Nom / Name : SOLUTIONS

No d'ét. / Stud. No.: BLANC

Test mi-session 1
Durée: 80 min
Place: LPR 155
22 octobre 2010
10:00–11:20

Prof.: Rémi Vaillancourt

MAT 2784 A

Midterm 1
Time: 80 min
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Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.
(b) *Répondre sur le questionnaire.*
Answer on the question sheets.
(c) *Les 6 questions sont d'égale valeur.*
The 6 questions have the same value.
(d) *Donner le détail de vos calculs.*
Show all computation.
(e) *Un formulaire se trouve à la fin du questionnaire.*
Formulae are at the end of the test sheets.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

L'équation différentielle homogène du 1er ordre, / The first-order homogeneous ODE,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet le facteur d'intégration / admits the integrating factor

$$\mu(x) = e^{\int f(x)\,dx} \qquad \text{si/if} \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x),$$

ou / or

$$\mu(y) = e^{-\int g(y)\,dy} \qquad \text{si/if} \qquad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y).$$

Qu. 1. (a) *Itérer 4 fois la récurrence de point fixe à 5 décimales près.*
Iterate 4 times the fixed point recurrence. Use at least 5 decimals.

$$x_{n+1} = g(x_n), \quad x_0 = 2, \quad \text{avec / with} \quad g(x) = \frac{5}{x^2} + 2.$$

et calculer l'erreur si : / and compute the error if: $p = g(p) = 2.690647$.

$x_1 = \boxed{3,25}$ $\quad x_1 - p = \boxed{0,559353}$

$x_2 = \boxed{2,473372781}$ $\quad x_2 - p = \boxed{-0,217274}$

$x_3 = \boxed{2,817317548}$ $\quad x_3 - p = \boxed{0,126671}$

$x_4 = \boxed{2,629938843}$ $\quad x_4 - p = \boxed{-0,060708}$

$x_5 = \boxed{2.722900}$ $\quad x_5 - p = \boxed{0.0322534}$

$g'(x) = -\frac{10}{x^3}$

(b) *Calculer* / Compute

$$g'(x_5) = \boxed{-0,495342} \approx g'(p) = -0.513369.$$

(c) *Quel est l'ordre p de convergence de la méthode ?*
What is the order p of convergence of the method?

$p = \boxed{1}$.

$0 < |g'(p)| < 1$

↳ ordre 1

Qu. 2. *Résoudre.* / Solve.

$$y'' + 4y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

✓

$L y_h = y'' + 4y = 0$

On pose : $y(x) = e^{\lambda x}$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda_{1,2} = \pm 2i$

$y_1(x) = A\cos 2x$

$y_2(x) = B\sin 2x$

$y_h = A\cos 2x + B\sin 2x$

On pose : $y_p = a\cos x + b\sin x$

$y'_p = -a\sin x + b\cos x$

$y''_p = -a\cos x - b\sin x$

$L y_p = y''_p + 4y_p = -a\cos x - b\sin x + 4a\cos x + 4b\sin x = 2\cos x$

$= 3a\cos x + 3b\sin x = 2\cos x$

$3a = 2$ $\quad \boxed{b = 0}$

$\boxed{a = 2/3}$

$y_p = \frac{2}{3}\cos x$

Alors,

Solution générale. $\boxed{y(x) = A\cos 2x + B\sin 2x + \frac{2}{3}\cos x}$

valeurs initiales : $y'(x) = -2A\sin 2x + 2B\cos 2x - \frac{2}{3}\sin x$

$1 = A + \frac{2}{3}$ $\quad \boxed{A = \frac{1}{3}}$

$0 = 2B \rightarrow B = 0$

Donc,

$\boxed{y(x) = \frac{1}{3}\cos 2x + \frac{2}{3}\cos x}$ Solution unique

✓

Qu. 3. *Trouver la solution générale.* / Find the general solution. ✓

$$y'' - 2y' + y = \frac{e^x}{x}.$$

$Ly_h = y'' - 2y' + y = 0$

On pose $y = e^{\lambda x}$

alors

$$\lambda^2 - 2\lambda + 1 = 0$$
$$\lambda^2 - \lambda - \lambda + 1 = 0$$
$$\lambda(\lambda - 1) - 1(\lambda - 1) = 0$$
$$(\lambda - 1)^2 = 0$$
$$\lambda_1 = \lambda_2 = 1$$

$y_1 = Ae^x$

$y_2 = Bxe^x$

$y_h = Ae^x + Bxe^x$

On pose $y_p = C_1(x)e^x + C_2(x)xe^x$

Wronskien :

$$e^{-x}\begin{pmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{pmatrix}\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^x/x \end{pmatrix}\cdot e^{-x}$$

$$L_2 - L_1 \to L_2 \quad \begin{pmatrix} 1 & x \\ 1 & 1+x \end{pmatrix}\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 1/x \end{pmatrix}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 1/x \end{pmatrix}$$

$$C_1' + xC_2' = 0$$
$$\boxed{C_2' = \frac{1}{x}} \to \boxed{C_1' = -1}$$

$C_1 = -x \quad C_2 = \ln|x|$

$$y_p = -xe^x + \ln|x|\,xe^x$$ ✓

$$\boxed{y(x) = Ae^x + Bxe^x - xe^x + \ln|x|\,xe^x}$$ Solution générale

Ou bien

$$y(x) = Ae^x + Cxe^x + \ln|x|\,xe^x \quad \text{où} \quad C \neq R$$

Qu. 4. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$y'' - 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

On pose $y = e^{\lambda x}$

alors,

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i$$

$y_1 = Ae^x \cos x$

$y_2 = Be^x \sin x$

Sol. générale $\boxed{y(x) = Ae^x \cos x + Be^x \sin x}$

Valeurs initiales:

$$y'(x) = A(e^x \cos x - e^x \sin x) + B(e^x \sin + e^x \cos x)$$

$$\cancel{e}^0 \times 1 = A \quad \rightarrow \boxed{A = 1}\cancel{/e}$$

$$2 = \cancel{e}A + Be \quad \boxed{B = 1}\cancel{/e}$$

Sol. unique. $\boxed{y(x) = e^x(\cos x + \sin x)}$

e (

2

Qu. 5. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$$\overset{M}{(1-x^2y)}\,dx+\overset{N}{\underset{x^2y-x^3}{x^2(y-x)}}\,dy=0,\qquad y(1)=2.$$

$M_y=-x^2 \neq N_x=2xy-3x^2$ pas Exacte!

Alors on trouve

$$f(x)=\frac{M_y-N_x}{N}=\frac{-x^2-2xy+3x^2}{x^2y-x^3}=\frac{-\cancel{x}(4x-2y)}{x^{\cancel{2}}(x-y)}$$

$$=-\frac{2}{x}$$

Facteur Integrant

$$\mu(x)=e^{\int\frac{-2}{x}dx}=x^{-2}$$

On multiple l'Eq. Diff. par x^{-2}:

$$(x^{-2}-y)\,dx+(y-x)\,dy=0$$

$M_y=-1 = N_x=-1$ Exacte

Donc

$$U(x,y)=\int N\,dy+T(x)$$

$$=\frac{y^2}{2}-xy+T(x)$$

$\frac{\partial U}{\partial x}=-y+T'(x)=M \rightarrow$ On déduit alors $T'(x)=x^{-2}$

$T(x)=-x^{-1}$

Alors, $U(x,y)=\boxed{\frac{y^2}{2}-xy-x^{-1}=C}$ sol. générale

Valeurs intiales:

$$\frac{4}{2}-2-1=c \quad \rightarrow c=-1$$

Solution unique: $\boxed{\frac{y^2}{2}-xy-\frac{1}{x}=-1}$ ✓

Qu. 6. (a) *Soit* / Consider $f(x) = x^3 - 5x^2 + 8x - 4.$
Calculer / Compute

$f'(x) = 3x^2 - 10x + 8$
$f''(x) = 6x - 10$

$f(2) = \boxed{0}$

$f'(2) = \boxed{0}$

$f''(2) = \boxed{2}$

(b) *Quelle est la multiplicité m du zéro $x = 2$ de $f(x)$?*
What is the multiplicity m of the zero $x = 2$ of $f(x)$?

$m = \boxed{2}$

(c) *Itérer deux fois à 6 décimales la méthode newtonienne modifiée avec m en (b) :*
Iterate twice to 6 decimals Newton's modified method with m in (b):

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 2.5.$$

$x_1 = 2,5 - 2\,\dfrac{(2,5^3 - 5(2,5)^2 + 8(2,5) - 4)}{(3(2,5)^2 - 10 \cdot 2,5 + 8)}$

$x_1 = \boxed{2,071429}$

$x_2 = \boxed{2,002304}$

$x_2 = x_1 - 2\,\dfrac{(x_1^3 - 5x_1^2 + 8x_1 - 4)}{(3x_1^2 - 10 \cdot x_1 + 8)}$

(d) *Quelle est l'ordre de convergence p de la méthode en (c) ?*
What is the order of convergence p of the method in (c)?

$p = \boxed{2}$

La Méthode Newton modifiée converge d'ordre 2.

Integration

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$

Laplace Transform: General Formulas

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$	Definition of Transform
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$	Differentiation of Transform
$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p

$$\sin x \sin y = \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]$$