Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

Nom / Name : Solution

No d'ét. / Stud. No.:

Test mi-session 2
Durée: 80 min
Place: LPR 155
19 novembre 2010
10:00–11:20

Prof.: Rémi Vaillancourt

MAT 2784 A

Midterm 2
Time: 80 min
Place: LPR 155
19 November 2010
10:00–11:20

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 6 questions sont d'égale valeur.*
The 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Un formulaire se trouve à la fin du questionnaire.*
Formulae are at the end of the test sheets.

1	/10
2	/10
3	/10
4	/10
5	/ ~~10~~ 5
6	/10
Total	/ ~~60~~ 55

+ boni 5

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Qu. 1. *Soit* / given

$$f(x) = x\,e^{x}.$$

Calculer $f'(2)$ par différence centrée,
Compute $f'(2)$ by centered difference,

$$f'(x_0) \approx \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right],$$

avec / with $h_1 = 0.2$, $h_2 = 0.1$ et / and $h_3 = 0.05$:

$$N_1(0.2) = \frac{1}{0.4}\left[f(2.2) - f(1.8)\right] = \boxed{22{,}41416066},$$

$$N_1(0.1) = \frac{1}{0.2}\left[f(2.1) - f(1.9)\right] = \boxed{22{,}22878688},$$

$$N_1(0.05) = \frac{1}{0.1}\left[f(2.05) - f(1.95)\right] = \boxed{22{,}18256486}.$$

Améliorer $f'(2)$ par extrapolation de Richardson :
Improve $f'(2)$ by Richardson's extrapolation :

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = \boxed{22{,}16699562},$$

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = \boxed{22{,}16715752},$$

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = \boxed{22{,}16716831}.$$

Qu. 2. *Résoudre.* / Solve.

$$y'' - y' = e^x \sin x.$$

$C_2 \cdot \frac{1}{2} e^x(-2c_1 + \sin +$

$y_h: \quad y'' - y' = 0$

équation caracteristique $\quad \lambda^2 - \lambda = 0$
$\lambda(\lambda - 1) = 0 \rightarrow \lambda = 0, \ \lambda = 1$

$y_h = c_1 + c_2 e^x$

$y_p = a e^x \sin x + b e^x \cos x$

Par une combinaison linéaire des dérivées de $e^x \sin x$

$y_p' = a e^x \cos x + a e^x \sin x - b e^x \sin x + b e^x \cos x$

$y_p' = (a+b) e^x \cos x + (a-b) e^x \sin x$

$y_p'' = a e^x \cos x - a e^x \sin x + a e^x \sin x + a e^x \cos x - b e^x \sin x - b e^x \cos x + b e^x \cos x - b e^x \sin x$

$y_p'' = 2a e^x \cos x - 2b e^x \sin x$

$y_p'' - y_p' = e^x \sin x$

$e^x \sin x = (2a - (a+b)) e^x \cos x - (2b + (a-b)) e^x \sin x$

$e^x \sin x = (a - b) e^x \cos x - (b + a) e^x \sin x$

$\left.\begin{array}{l} a - b = 0 \\ b + a = -1 \end{array}\right\} \quad a = b, \quad a = -\frac{1}{2} \quad b = -\frac{1}{2}$

$y_p = -\frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x$

$y_g = y_h + y_p$

$$\boxed{y(x) = c_1 + c_2 e^x - \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x}$$

✓

✓

Qu. 2. *Résoudre.* / Solve.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y'' - y' = e^x \sin x.$$

On pose $y = e^{\lambda x}$.
On obtient

$\lambda^2 - \lambda = 0$ $\lambda = \frac{1 \pm \sqrt{1+4.0}}{2}$ $\lambda_{1,2} = 1, 0$

$\lambda(\lambda - 1) = 0 \Rightarrow$ $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases}$

On obtient $y_{homogène}$

$y_h = C_1 e^x + C_2 e^{0x} = C_1 e^x + C_2$

On a $y_p = A(x) e^x + B(x).1$

Par variation des paramètres

$$\begin{bmatrix} e^x & 1 \\ e^x & 0 \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \sin x \end{bmatrix}$$

voir table d'intégration p. 8

$A'e^x + B' = 0$ $B' = -A'e^x$

$A'e^x + 0 = e^x \sin x$ $A = \int A'$ et $B = \int B'$

$A' = \sin x$. $A = \int \sin x \, dx = -\cos x$

$B' = -e^x \sin x$ $B = -\int e^x \sin x \, dx = -\frac{e^x}{2}(\sin x - \cos x)$

Donc $y_p = -e^x \cos x - \frac{e^x}{2}(\sin x - \cos x)$

On sait que $y_g = y_h + y_p$

Donc $\boxed{y_g = C_1 e^x + C_2 - e^x \cos x - \frac{e^x}{2}(\sin x - \cos x)}$

$y_g \; y_p \; y_h = C_1 e^x + C_2 - \frac{e^x \cos x}{2} - \frac{e^x \sin x}{2}$

indices inférieurs

Qu. 3. *Résoudre par Laplace.* / Solve by Laplace transform.

$$y'' + 4y' = u(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

$$s^2 Y(s) - sY(0) - Y'(0) + 4(sY(s) - Y(0)) = \frac{e^{-s}}{s}$$

$$s^2 Y(s) + 4sY(s) = \frac{e^{-s}}{s}$$

$$Y(s)(s^2+4s) = \frac{e^{-s}}{s}$$

$$Y(s) = \frac{e^{-s}}{s^2(s+4)}$$

Par fraction partielle: $\frac{1}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}$

$$1 = As(s+4) + B(s+4) + Cs^2$$

$$1 = As^2 + 4As + Bs + 4B + Cs^2$$

On forme la matrice pour trouver A, B, C

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/16 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/16 \end{array}\right] \Rightarrow \begin{array}{l} A = -\frac{1}{16} \\ B = \frac{1}{4} \\ C = \frac{1}{16} \end{array}$$

On a donc

$$Y(s) = e^{-s}\left(-\frac{1}{16s} + \frac{1}{4s^2} + \frac{1}{16(s+4)}\right)$$

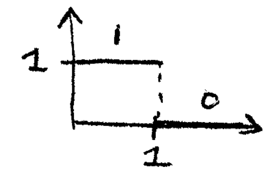
On trouve ainsi par $\mathcal{L}^{-1}$

$$\boxed{y(t) = u(t-1)\left(\frac{-1}{16} + \frac{(t-1)}{4} + \frac{1e^{-4(t-1)}}{16}\right)}$$

Qu. 4. *Résoudre par Laplace.* / Solve by Laplace transform.

$$y'' + 4y = \begin{cases} 1, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \qquad y(0) = 0, \quad y'(0) = -1.$$

$g(t) = 1 - u(t-1) \cdot 1$



$G(s) = \frac{1}{s} - \frac{e^{-s}}{s}$

on sait que

$s^2 Y(s) - s\,Y(0) - Y'(0) + 4Y(s) = \frac{1 - e^{-s}}{s}$

$s^2 Y(s) + 1 + 4Y(s) = \frac{1 - e^{-s}}{s}$

$(s^2+4)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} - 1$

$Y(s) = \frac{1}{(s^2+4)s} - \frac{e^{-s}}{(s^2+4)s} - \frac{1}{(s^2+4)}$

Par fraction partielle :

Pour $\frac{1}{(s^2+4)s} = \frac{As+B}{(s^2+4)} + \frac{C}{s} \Rightarrow s(As+B) + C(s^2+4) = 1$

$As^2 + Bs + Cs^2 + 4C = 1$

On obtient la matrice

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \Rightarrow \begin{array}{l} A = -1/4 \\ B = 0 \\ C = 1/4 \end{array}$$

donc $\frac{1}{(s^2+4)s} = \frac{-1s}{4(s^2+4)} + \frac{1}{4s}$

On obtient donc

$Y(s) = \frac{-s}{4(s^2+4)} + \frac{1}{4s} - e^{-s}\left(\frac{-s}{4(s^2+4)} + \frac{1}{4s}\right) - \frac{1}{s^2+4}$

Donc Par $\mathcal{L}^{-1}$

$$\boxed{y(t) = -\frac{1}{4}\cos(2t) + \frac{1}{4} - u(t-1)\left(-\frac{1}{4}\cos(2(t-1)) + \frac{1}{4}\right) - \frac{1}{2}\sin(2t)}$$

✓

Qu. 5. *Trouver les transformées de Laplace.* / Find the Laplace transforms.

(a) $f(t) = u(t-1)t^2$.

$f(t) = u(t-1)((t-1)+1)^2 = u(t-1)((t-1)^2 + 2(t-1) + 1)$

$$\boxed{\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)}$$

✓

(b) $f(t) = 1 * e^{-2t}$.

$f(t) = \underset{(f*g)(t)}{(1*e^{-2t})(t)} = \int_0^t 1\, e^{-2(t-\tau)}\,d\tau$

On sait que

$\mathcal{L}\{(f*g)\} = \mathcal{L}(f)\mathcal{L}(g)$

$\mathcal{L}\{1*e^{-2t}\} = \mathcal{L}\{1\} * \mathcal{L}\{e^{-2t}\}$

$$= \boxed{\frac{1}{s} \cdot \frac{1}{(s+2)}}$$

✓

Qu. 6. *Trouver h et n pour approcher l'intégrale à 10^{-4} près,*
Find h and n to approximate the integral to 4 decimals,

$$\int_1^2 x^2 \ln x \, dx,$$

$$\frac{(b-a)h^2}{12} f''(\xi)$$

par la méthode des trapèzes / by the composite trapezoidal rule :

$$\int_a^b f(x)\,dx = \frac{h}{2}\sum_{i=1}^{n}[f(x_{i-1}) + f(x_i)] - \frac{h^3}{12}\cancel{\sum_{i=1}^{n}} f''(\xi_i).$$

$f = x^2 \ln x \, dx$

$f' = 2x \ln x + x$

$f'' = 2 \ln x + 2 + 1$

Max sur 1 à 2 : $2\ln 2 + 3 = M$

On sait que $b = 2$ et $a = 1$

donc

$$\frac{(b-a)h^2 . M}{12} < 10^{-4}$$

$$\frac{h^2(2\ln 2+3)}{12} < 10^{-4}$$

$h < 0{,}0165402375$

On sait que $nh = b - a$

$$n = \frac{1}{0{,}0165402375} = 60{,}486247$$

Donc on prend $\boxed{n = 61}$

$$\boxed{h = \frac{1}{n} = \frac{1}{61}} = 0{,}016393$$

Integration

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$

Laplace Transform: General Formulas

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$	Definition of Transform
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$	Differentiation of Transform
$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p

$$\sin x \sin y = \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^2}$	te^{at}
$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1}e^{at}$
$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$
$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(e^{at} - e^{bt})$
$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega} \sin \omega t$
$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a} \sinh at$
$\dfrac{s}{s^2-a^2}$	$\cosh at$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega} e^{at} \sin \omega t$
$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$
$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$
$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$
$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$
$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$
$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$
$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$
$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$
$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2} t\right)$
$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$
$\dfrac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)} \left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$
$\dfrac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$
$\dfrac{1}{\sqrt{s}} e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$
$\dfrac{1}{s^{3/2}} e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$
$e^{-k\sqrt{s}} \quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$
$\dfrac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$
$\ln \dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt} - e^{at})$

INTEGRATING FACTORS

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \quad \mapsto \quad \exp\left(\int f(x)\,dx\right)$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y) \quad \to \quad \exp\left(-\int g(y)\,dy\right)$$

FOR $\quad M(x,y)\,dx + N(x,y)\,dy = 0.$