MAT 2784 A SOLUTIONS

D 2.1

MAT 2784 A

DEVOIR 2

RÉMI VAILLANCOURT

Exercice no 1.35

$x(\ln x)y' + y = 2\ln x \quad (1)$

$x(\ln x)\,dy + (y - 2\ln x)\,dx = 0$

$(y - 2\ln x)\,dx + x\ln x\,dy = 0$

$M\,dx + N\,dy = 0$

$M_y = 1$ $\qquad M_y \neq N_x$ donc (1) n'est pas exacte

$N_x = \ln x + 1$

$$\frac{M_y - N_x}{N} = \frac{1 - \ln x - 1}{x\ln x} = \frac{-\ln x}{x\ln x} = \frac{-1}{x} = f(x)$$

$$\mu(x) = g(x) = e^{\int \frac{-1}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

En multipliant (1) par $\mu(x)$ on a :

$$\frac{1}{x}\left[(y - 2\ln x)\,dx + x\ln x\,dy\right] = 0$$

$$\left(\frac{y - 2\ln x}{x}\right)dx + \ln x\,dy = 0 \qquad \text{(exacte)}$$

$$u_x\,dx + u_y\,dy = 0$$

$u_y = \ln x \Rightarrow u(x,y) = \int \ln x\,dy + T(x)$

$u(x,y) = y\ln x + T(x)$

$u_x = y\frac{1}{x} + T'(x)$

$u_x = \frac{y - 2\ln x}{x}$

$\Rightarrow \frac{y}{x} + T'(x) = \frac{y}{x} - 2\frac{\ln x}{x}$

$T'(x) = -2\frac{\ln x}{x}$

$T(x) = -2\int \frac{\ln x}{x}\,dx$ posons $u = \ln x$

D 2 : 2

$T(x) = -2\int \frac{\ln x}{x}\,dx = -2\int u\,du = -2\left(\frac{1}{2}u^2\right) = -u^2$

$\Leftrightarrow T(x) = -(\ln x)^2$

$u(x,y) = y\ln x + -(\ln x)^2 = C$

Solution générale $y\ln x - (\ln x)^2 = C \Leftrightarrow \boxed{y\ln x - (\ln x)^2 = C}$

Posons $C=0$ on a: $y\ln x - (\ln x)^2 = 0$

$\boxed{y = \ln x}$

Vérification

$x(\ln x)y' + y = 2\ln x \quad y = \ln x$

$x(\ln x)(\ln x)' + \ln x = x\ln x\left(\frac{1}{x}\right) + \ln x$

$= 2\ln x$

D2.3

Exercice n° 1.45

$x^2 - y^2 = c^2$

On trouve la famille des courbes orthogonales (la pente) en dérivant par x.

$$x^2 - y^2 = c^2$$

d_x : $\quad 2x - 2yy' = 0$

$$y' = \frac{2y}{2x} = \frac{y}{x} = m$$

L'équation de la famille orthogonale satisfait l'équation

$$y_{orth} = \frac{-y_{orth}}{x} \quad \text{et on a} \quad y = \frac{-y}{x}$$

$$y\,dx + x\,dy = 0$$
$$M\,dx + N\,dy = 0$$
$$M_y = 1 \text{ et } N_x = 1 \qquad M_y = N_x \text{ donc exacte}$$

$$u_y = x$$
$$u(x,y) = \int x\,dy + T(x)$$
$$= xy + T(x)$$

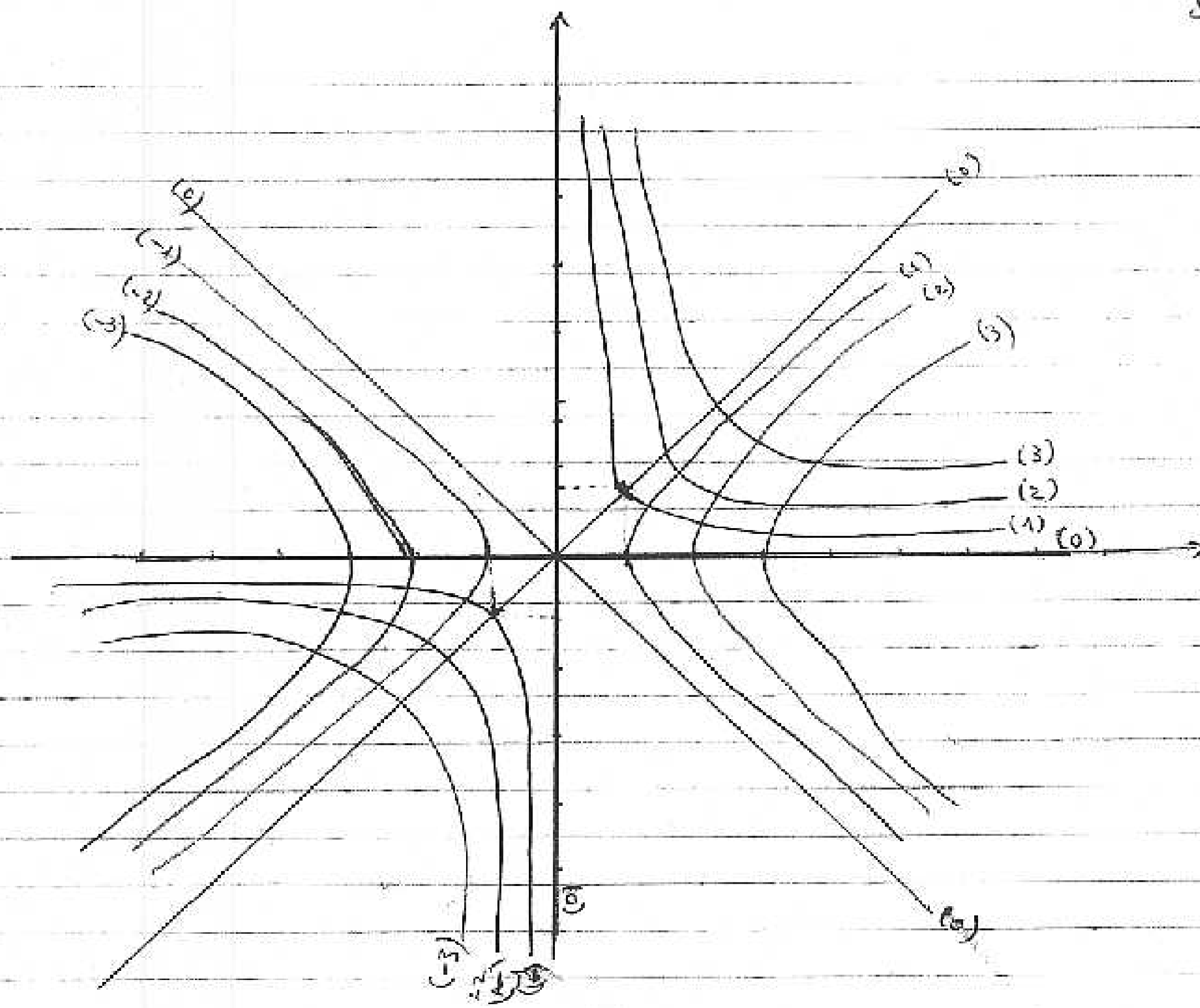
$$u_x = y + T'(x) \qquad \text{or } u_x = y$$
$$y = y + T'(x) \Rightarrow T'(x) = 0$$
$$\Rightarrow T(x) = \int 0\,dx = C \quad (\text{cte})$$

$$u(x,y) = xy + c$$

D'où $\boxed{xy = \cancel{c}}\ k$ (c est déjà employé)

On fait varier k ~~c~~ en lui donnant les valeurs 0, 1, 2, ...

D2: 4



~ famille de la courbe
~ famille orthogonale

D2.5

Exercice n° 1.47

$e^x \cos y = c$

$-e^x \sin y\, y' + \cos y\, e^x = 0$

$e^x \sin y\, y' = \cos y\, e^x$

$y' = \dfrac{\cos y}{\sin y}$

$y' = \dfrac{\cos y}{\sin y} = m \qquad y'_{orth} = -\dfrac{1}{m} = \dfrac{-\sin y}{\cos y}$

$\dfrac{dy}{dx} = \dfrac{-\sin y}{\cos y}$

$\cos y\, dy = -dx \sin y$

$\dfrac{\cos y}{\sin y}\, dy = -dx$

$\displaystyle\int \frac{\cos y}{\sin y}\, dy = -\int dx$

$\ln|\sin y| = -x + c_1$

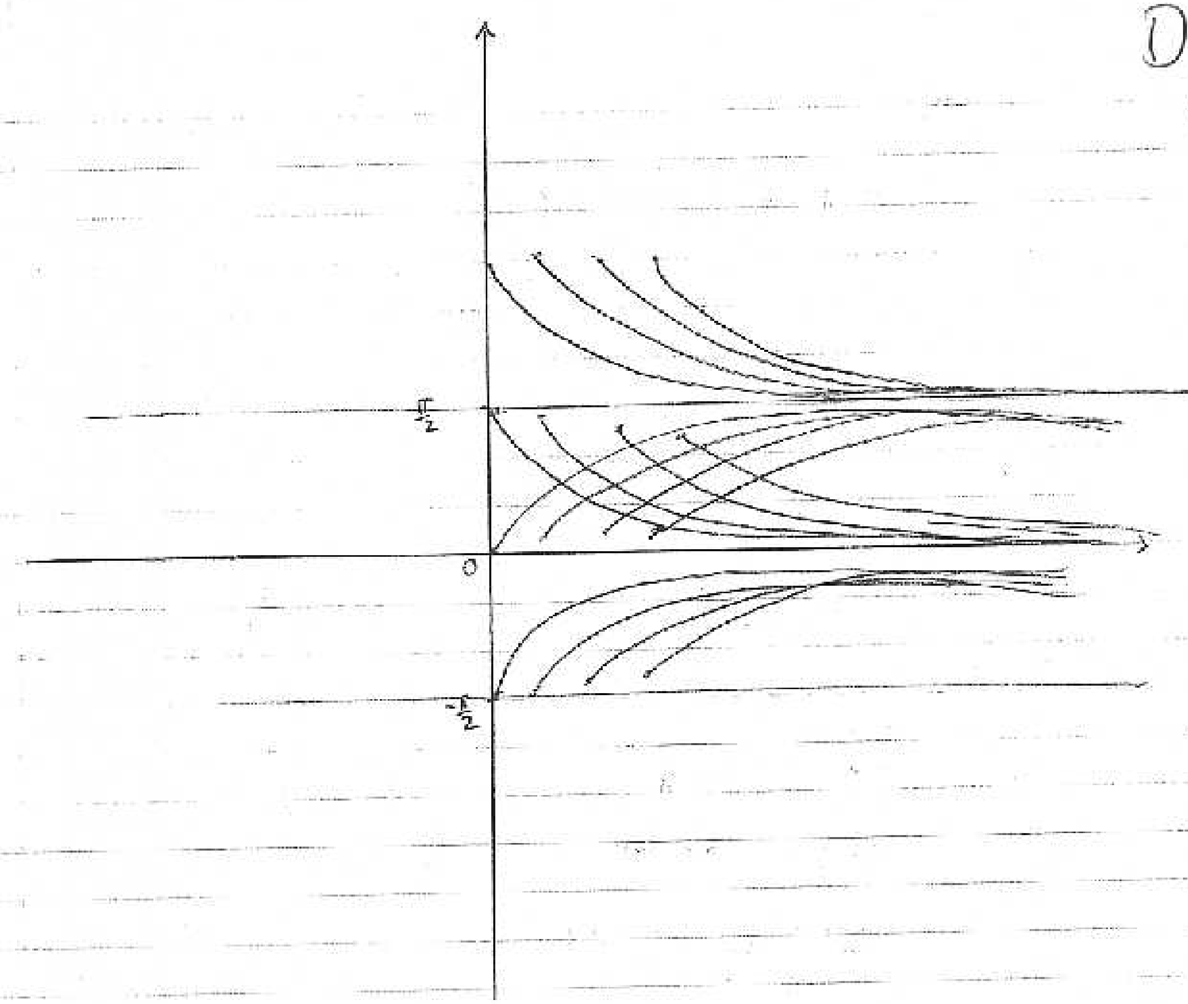
$\sin y = e^{-x+c_1}$

$\sin y = e^{-x} \cdot e^{c_1}$

$\sin y = c\, e^{-x}$

$\boxed{y = \operatorname{Arc\,sin}(c\, e^{-x})}$

D2.6



~ famille orthogonale : $\mathrm{Arc\,sin}(c\,e^{-x}) = y$

~ famille de la courbe : $\mathrm{Arc\,cos}(c\,e^{x}) = y$

Exercice n° 2.4

$$y'' + y' + \frac{1}{4}y = 0 \quad (1) \quad y(2) = 1 \quad ; \quad y'(2) = 1$$

Posons $\quad y = e^{\alpha x} \quad , \quad y' = \alpha e^{\alpha x} \quad , \quad y'' = \alpha^2 e^{\alpha x}$

(1) devient :

$$\alpha^2 e^{\alpha x} + \alpha e^{\alpha x} + \frac{1}{4} e^{\alpha x} = 0$$

$$e^{\alpha x}\left(\alpha^2 + \alpha + \frac{1}{4}\right) = 0$$

$$\alpha^2 + \alpha + \frac{1}{4} = 0$$

$$\Delta = 1 - 4 \cdot \frac{1}{4} = 0$$

$$\alpha_{1,2} = \frac{-1}{2}$$

$$y_1(x) = e^{-\frac{1}{2}x}$$

$$y_2(x) = x\, e^{-\frac{1}{2}x}$$

Solution générale est : $\quad y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$

Solution particulière en prenant les conditions initiales en compte :

$$y(2) = 1 \Rightarrow \quad 1 = c_1 e^{-1} + 2 c_2 e^{-1}$$

$$y'(2) = 1 \Rightarrow \quad 1 = -\frac{1}{2} c_1 e^{-1} + \left[c_2 e^{-1} - \frac{c_2}{2} e^{-1} \right]$$

$$1 = -\frac{1}{2} c_1 e^{-1}$$

$$c_1 = \frac{-2}{e^{-1}} \Rightarrow \quad c_1 = -2e^{-1}$$

$$\boxed{c_1 = -2e}$$

$$1 = -2e\, e^{-1} + 2 c_2 e^{-1}$$

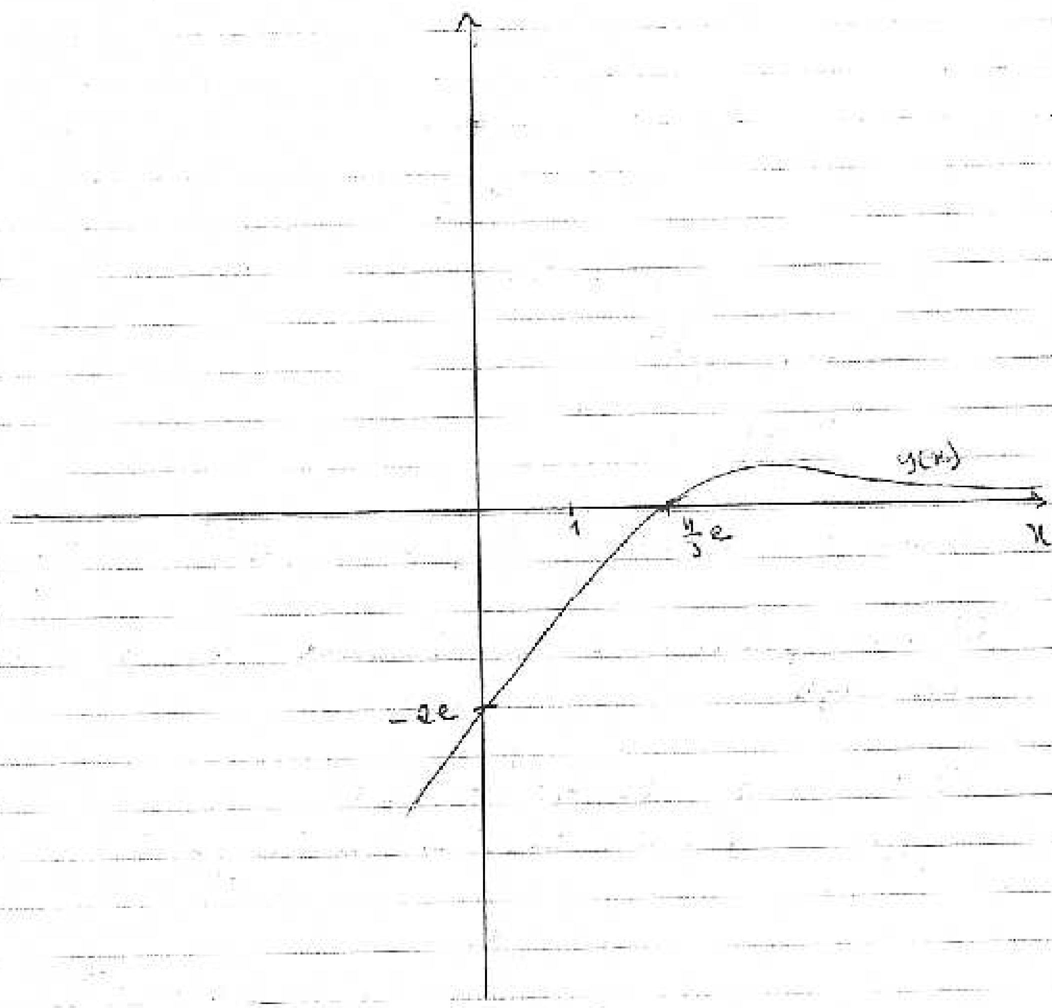
$$\boxed{c_2 = \frac{3}{2} e}$$

D 2,8

et la solution est :

$$y(x) = -2e^{1}e^{-\frac{1}{2}x} + \frac{3}{2}xe^{-\frac{1}{2}x}e^{1}$$

$$\boxed{y(x) = -2e^{-\frac{1}{2}x+1} + \frac{3}{2}xe^{-\frac{1}{2}x+1}}$$



D2.9

Exercice n° 2.5

$y'' + 9y = 0 \quad (1) \qquad y(0) = 0 \qquad y'(0) = 1$

Posons $\quad y = e^{\alpha x} \qquad y'' = \alpha^2 e^{\alpha x}$

$$e^{\alpha x}(\alpha^2 + 9) = 0$$

$$\alpha_1 = 0 + 3i$$

$$\alpha_2 = 0 - 3i$$

$$y(x) = C_1 e^0 \cos 3x + C_2 e^0 \sin 3x$$

$y(0) = 0 \Rightarrow C_1 = 0$

$y'(0) = 1 \Rightarrow 3C_2 \cos 3x = 1 \Rightarrow 3C_2 = 1$

$\Rightarrow C_2 = \frac{1}{3}$

D'où $\quad \boxed{y(x) = \frac{1}{3} \sin 3x}$

Vérification

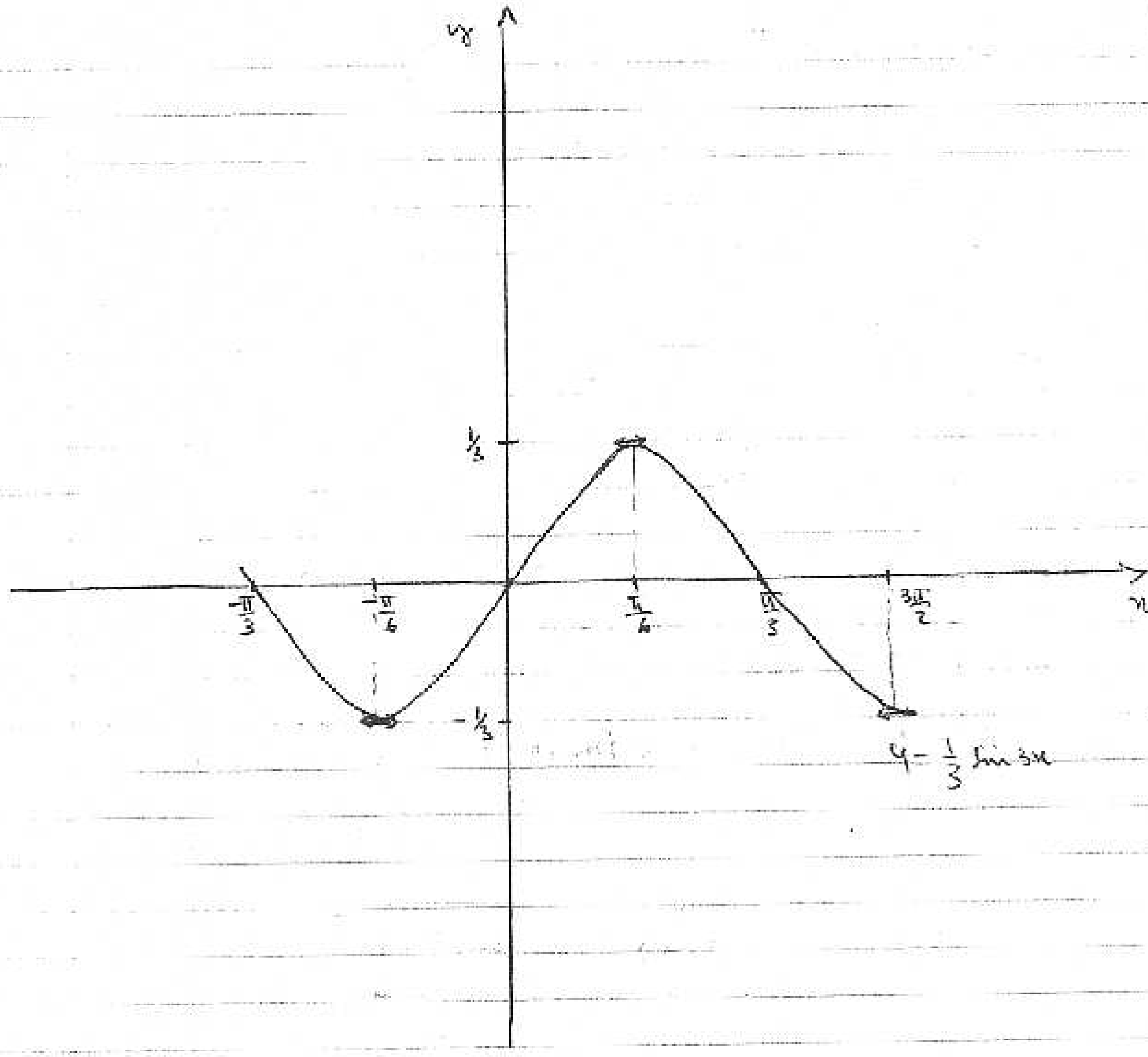
$$y'(x) = \cos 3x \quad \text{et} \quad y''(x) = -\frac{1}{3} \sin 3x$$

$$-\frac{1}{3}\sin 3x + 9\left(\frac{1}{3}\sin 3x\right) = -\frac{1}{3}\sin 3x + \frac{1}{3}\sin 3x = 0$$

Tracer la fonction

- $y(0) = 0$
- l'amplitude est $\frac{1}{3}$
- $\frac{1}{3}\sin 3x = \frac{1}{3} \Rightarrow 3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

D 2.10


y
$\frac{1}{3}$
$-\frac{1}{3}$
$-\frac{\pi}{3}$
$-\frac{\pi}{6}$
$\frac{\pi}{6}$
$\frac{\pi}{3}$
$\frac{3\pi}{2}$
x
$y = \frac{1}{3}\sin 3x$

D2,11

Exercice n° 2.7

$y'' - 2y' + 3y = 0 \qquad y(0) = 1 \;,\; y'(0) = 3$

$y = e^{\lambda x} \quad ; \quad y' = \lambda e^{\lambda x} \quad ; \quad y = \lambda^2 e^{\lambda x}$

$e^{\lambda x}(\lambda^2 - 2\lambda + 3) = 0$

$\lambda^2 - 2\lambda + 3 = 0$

$\lambda_1 = \dfrac{2 + \sqrt{4-12}}{2} = 1 + i\sqrt{2}$

$\lambda_{1,2} = \alpha \pm i\beta$

$\lambda_2 = \dfrac{2 - \sqrt{4-12}}{2} = 1 - i\sqrt{2}$

$y(x) = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$

$y(x) = C_1 e^{x}\cos x\sqrt{2} + C_2 e^{x}\sin x\sqrt{2}$

$y(0) = 1 \iff C_1 e^0 \cos 0 + \underset{0}{\underbrace{C_2 e^0 \sin 0}} = 1$

$\boxed{C_1 = 1}$

$y'(0) = 3$

$y'(x) = C_1 e^{x}\cos(x\sqrt{2}) - \sqrt{2}\, C_1 e^{x}\sin(x\sqrt{2}) + C_2 e^{x}\sin x\sqrt{2} + C_2\sqrt{2}\cos x\sqrt{2}$

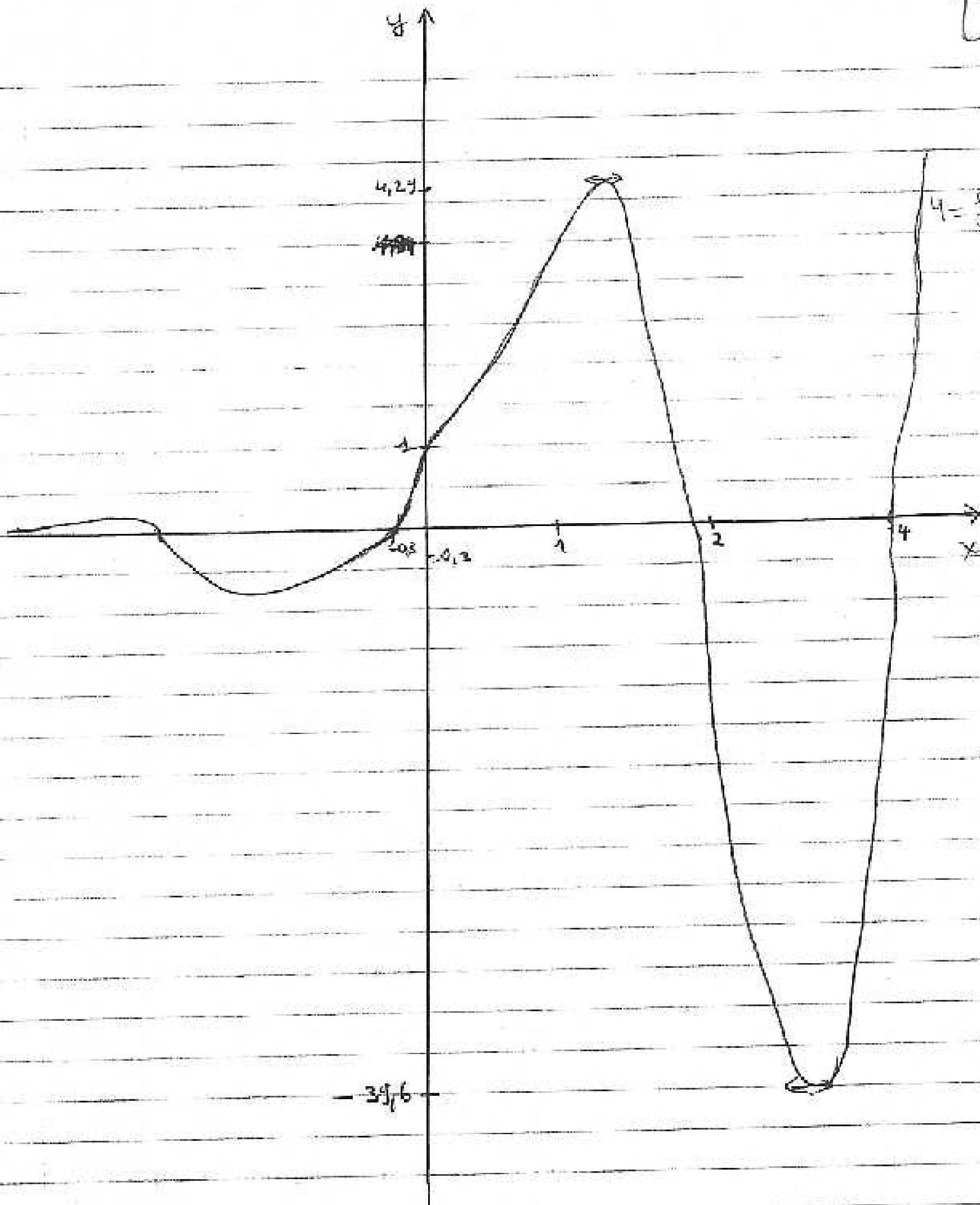
$C_1 + \sqrt{2}\, C_2 = 3$

$\sqrt{2}\, C_2 = 3 - C_1$

$C_2 = \dfrac{3 - C_1}{\sqrt{2}} = \dfrac{3-1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

$y(x) = e^{x}\cos(x\sqrt{2}) + \sqrt{2}\, e^{x}\sin(x\sqrt{2})$

D2.12



$$y = e^{x}\cos x\sqrt{2} + \sqrt{2}\, e^{x}\sin x\sqrt{2}$$

D2.13

Exercice n° 8.9

$x_{n+1} = g(x_n)$

$g(x) = 1 + \sin^2 x$

$x_0 = 1$

n	$x_{n+1} = g(x)$	$x_{n+1} - x_n$	$\varepsilon_{n+1} / \varepsilon_n^2$
0	1		
1	1,708073	0,708073	
2	1,981273	0,2732	
3	1,840762	−0,140511	
4	1,928872	0,08811	
5	1,877169	−0,051783	

Nous constatons que l'ordre de convergence est 1.

Vérification

$g'(x) = 2 \sin x \cos x$

$g'(p) = 2 \sin p \cos p \qquad p \simeq 1.9$

$\neq 0$

Donc l'ordre de convergence est bien 1.

#8.10) $f(x) = 2x - \tan x$ $\qquad f(x) = 0$

$f(x) = 2x - \tan(x)$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 2 - \sec^2(x)$

$\therefore x_{n+1} = x_n - \dfrac{2x_n - \tan(x_n)}{2 - \sec^2(x_n)}$

$x_0 = 1$

$x_1 = 1,310478$
$x_2 = 1,223929$
$x_3 = 1,176051$
$x_4 = 1,165927$
$x_5 = 1,165561$
$x_6 = 1,165561$

∴ la racine est 1.165561 et l'ordre de convergence est 2 (car la la dérivé est de f(x) non null à la 2ème valeur de x)