C_4 = 1$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1 - C_2 + C_4 = 0 \\ C_1 + C_2 - C_3 = 0 \\ C_1 - C_2 - C_4 = 1 \end{cases}$$

syst de ~~3~~ 4 équ. à 4 incon.

Suite 3-6 :

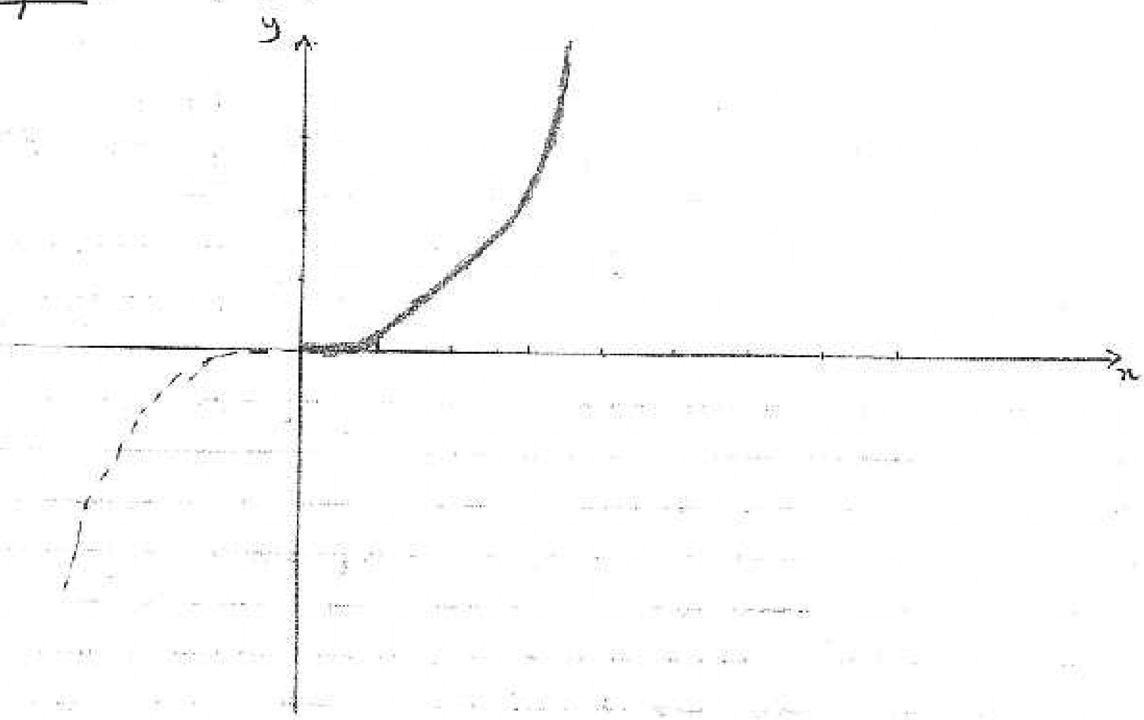
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{2} & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right)$$

$$\Rightarrow \begin{cases} C_1 + C_2 + C_3 = 0 & \Rightarrow C_1 = -C_2 - C_3 = \frac{1}{4} - 0 \Rightarrow C_1 = \frac{1}{4} \\ 2C_2 + C_3 - C_4 = 0 & \Rightarrow 2C_2 = C_4 - C_3 = \frac{-1}{2} - 0 \Rightarrow C_2 = \frac{-1}{4} \\ 2C_3 = 0 & \Rightarrow C_3 = 0 \\ -2C_4 = 1 & \Rightarrow C_4 = \frac{-1}{2} \end{cases}$$

$$\boxed{y(x) = \frac{1}{4} e^{x} - \frac{1}{4} e^{-x} - \frac{1}{2} \sin x}$$ est la solution unique

le graphe : $x \geq 0$.



Voir verso.

3-8 : $y''' - 2y'' + 4y' - 8y = 0$; $y(0) = 2$; $y'(0) = 0$; $y''(0) = 0$

On pose : $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$y''' = \lambda^3 e^{\lambda x}$

$e^{\lambda x}(\lambda^3 - 2\lambda^2 + 4\lambda - 8) = 0$

$e^{\lambda x} \neq 0 \Rightarrow \lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$ c'est 1 équ. carac.

* pour $\lambda_1 = 2 \Rightarrow P(\lambda) = \lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$

$$\begin{array}{r|l} \lambda^3 - 2\lambda^2 + 4\lambda - 8 & \lambda - 2 \\ \hline -\lambda^3 \pm 2\lambda^2 & \lambda^2 + 4 \\ 0 + 4\lambda - 8 & \\ -4\lambda \pm 8 & \\ 0 \quad 0 & \end{array}$$

$\Rightarrow P(\lambda) = (\lambda - 2)(\lambda^2 + 4)$.

On a : $\lambda_1 = 2$ & $\lambda^2 + 4 = 0$

$\lambda^2 = -4 = 4i^2$

$\lambda_2 = 2i$; $\lambda_3 = -2i$

2 racines complexes.

$y_2(x) = e^{\alpha x} \cos \beta x$

$y_3(x) = e^{\alpha x} \sin \beta x$

$\Rightarrow y_2(x) = \cos 2x$

$y_3(x) = \sin 2x$.

$\Rightarrow$ $\boxed{y(x) = C_1 e^{2x} + C_2 \cos 2x + C_3 \sin 2x}$ est l'équ. générale.

* pour $x = 0$, $y = 2 \Rightarrow C_1 + C_2 = 2$
* $y'(x) = 2C_1 e^{2x} - 2C_2 \sin 2x + 2C_3 \cos 2x$.

pour $x = 0$, $y' = 0 \Rightarrow 2C_1 + 2C_3 = 0$.

* $y''(x) = 4C_1 e^{2x} - 4C_2 \cos 2x - 4C_3 \sin 2x$.

pour $x = 0$; $y'' = 0 \Rightarrow 4C_1 - 4C_2 = 0$

- Voir la suite de l'autre feuille.

Suite 3.8 :

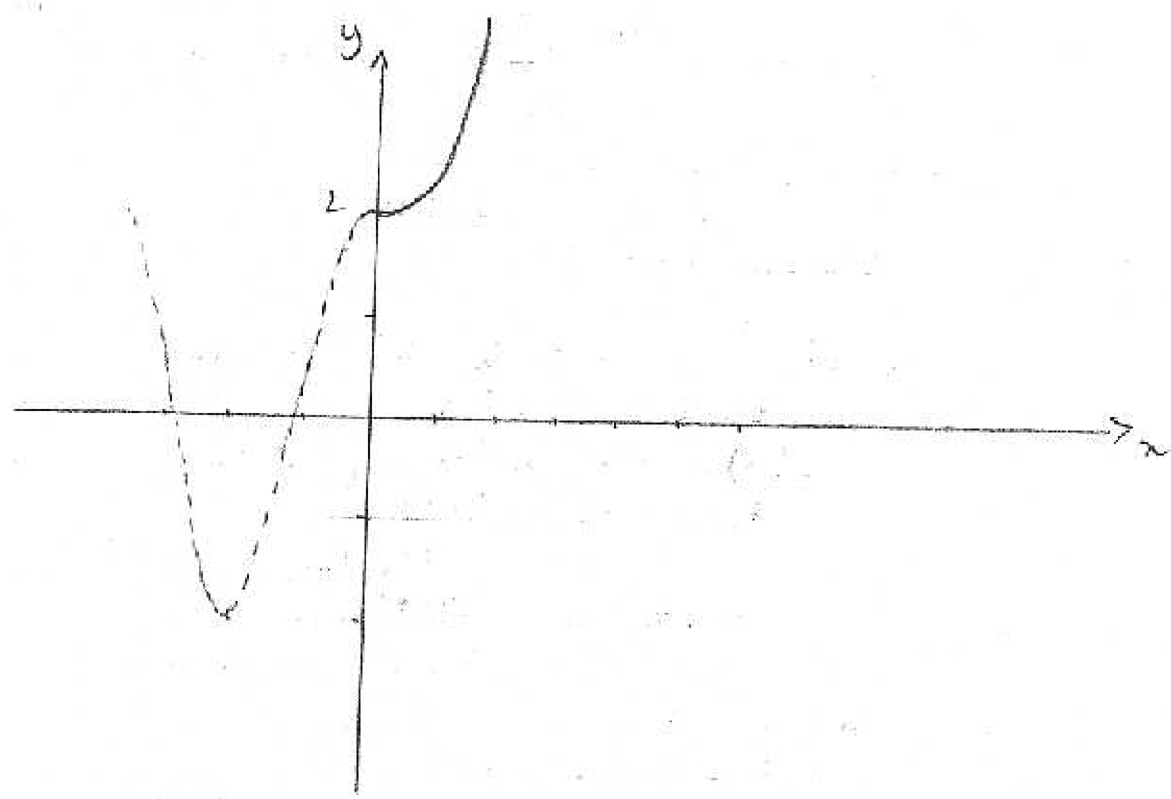
$$\begin{cases} C_1 + C_2 = 2 & \text{—— ①} \\ C_1 + C_3 = 0 & \text{—— ②} \\ C_1 - C_2 = 0 & \text{—— ③} \end{cases}$$

① + ③ $\Rightarrow 2C_1 = 2 \Rightarrow C_1 = 1$ et $C_2 = 1$.

② $\Rightarrow C_3 = -C_1 = -1$.

$\Rightarrow$ $\boxed{y(x) = e^{2x} + \cos 2x - \sin 2x}$ est la solution unique.

- le graphe : $x \geq 0$



— Voir verso —

8.16 :

n	x_n	Δx_n	$\Delta^2 x_n$
1	$x_1 = 4{,}000$		
		$0{,}6834$	
2	$x_2 = 3{,}3166$		$9{,}4705$
		$0{,}2129$	
3	$x_3 = 3{,}1037$		

$x_{n+1} = g(x_n)$

$g(x_n) = \sqrt{2x_n+3}$

$x_2 = \sqrt{2x_1+3} = \sqrt{2.4{,}000+3} = 3{,}3166$

$a_1 = x_1 - \dfrac{(\Delta x_1)^2}{\Delta^2 x_1} = 4{,}000 - \dfrac{(0{,}6834)^2}{9{,}4705} \Rightarrow \boxed{a_1 = 3{,}0074}$

$* \ \Delta x_1 = x_{n+1} - x_n$

$= x_2 - x_1 = 3{,}3166$

$* \ \Delta^2 x_n = \Delta x_2 - \Delta x_1$

8.17 : Méthode de Steffensen.

On a : $p \simeq 1{,}9$.

$g(x) = 1 + \sin^2 x \quad ; \quad x_0 = 1.$

$a_0 = x_0 - \dfrac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$

$a_n = x_n - \dfrac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}$

$S_0 = x_0 = 1.$

$S_1 = a_0 = 2{,}152905.$

$S_2 \begin{cases} S_1 \\ z_1 = g(S_1) \\ z_2 = g(z_1) \end{cases} \Rightarrow S_2 \begin{cases} S_1 = 2{,}152905 \\ z_1 = g(S_1) = 1 + \sin^2 2{,}152905 = 1{,}697734 \\ z_2 = g(z_1) = 1 + \sin^2 (1{,}697734) = \\ = 1{,}993997 \end{cases}$

$S_3 \begin{cases} S_2 = 1{,}873473 \\ z_1 = g(S_2) = 1{,}911152 \\ z_2 = g(z_1) = 1{,}88856 \end{cases}$

n	x_n	a_n	S_n
0	1	2,152905	1
1	1,708073	1,884486	2,152905
2	1,981273	1,894914	1,873473
3	1,840462	1,896288	1,897027
4	1,928872		1,897133
5	1,877169		1,897118

$S_4 = \begin{cases} S_3 = 1,897027 \\ z_1 = g(S_3) = 1,897295 \\ z_2 = g(z_1) = 1,897132 \end{cases}$

$S_5 = \begin{cases} S_4 = 1,897193 \\ z_1 = g(S_4) = 1,897194 \\ z_2 = g(z_1) = 1,897194 \end{cases}$

D'après la méthode de Steffensen : $p \simeq 1,897193$.
elle converge d'ordre 2,

car $\varepsilon_{n+1}/\varepsilon_n^2 \simeq$ cte

et $g'(1.877169) = 2 \sin \bar{x} \cos \bar{x}$

$= \sin 2\bar{x}$

$\neq 0$

où $\bar{x} = 1.877169$

Puisque $x_{n+1} = g(x_n)$
converge d'ordre 1
alors Steffensen converge d'ordre 2.