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$\boxed{1.8}$ $x \sin y\, dx + (x^2+1)\cos y\, dy = 0 \qquad y(1) = \pi/2$

$M = x \sin y \qquad N = (x^2+1)\cos y$

$M_y = x \cos y \qquad N_x = 2x \cos y$

$M_y \neq N_x$ l'équation n'est pas exacte.

Méthode du facteur d'intégration

$$\mu = \frac{M_y - N_x}{N} = \frac{x\cos y - 2x\cos y}{(x^2+1)\cos y}$$

$$= \frac{-x}{x^2+1} \Rightarrow \mu(x) = e^{\int f(x)\,dx}$$

$$\mu(x) = e^{\int \frac{-x}{x^2+1}\,dx}$$

$$= e^{-\frac{1}{2}\ln|x^2+1|}$$

$$= e^{\ln(x^2+1)^{-\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{(x^2+1)}} = f(x)$$

Intégration par substitution

$$-\int \frac{x}{x^2+1}\,dx \qquad u = x^2+1,\quad du = 2x\,dx,\quad dx = \frac{du}{2x}$$

$$-\int \frac{x}{u}\frac{du}{2x}$$

$$= -\frac{1}{2}\frac{du}{u} = -\frac{1}{2}\ln|u|$$

$$M = \frac{x \sin y\, dx}{\sqrt{(x^2+1)}} \qquad N = \frac{(x^2+1)\cos y\, dy}{\sqrt{(x^2+1)}}$$

$$\mu M_y = \frac{x\cos y}{\sqrt{(x^2+1)}} \qquad \mu N_x = \frac{\cancel{2}x\cos y}{\cancel{2}\sqrt{(x^2+1)}}$$

$M_y = N_x$ l'équation est exacte

$\int U_x\,dx = \int M\,dx$

$$U = \int \frac{x \sin y}{\sqrt{(x^2+1)}}\, dx = \sin y \int \frac{x}{\sqrt{(x^2+1)}}$$

$$= \sin y \left[(x^2+1)^{1/2}\right] + T(y)$$

$$U_y = \cos y \left[(x^2+1)^{1/2}\right] + T'(y)$$

$$\mu N_y = U_y$$

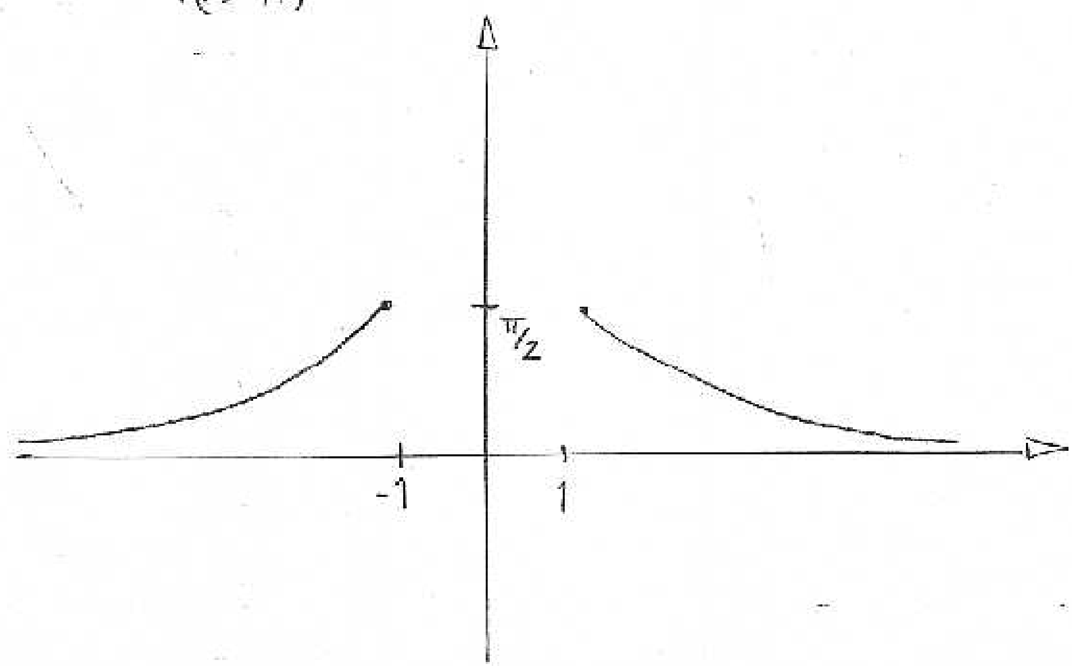
$$\cos y \left[(x^2+1)^{1/2}\right] + T'(y) = \frac{(x^2+1)\cos y}{\sqrt{(x^2+1)}}$$

$$T'(y) = 0 \qquad y(1) = \pi/2$$

$$U(x,y) = \sin y \sqrt{(x^2+1)} = c$$
$$\sin \tfrac{\pi}{2} \sqrt{2} = \sqrt{2}$$

$$U(x,y) = \sin y \sqrt{(x^2+1)} = \sqrt{2}$$

$$\sin y = \frac{\sqrt{2}}{\sqrt{(x^2+1)}} \longrightarrow y = \arcsin \sqrt{\frac{2}{x^2+1}}$$



$\boxed{1.11}$ $xy' = y + \sqrt{y^2 - x^2}$

$$\frac{x}{y}\frac{dy}{dx} = \frac{y + \sqrt{y^2 - x^2}}{y}$$

$$= 1 + \sqrt{\frac{y^2}{y^2} - \frac{x^2}{y^2}}$$

substitution $u = \frac{x}{y}$

$x = y \cdot u$

$dx = y\,du + u\,dy$

$$= 1 + \sqrt{1 - \frac{x^2}{y^2}}$$

$$(u)\,dy = 1 + \sqrt{1 - (u)^2}\,dx$$

$$= 1 + \sqrt{1 - (u)^2}\,[y\,du + u\,dy]$$

$$= 1 + \sqrt{1 - (u)^2}\,[y\,du] + 1 + \sqrt{1 - (u)^2}\,[u\,dy]$$

$$\cancel{u\,dy} = \left[y + y\sqrt{1 - (u)^2}\right]du + \cancel{u\,dy} + u\sqrt{1 - (u)^2}\,dy$$

$$-u\sqrt{1 - (u)^2}\,dy = \left[y + y\sqrt{1 - (u)^2}\right]du$$

$$-\int u\sqrt{1 - (u)^2}\,dy = \int y\,du + \int y\sqrt{1 - (u)^2}\,du$$

$$-u\sqrt{1 - (u)^2}\,y = uy + y\left[\frac{1}{2}u\sqrt{1 - u^2} + \frac{1}{2}\arcsin(u)\right]$$

$$-x\sqrt{1 - \left(\frac{x}{y}\right)^2} = x + y\left[\frac{x}{2y}\sqrt{1 - \left(\frac{x}{y}\right)^2} + \frac{1}{2}\arcsin\left(\frac{x}{y}\right)\right]$$

1.14 $(3x^2+9xy+5y^2)\,dx-(6x^2+4xy)\,dy=0$

$y(2)=-6.$

Rés.

$M_y = 9x+10y \qquad M_y \neq N_x$

$N_x = -12x-4y$

$$\frac{M_y - N_x}{N} = \frac{21x+14y}{-2x(3x+2y)} = \frac{7(3x+2y)}{-2x(3x+2y)} = -\frac{7}{2x} = f(x)$$

$$\mu(x) = e^{-\frac{7}{2}\int \frac{1}{x}dx} = e^{\ln x^{-7/2}} = x^{-7/2}$$

$$(3x^{-3/2}+9x^{-5/2}y+5x^{-7/2}y)\,dx-(6x^{-3/2}+4x^{-5/2}y)\,dy=0$$

$$u(x,y) = -\int(6x^{-3/2}+4x^{-5/2}y)\,dy + T(x)$$

$$= -6x^{-3/2}y-2x^{-5/2}y^2+T(x)$$

$$u_x = 9x^{-5/2}y+5x^{-7/2}y^2+T'(x)$$

$$= 3x^{-3/2}+9x^{-5/2}y+5x^{-7/2}y$$

$$\Rightarrow T'(x) = 3x^{-3/2} \Rightarrow \boxed{T(x) = -6x^{-1/2}}$$

$$u(x,y) = -6x^{-3/2}y-2x^{-5/2}y^2-6x^{-1/2}$$

Sol gén: $\boxed{-6x^{-3/2}y-2x^{-5/2}y^2-6x^{-1/2} = C}$

$y(2)=-6 \Rightarrow$

$$C = -6\cdot 2^{-3/2}(-6)-2\cdot 2^{-5/2}(-6)^2-6\cdot 2^{-1/2}$$

$\boxed{C = -4.2426}$

La sol.:

$$-6x^{-3/2}y-2x^{-5/2}y^2-6x^{-1/2} = -4.2426$$

$\boxed{1018}$ $(3x^2y^2 - 4xy)y' + 2xy^3 - 2y^2 = 0$

$M = 2xy^3 - 2y^2$ $\quad\quad N = 3x^2y^2 - 4xy$

$M_y = 6xy^2 - 4y$ $\quad\quad N_x = 6xy^2 - 4y$

l'équation est exacte

$dU = Mdx + Ndy$

$U = \int N\,dy + T(x)$

$= \int (3x^2y^2 - 4xy)\,dy + T(x)$

$= x^2y^3 - 2xy^2 + T(x)$

$U_x = 2xy^3 - 2y^2 + T(x)$ et $U_x = M$

$2xy^3 - 2y^2 = 2xy^3 - 2y^2 + T'(x)$

$T'(x) = 0$

$T(x) = 0$

$\cancel{U(x,y) = 2xy^3 - 2y^2 = c}$

$u(x,y) = x^2y^3 - 2xy^2 = c$

Sol. gén.

$x^2y^3 - 2xy^2 = c$

1.22 $\frac{2x}{y^3}dx + \frac{(y^2-3x^2)}{y^4}dy = 0$, $y(1)=1$

$2xy^{-3}dx + (y^{-2} - 3x^2y^{-4})dy = 0$

$M_y = -6xy^{-4} = N_x = -6xy^{-4} \rightarrow$ exacte

$u(x,y) = \int 2xy^{-3}dx + T(y)$

$= \frac{\cancel{2}x^2y^{-3}}{\cancel{2}} + T(y)$

$= x^2y^{-3} + T(y)$

$\frac{du}{dy} = -3x^2y^{-4} + t(y) = N = y^{-2} - 3x^2y^{-4}$

$t(y) = y^{-2} \longrightarrow T(y) = -y^{-1}$

$u(x,y) = x^2y^{-3} - y^{-1}$

solution générale:

$u(x,y) = \frac{x^2}{y^3} - \frac{1}{y} = C$

$\frac{1^2}{1^3} - \frac{1}{1} = 0 \longrightarrow C = 0$ pour $y(1)=1$

solution unique:

$\frac{x^2}{y^3} - \frac{1}{y} = 0$

$\boxed{1.28}$ $(1-x^2y)dx + x^2(y-x)\,dy = 0$

$M = 1 - x^2y \qquad N = x^2(y-x)$

$M_y = -x^2 \qquad N_x = 2xy - 3x^2$

l'équation n'est pas exacte

* Méthode du facteur d'intégration:

$$\frac{M_y - N_x}{N} = \frac{-x^2 - (2xy - 3x^2)}{x^2(y-x)} = \frac{2x^2 - 2xy}{x^2(y-x)}$$

$$= \frac{2x}{-x^2}\frac{\cancel{(x-y)}}{\cancel{(x-y)}} = \frac{-2}{x} = f(x)$$

$$\mu(x) = e^{\int f(x)dx}$$

$$= e^{-2\ln|x|} = e^{\ln|x^{-2}|} = x^{-2} = \frac{1}{x^2}$$

* $\mu M = \frac{1}{x^2} - y \qquad \mu N = y - x$

$\mu M_y = -1 \qquad \mu N_x = -1$ l'équation est exacte

$dU = \mu M dx + \mu N dy$

$$U = \int N dy + T(x)$$

$$= \int (y-x)\,dy + T(x)$$

$$= \frac{y^2}{2} - xy + T(x)$$

$$U_x = -y + T(x) \qquad U_x = uM$$

$$-y + T(x) = \frac{1}{x^2} - y$$

$$T'(x) = \frac{1}{x^2} \qquad \int x^{-2}\,dx \rightarrow \frac{x^{-1}}{-1} = \frac{-1}{x}$$

$$T(x) = \frac{-1}{x}$$

$$u(x,y) = \frac{y^2}{2} - xy - \frac{1}{x} = C$$

$\boxed{8.5}$ Montrez que le point fixe $x_{n+1} = \sqrt{2x_n + 3}$ pour $f(x) = x^2 - 2x - 3 = 0$ converge sur l'intervalle $[2,4]$.

$$f(x) = x^2 - 2x - 3 = 0 \rightarrow (x+1)(x-3) = 0$$

$$\downarrow$$

$$\sqrt{2x_n + 3} = g(x) \qquad \text{racines} = -1 \text{ et } 3$$

Théoreme

1- la dérivée est croissante $x_n = 2 \rightarrow$ minimum

$x_n = 4 \rightarrow$ maximum

$x_n = 2 \quad \sqrt{7} > 2$

$x_n = 4 \quad \sqrt{11} < 4 \quad$ donc contenu dans $[2,4]$

2- $g'(x) = \dfrac{1}{\sqrt{(2x_n + 3)}}$ existe $\forall\, x \in [2,4]$

3- $K = \dfrac{1}{\sqrt{7}}$ la valeur maximale

$0 < K < 1 \rightarrow$ converge

8.8

n	x_n	E_n	E_n/E_{n-1}
0	2	1	1
1	0,540 302	0,459698	0,459698
2	0,896 187	0,103813	0,225829
3	0,994 616	0,005384	0,051862
4	0,999 986	0,000014	0,002600

$\downarrow$ tend vers 1

$p = 1$

$g(x) = \cos(x-1)$

$g'(x) = -\sin(x-1)$ $\quad g'(1) = 0$

$g''(x) = -\cos(x-1)$ $\quad g''(1) = -1 \neq 0$

la 2^e dérivée est non nulle, donc on a convergence d'ordre 2 stricte