Nom / Name : SOLUTIONS

No d'ét. / Stud. No.: JAUNES

Test mi-session 1
Durée: 80 min
Place: LPR 155
22 octobre 2010
10:00–11:20

Prof.: Rémi Vaillancourt

MAT 2784 A

Midterm 1
Time: 80 min
Place: LPR 155
22 October 2010
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Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 6 questions sont d'égale valeur.*
The 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Un formulaire se trouve à la fin du questionnaire.*
Formulae are at the end of the test sheets.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

L'équation différentielle homogène du 1er ordre, / The first-order homogeneous ODE,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet le facteur d'intégration / admits the integrating factor

$$\mu(x) = e^{\int f(x)\,dx} \qquad \text{si/if} \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x),$$

ou / or

$$\mu(y) = e^{-\int g(y)\,dy} \qquad \text{si/if} \qquad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y).$$

Qu. 1. (a) *Soit* / Consider $f(x) = x^3 - 5x^2 + 8x - 4.$
Calculer / Compute

$f(2) = \boxed{0}$

$f'(2) = \boxed{0}$

$f''(2) = \boxed{2}$

$f'(x) = 3x^2 - 10x + 8$

$f''(x) = 6x - 10$

(b) *Quelle est la multiplicité* m *du zéro* $x = 2$ de $f(x)$?
What is the multiplicity m of the zero $x = 2$ of $f(x)$?

$m = \boxed{2}$

(c) *Itérer deux fois à 6 décimales la méthode newtonienne modifiée avec* m *en (b)* :
Iterate twice to 6 decimals Newton's modified method with m in (b):

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 2.5.$$

$x_1 = \boxed{2.071429}$

$x_2 = \boxed{2.002304}$

(d) *Quelle est l'ordre de convergence* p *de la méthode en (c)* ?
What is the order of convergence p of the method in (c)?

$p = \boxed{2}$

✓

Qu. $\cancel{3}$ 2. *Trouver la solution générale.* / Find the general solution.

$$y'' - 2y' + y = \frac{e^x}{x}.$$

$y_h \quad y = e^{\lambda}$

$$\lambda^2 e^{\lambda} - 2\lambda e^{\lambda} + e^{\lambda} = 0$$
$$e^{\lambda}(\lambda^2 - 2\lambda + 1) = 0$$
$$(\lambda - 1)^2 = 0$$
$$\lambda_{1,2} = 1$$

$$y_h = Ae^x + Bxe^x$$

$$y_p = C_1(x)e^x + C_2(x)xe^x$$

$$\begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} \begin{vmatrix} C_1'(x) \\ C_2'(x) \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{e^x}{x} \end{vmatrix}$$

$\ell_2 - \ell_1$
$$\begin{vmatrix} e^x & xe^x \\ 0 & e^x \end{vmatrix} \begin{vmatrix} C_1'(x) \\ C_2'(x) \end{vmatrix} = \begin{matrix} 0 \\ \frac{e^x}{x} \end{matrix}$$

$$e^x \cdot C_2'(x) = \frac{e^x}{x}$$
$$C_2'(x) = \frac{1}{x}$$
$$C_2(x) = \ln|x|$$

$$C_1'(x)e^x + C_2'(x)xe^x = 0$$
$$C_1'(x)e^x + e^x = 0$$
$$C_1'(x) = -1$$
$$C_1(x) = -x$$

sl gen.

$$\boxed{y = Ae^x + Bxe^x + xe^x(\ln|x| - 1)}$$ ✓

Qu. 3. *Résoudre.* / Solve.

$$y'' + 9y = 3\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

$y = e^{\lambda}$

$\lambda^2 e^{\lambda} + 9e^{\lambda} = 0$

$e^{\lambda}(\lambda^2 + 9) = 0$

$\lambda = \pm 3i$

$Y_h = A\cos 3x + B\sin 3x$

$Y_p = a\cos x + b\sin x$

$Y_p' = -a\sin x + b\cos x$

$Y_p'' = -a\cos x - b\sin x$

$-a\cos x - b\sin x + 9(a\cos x + b\sin x)$

$8a\cos x + 8b\sin x = 3\cos x$

$a = \frac{3}{8} \quad b = 0$

$Y_p = \frac{3}{8}\cos x$

$Y = A\cos 3x + B\sin 3x + \frac{3}{8}\cos x$

$y(0) = 1$

$1 = A + \frac{3}{8}$

$A = \frac{5}{8}$

$Y'(x) = -3A\sin 3x + 3B\cos 3x - \frac{3}{8}\sin x$

$Y'(0) = 0$

$0 = 3B$

$B = 0$

Sol: $\boxed{Y = \frac{5}{8}\cos 3x + \frac{3}{8}\cos x}$ ✓

Qu. 4. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$y'' - 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 3.$$

$y = e^{\lambda}$

$\lambda^2 e^{\lambda} - 2\lambda e^{\lambda} + 2e^{\lambda} = 0$

$e^{\lambda}(\lambda^2 - 2\lambda + 2) = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4(2)}}{2}$

$\lambda = 1 \pm i$

$y = C_1 e^{x} \cos x + C_2 e^{x} \sin x$

$y(0) = 1$

$1 = C_1(1)(1) + C_2(1)(0)$

$C_1 = 1$

$y'(x) = C_1(e^{x}\cos x - e^{x}\sin x) + C_2(e^{x}\sin x + e^{x}\cos x)$

$y'(0) = 3$

$3 = C_1(1 - 0) + C_2(0 + 1)$

$3 = 1 + C_2$

$C_2 = 2$

$\boxed{Y = e^{x}\cos x + 2e^{x}\sin x}$ ✓

Qu. 5. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$$(1 - x^2y)\,dx + x^2(y - x)\,dy = 0, \qquad y(1) = 2.$$

$M_y = -x^2 \qquad N_x = 2xy - 3x^2$

$$f(x) = \frac{M_y - N_x}{N} = \frac{2x^2 - 2xy}{x^2y - x^3}$$

$$f(x) = \frac{2x(x-y)}{-x^2(x-y)}$$

$$f(x) = \frac{-2}{x}$$

$$\mu(x) = e^{-2\int \frac{1}{x}dx}$$

$$\mu(x) = \frac{1}{x^2}$$

$$\left(\frac{1}{x^2} - y\right)dx + (y-x)\,dy = 0$$

$M_y = -1 \qquad N_x = -1$

$du = Mx + Ny$

$$u = \int\left(\frac{1}{x^2} - y\right)dx + T(y)$$

$$u = \frac{-1}{x} - xy + T(y)$$

$u_y = -x + T'(y)$

$u_y = N \therefore T'(y) = y$

$$T(y) = \frac{y^2}{2}$$

Sol gen.

$$C = \frac{y^2}{2} - \frac{1}{x} - xy$$

Qu. 6. (a) *Itérer 4 fois la récurrence de point fixe à 5 décimales près.*
Iterate 4 times the fixed point recurrence. Use at least 5 decimals.

$$x_{n+1} = g(x_n), \quad x_0 = 2, \quad \text{avec / with} \quad g(x) = \frac{5}{x^2} + 2.$$

et calculer l'erreur si : / and compute the error if: $p = g(p) = 2.690647$.

$x_1 = \boxed{3.25}$ $x_1 - p = \boxed{0.559353}$

$x_2 = \boxed{2.47337}$ $x_2 - p = \boxed{-0.217274}$

$x_3 = \boxed{2.81732}$ $x_3 - p = \boxed{0.126671}$

$x_4 = \boxed{2.629939}$ $x_4 - p = \boxed{-0.060708}$

$x_5 = \boxed{2.722900}$ $x_5 - p = \boxed{0.0322534}$

$g'(x) = \frac{-10}{x^3}$

(b) *Calculer* / Compute

$g'(x_5) = \boxed{-0.495342} \approx g'(p) = -0.513369.$

$0 < g'(x_5) < 1$

(c) *Quel est l'ordre p de convergence de la méthode ?*
What is the order p of convergence of the method?

$p = \boxed{1}$.

✓