D7.1

Vendredi 26 novembre 2010
MAT 2784 A.

RÉMI
VAILLANCOURT

Devoir 7

5.43 (convolution) $y'' + y = \sin t$ $y(0) = 0, y'(0) = 0$

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin t)$

$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$

$Y(s)(s^2+1) = \frac{1}{s^2+1}$ $Y(s) = \frac{1}{(s^2+1)(s^2+1)}$

$y(t) = \sin t * \sin t$

$\int_0^t \sin\tau \sin(t-\tau)\, d\tau$ → identité trigonométrique :
$\sin x \sin y = \frac{1}{2}\left[-\cos(x+y) + \cos(x-y)\right]$

$\frac{1}{2}\int_0^t -\cos(t) + \cos(2\tau - t)\, d\tau$

$u = 2\tau - t$
$\frac{du}{2} = d\tau$

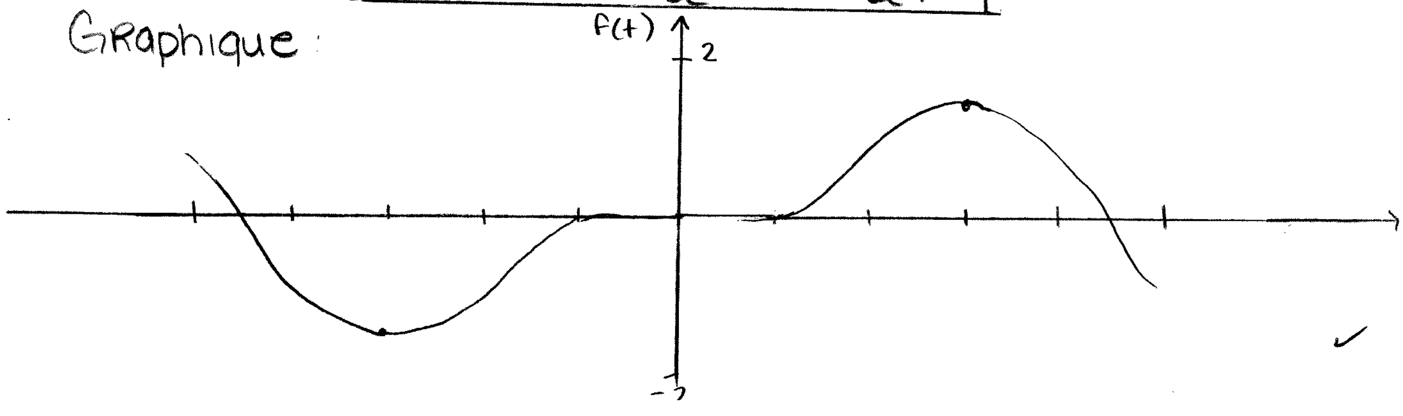
$\left. \frac{-\tau\cos t}{2} + \frac{\sin(2\tau - t)}{4} \right|_0^t$

$-\frac{t\cos t}{2} + \frac{\sin t}{4} - \frac{\sin(-t)}{4}$

$-\frac{t\cos t}{2} + \frac{\sin t}{4} + \frac{\sin t}{4} \to -\frac{t\cos t}{2} + \frac{\cancel{2}\sin t}{\cancel{4}\,2}$

$\mathcal{L}^{-1}(F(s)) = \boxed{f(t) = -\frac{t\cos t}{2} + \frac{\sin t}{2}}$ v. Table

Graphique :



5.52. $y'' + 3y' + 2y = 1 - u(t-1) \quad y(0)=0 \quad y'(0)=1$

$\mathcal{L}(y'') + \mathcal{L}(3y') + \mathcal{L}(2y) = \mathcal{L}(1) - \mathcal{L}(u(t-1))$

$S^2Y(S) - Sy(0) - y'(0) + 3SY(S) - 3y(0) + 2Y(S) = G(S)$

$Y(S)(S^2+3S+2) - 1 = G(S)$

$G(S) = \frac{1}{S} - \frac{e^{-S}}{S}$

$$Y(S) = \frac{\frac{1}{S} - \frac{e^{-S}}{S} + 1}{(S^2+3S+2)}$$

$$Y(S) = \frac{1}{S(S+2)(S+1)} - \frac{e^{-S}}{S(S+2)(S+1)} + \frac{1}{(S+2)(S+1)}$$

Fractions Simples: $\frac{A}{S} + \frac{B}{S+2} + \frac{C}{S+1} = \frac{1}{S(S+2)(S+1)}$

$A(S^2+3S+2) + B(S^2+S) + C(S^2+2S) = 1$

S^2: $A+B+C=0$ $\qquad B = -C - 1/2$

S^1: $3A + B + 2C = 0$ $\qquad 3/2 - C - 1/2 + 2C = 0$

S^0: $2A = 1$ $\qquad \boxed{A = 1/2}$ $\qquad 2/2 = -C$ $\qquad \boxed{C=-1}$

$\boxed{B = 1/2}$

$\frac{D}{(S+2)} + \frac{E}{(S+1)} = \frac{1}{(S+2)(S+1)}$ $\qquad DS + D + ES + 2E = 1$

$\boxed{D=-1}$ $\boxed{E=1}$

$$Y(S) = \left(\frac{1}{2S} + \frac{1}{2(S+2)} - \frac{1}{(S+1)}\right) - e^{-S}\left(\frac{1}{2S} + \frac{1}{2(S+2)} - \frac{1}{S+1}\right) + \left(\frac{1}{S+1} - \frac{1}{S+2}\right)$$

$\mathcal{L}^{-1}(Y(S)) = y(t) =$

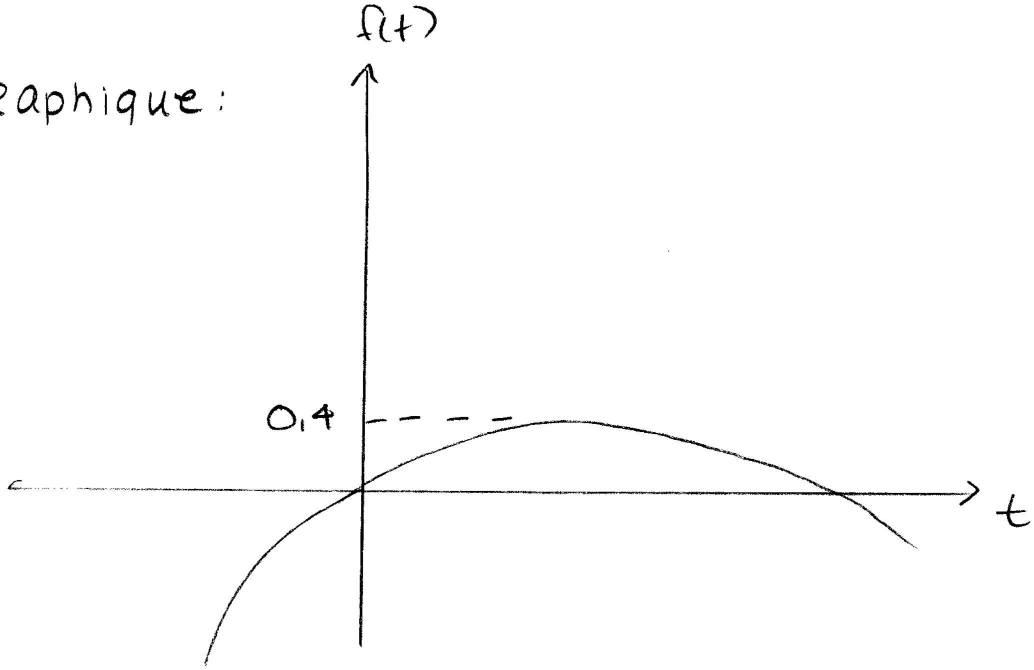
$$\left(\frac{1}{2} + \frac{e^{-2t}}{2} - \cancel{e^{-t}}\right) - u(t-1)\left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right) + \cancel{e^{-t}} - e^{-2t}$$

Réponse:

$$\boxed{\frac{1}{2} - \frac{e^{-2t}}{2} - u(t-1)\left(\frac{1}{2} + \frac{e^{-2(t-1)}}{2} - e^{-(t-1)}\right)}$$

✓

Graphique:



5.53 $y'' - y = \sin t + \delta(t - \pi/2)$ $\quad y(0) = 3,5$
$y'(0) = -3,5$.

$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(\sin t) + \mathcal{L}(\delta(t-\pi/2))$

$s^2 Y(s) - s\,y(0) - y'(0) - Y(s) = G(s)$

$Y(s)(s^2-1) - 3{,}5s + 3{,}5 = G(s)$

$G(s) = \dfrac{1}{s^2+1} + e^{-(\pi/2)s}$

$$Y(s) = \frac{3{,}5s}{(s^2-1)} - \frac{3{,}5}{(s^2-1)} + \frac{1}{(s^2+1)(s^2-1)} + \frac{e^{-(\pi/2)s}}{(s^2-1)}$$

Fractions simples $\rightarrow$

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1} = \frac{1}{(s^2+1)(s^2-1)}$$

$(As+B)(s^2-1) + (Cs+D)(s^2+1) = 1$

$As^3 - As + Bs^2 - B + Cs^3 + Cs + Ds^2 + D = 1$

$s^3: A + C = 0 \qquad s^1: -A + C = 0 \qquad 2B = -1 \qquad \boxed{D = \frac{1}{2}}$

$s^2: B + D = 0 \qquad s^0: -B + D = 1 \qquad D = B + 1$

$A = -C \qquad 2C = 0 \rightarrow \boxed{A = 0, C = 0} \qquad \boxed{B = \frac{-1}{2}}$

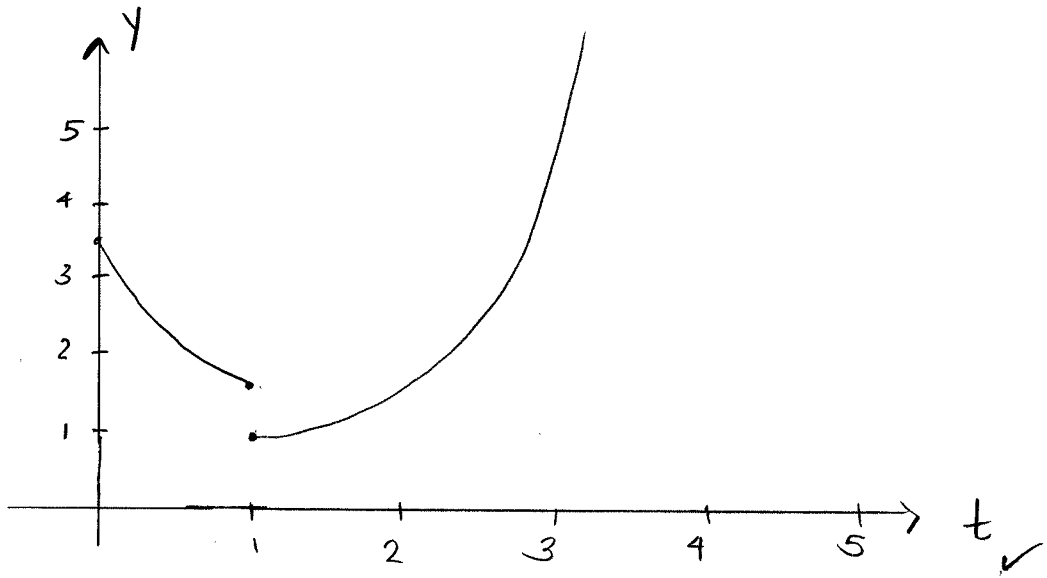
$$Y(s) = \frac{3{,}5s}{s^2-1} - \frac{3{,}5}{s^2-1} - \frac{1}{2(s^2+1)} + \frac{1}{2(s^2-1)} + \frac{e^{-(\pi/2)s}}{s^2-1}$$

$\mathcal{L}^{-1}(Y(s)) = y(t) \rightarrow$

ensemble

$$\boxed{y(t) = 3{,}5\cosh t - 3{,}5\sinh t - \frac{\sin t}{2} + \frac{\sinh t}{2} + u\left(t-\frac{\pi}{2}\right)\sinh\left(t-\frac{\pi}{2}\right)}$$

Graphique:



5,56. $y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$.

$\mathcal{L}(\sin t) + \mathcal{L}\left(\int_0^t y(\tau)\sin(t-\tau)\,d\tau\right)$

$= \frac{1}{s^2+1} + y * \sin t$

$Y(s) = \frac{1}{s^2+1} + Y(s)\cdot\frac{1}{s^2+1}$

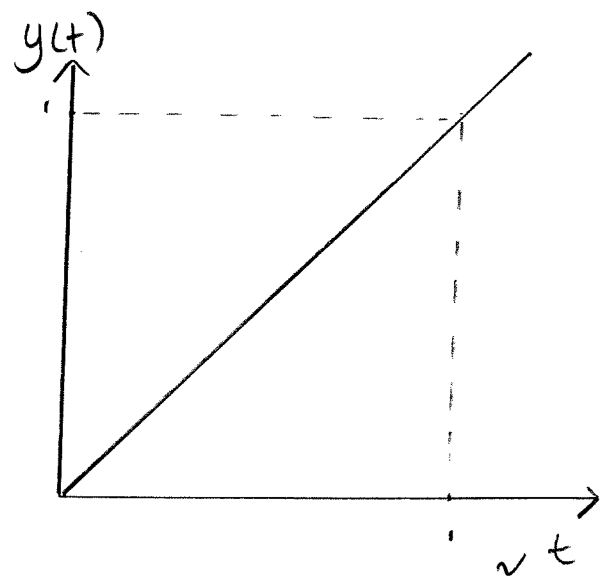
$Y(s) - \frac{Y(s)}{s^2+1} = \frac{1}{s^2+1}$

$\frac{(s^2+1)Y(s) - Y(s)}{\cancel{s^2+1}} = \frac{1}{\cancel{s^2+1}}$

$Y(s)(s^2+1-1) = 1$

$Y(s) = \frac{1}{s^2}$

$\mathcal{L}^{-1}(Y(s)) = y(t)$.



$\boxed{\text{Réponse: } y(t) = t}$

5.60. $f(t) = \pi - t \quad 0 < t < 2\pi.$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}(\pi - t)\,dt$$

Intégration par parties $\int u\,dv = uv - \int v\,du$

$u = \pi - t \quad du = -dt \rightarrow -du = dt$
$dv = e^{-st}dt \quad v = -\frac{e^{-st}}{s}$

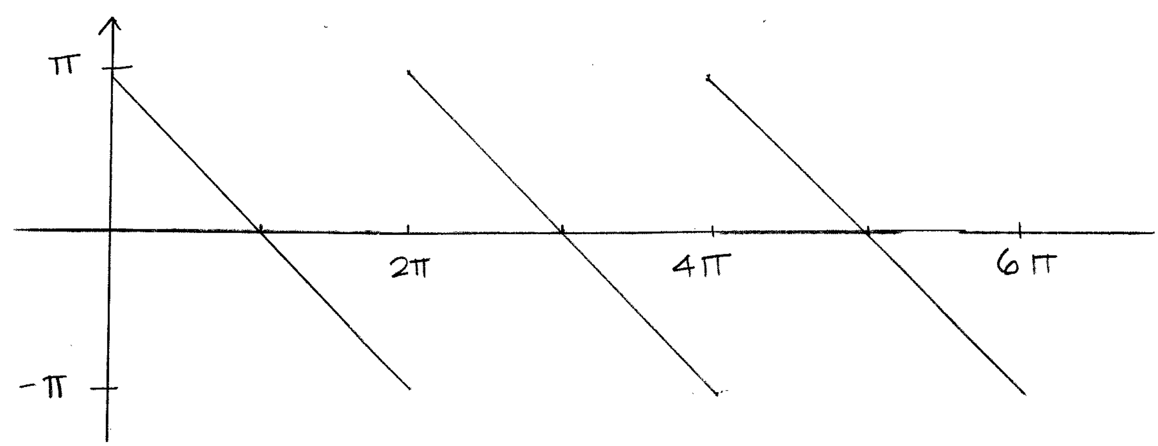
$$\int_0^{2\pi} e^{-st}(\pi - t)dt = (\pi - t)\left(-\frac{e^{-st}}{s}\right) - \int_0^{2\pi} \frac{e^{-st}}{s}dt$$

$$(\pi - t)\left(-\frac{e^{-st}}{s}\right)\Big|_0^{2\pi} + \frac{e^{-st}}{s^2}\Big|_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left(+\frac{\pi e^{-2\pi s}}{s} + \frac{\pi}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{1}{s^2}\right)$$

$$\boxed{F(s) = \frac{1}{1-e^{-2\pi s}}\left(+\frac{\pi e^{-2\pi s}}{s} + \frac{\pi}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{1}{s^2}\right)}$$

Graphique



5.63. $f(t) = \begin{cases} t, & \text{si } 0 < t < \pi \\ \pi - t, & \text{si } \pi < t < 2\pi \end{cases}$ Fonction périodique

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-2\pi s}}\left(\underbrace{\int_0^{\pi} e^{-st}t\,dt}_{1} + \underbrace{\int_{\pi}^{2\pi} e^{-st}(\pi - t)dt}_{2}\right)$$

1 → par parties

$u = t \quad du = dt \quad dv = e^{-st}dt \quad v = -\frac{e^{-st}}{s}$

$$-\frac{te^{-st}}{s}\Bigg|_0^{\pi} + \int_0^{\pi} \frac{e^{-st}}{s}\,dt = -\frac{te^{-st}}{s}\Bigg|_0^{\pi} - \frac{e^{-st}}{s^2}\Bigg|_0^{\pi}$$

$$= -\frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2}$$

2 → par parties: $u = \pi - t \quad du = -dt \rightarrow -du = dt$

$dv = e^{-st}\,dt \quad v = -\frac{e^{-st}}{s}$

$$\int_{\pi}^{2\pi} e^{-st}(\pi - t)\,dt = (\pi - t)\left(-\frac{e^{-st}}{s}\right) - \int_{\pi}^{2\pi} \frac{e^{-st}}{s}\,dt$$

$$(\pi - t)\left(-\frac{e^{-st}}{s}\right)\Bigg|_{\pi}^{2\pi} + \frac{e^{-st}}{s^2}\Bigg|_{\pi}^{2\pi}$$

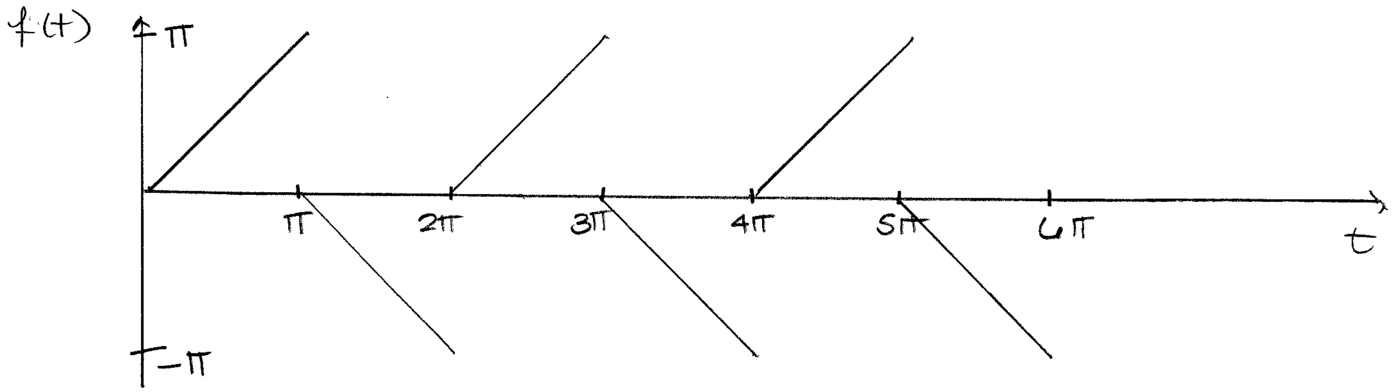
$$= +\frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}$$

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-2\pi s}}\left(-\frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}\right)$$

Réponse:

$$\boxed{F(s) = \frac{1}{1-e^{-2\pi s}}\left(-\frac{\pi e^{-\pi s}}{s} - \frac{2e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2}\right)}$$

Graphique:



10.9. $y' = x^2 + y^2 \quad y(0) = 1$

4 décimales près sur $0 \leq x \leq 1$
méthode d'Euler améliorée
4 pas. $h = 0{,}1$.

$F = x^2 + y^2$
$x_n = x_0 + hn$
$= 0 + 0{,}1n = 0{,}1n \quad y_0^c = 1$

algorithme :
$y_{n+1}^p = y_n^c + h f(x_n, y_n^c)$
$y_{n+1}^c = y_n^c + \frac{1}{2} h [f(x_n, y_n^c) + f(x_{n+1}, y_{n+1}^p)]$

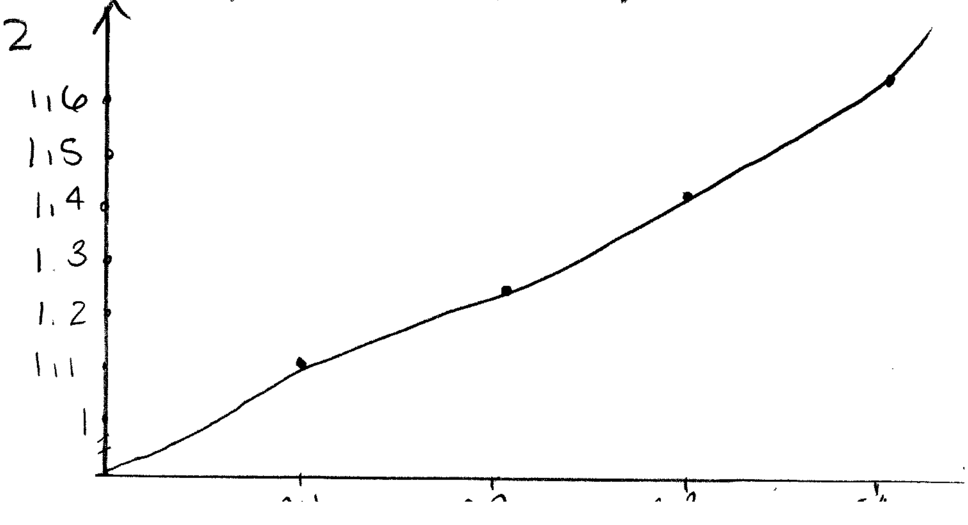
n	x_n	y_n^p	y_n^c
0	0	—	1,0000
1	0,1	1,1000	1,1110
2	0,2	1,2354	1,2515
3	0,3	1,4121	1,4360
4	0,4	1,6512	1,6879

$x_0 = 0 \quad x_1 = 0{,}1(1) = 0{,}1 \quad x_2 = 0{,}1(2) = 0{,}2$
$x_3 = 0{,}1(3) = 0{,}3 \quad x_4 = 0{,}1(4) = 0{,}4$

$y_1^p = 1 + 0{,}1 f(0,1) = 1{,}1000$
$y_1^c = 1 + \frac{0{,}1}{2}[f(0,1) + f(0{,}1, 1{,}1)] =$
$y_2^p = 1{,}111 + 0{,}1 f(0{,}1, 1{,}111) = 1{,}2354321$
$y_2^c = 1{,}111 + 0{,}05[f(0{,}1, 1{,}111) + f(0{,}2, 1{,}2354)] = 1{,}25152670$

$y_3^p = 1{,}2515 + 0{,}1 f(0{,}2, 1{,}2515) = 1{,}412125225$
$y_3^c = 1{,}2515 + 0{,}05[f(0{,}2, 1{,}2515) + f(0{,}3, 1{,}4121)]$
$= 1{,}4360013933$
$y_4^p = 1{,}4360 + 0{,}1 f(0{,}3, 1{,}4360) = 1{,}6512096.$
$y_4^c = 1{,}4360 + 0{,}05[f(0{,}3, 1{,}4360) + f(0{,}4, 1{,}6512)]$
$= 1{,}6879927872$

Graphique :



10.12. $y' = x + \sin y$ $\quad y(0) = 0$ $\quad$ 4 décimales près $0 \leq x \leq 1$

méthode de Runge-Kutta ordre 4 avec $h = 0{,}1$.

$f(x, y) = x + \sin y$

$x_n = 0 + 0{,}1 n$

$n \leq 4$. (4 pas)

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

	C	A			
K_1	0	0			
K_2	1/2	1/2	0		
K_3	1/2	0	1/2	0	
K_4	1	0	0	1	0
y_{n+1}	b^T	1/6	2/6	2/6	1/6

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + c_2 h, y_n + a_{21} K$

$K_3 = hf(x_n + c_3 h, y_n + a_{32} K$

$K_4 = hf(x_n + c_4 h, y_n + a_{43} K_3$

n	x_n	y_n	K_1	K_2	K_3	K_4
0	0	0	0	0,0050	0,0052	0,0105
1	0,1	0,0052	0,0105	0,0160	0,0163	0,0221
2	0,2	0,0214	0,0221	0,0282	0,0285	0,0350
3	0,3	0,0498	0,0350	0,0417	0,0421	0,0492
4	0,4	0,0918	0,0492	0,0566	0,0570	0,0648

$n = 0$ $\quad K_1 = 0{,}1 f(0, 0) = 0$

$K_2 = 0{,}1 f(0{,}05, 0) = 0{,}005$

$K_3 = 0{,}1 f(0{,}05, 0{,}0025) = 0{,}005249999$

$K_4 = 0{,}1 f(0{,}1, 0{,}005) = 0{,}010499997$

$n = 1$ $\quad K_1 = 0{,}1 f(0{,}1, 0{,}0052) = 0{,}010519997$

$K_2 = 0{,}1 f(0{,}15, 0{,}01045) = 0{,}016004498$

$K_3 = 0{,}1 f(0{,}15, 0{,}0132) = 0{,}016031996$

$K_4 = 0{,}1 f(0{,}2, 0{,}0215) = 0{,}022149834$

Exemple de calculs.

Graphique: