12 novembre 2010
MAT 2784 A.

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Devoir 6

5.22. $F(S) = \dfrac{1}{(S-2)(S^2+4S+3)} = \dfrac{1}{(S-2)(S+1)(S+3)}$

$$\frac{A}{(S-2)} + \frac{B}{(S+1)} + \frac{C}{(S+3)} = \frac{1}{(S-2)(S+1)(S+3)}$$

$$(S+1)(S+3)A + (S-2)(S+3)B + (S-2)(S+1)C = 1$$
$$(S^2+4S+3)A + (S^2+S-6)B + (S^2-S-2)C = 1$$
$$AS^2 + BS^2 + CS^2 = 0 \quad A+B+C=0 \quad 3A-6B-2C=1$$
$$4SA + BS - CS = 0 \quad 4A+B-C=0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 1 & -1 & 0 \\ 3 & -6 & -2 & 1 \end{array}\right] \begin{array}{l} L_3-3L_1 \to L_3 \\ L_2-4L_1 \to L_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & -9 & -5 & 1 \end{array}\right] L_3-3L_2 \to L_3 \to$$

o.k.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & 10 & 1 \end{array}\right] \quad C = 1/10 \quad B = -1/6 \quad A = 1/15$$

$$F(S) = \frac{1}{15(S-2)} - \frac{1}{6(S+1)} + \frac{1}{10(S+3)}$$

Réponse

$$\mathcal{L}^{-1}(F(S)) = f(t) = \frac{e^{2t}}{15} - \frac{e^{-t}}{6} + \frac{e^{-3t}}{10}$$

✓

5.34. $f(t) = t\sin 3t$.

$$\mathcal{L}\{t\sin 3t\} = -\frac{d}{ds}\mathcal{L}(\sin 3t) = -\frac{d}{ds}\left(\frac{3}{S^2+9}\right)$$
$$= -\left(\frac{-3(2S)}{(S^2+9)^2}\right) = \frac{6S}{(S^2+9)^2}$$

Réponse: $F(S) = \dfrac{6S}{(S^2+9)^2}$

✓

5.37. $f(t) = \int_0^t \tau e^{t-\tau} d\tau$

Intégration par parties

$\int u\,dv = uv - \int v\,du$

ou

$u = \tau \quad du = 1\,d\tau$

$dv = e^{t-\tau} d\tau$

$v = -e^{t-\tau}$

$$\int u\,dv = -\tau e^{t-\tau} + \int_0^t e^{t-\tau} d\tau$$

$$= \left(-\tau e^{t-\tau} - e^{t-\tau}\right)\Big|_0^t$$

$$= (-te^0 - e^0) - (0 - e^t) \rightarrow -t - 1 + e^t$$

$$\frac{1}{S}\mathcal{L}(f(t)) = \frac{1}{S}\left(-\frac{1}{S^2} - \frac{1}{S} + \frac{1}{S-1}\right)$$

Réponse

$$F(S) = -\frac{1}{S^3} - \frac{1}{S^2} + \frac{1}{S(S-1)}$$

$\blacktriangleright\ L\left\{\int_0^t \tau e^{t-\tau} d\tau\right\} = \frac{1}{S} \ldots$

5.42. $y'' + y = \sin 3t \qquad y(0) = 0 \quad y'(0) = 0.$

$S^2 + 1 = 0$

$L\{y''\} + L\{y\} = L\{\sin 3t\}$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{3}{S^2+9}$$

$$Y(s)(S^2+1) = \frac{3}{S^2+9} \rightarrow Y(s) = \frac{3}{(S^2+1)(S^2+9)}$$

$$\frac{As+B}{(S^2+1)} + \frac{Cs+D}{(S^2+9)} = \frac{3}{(S^2+1)(S^2+9)}$$

$$(As+B)(S^2+9) + (Cs+D)(S^2+1) = 3$$

$$AS^3 + 9As + BS^2 + 9B + CS^3 + Cs + DS^2 + D = 3.$$

$S^3: (A+C) = 0 \qquad S: (9A+C) = 0$

$S^2: (B+D) = 0 \qquad S^0: (9B+D) = 3$

$A = -C, \quad -9C + C = 0 \rightarrow C = 0 \quad A = 0$

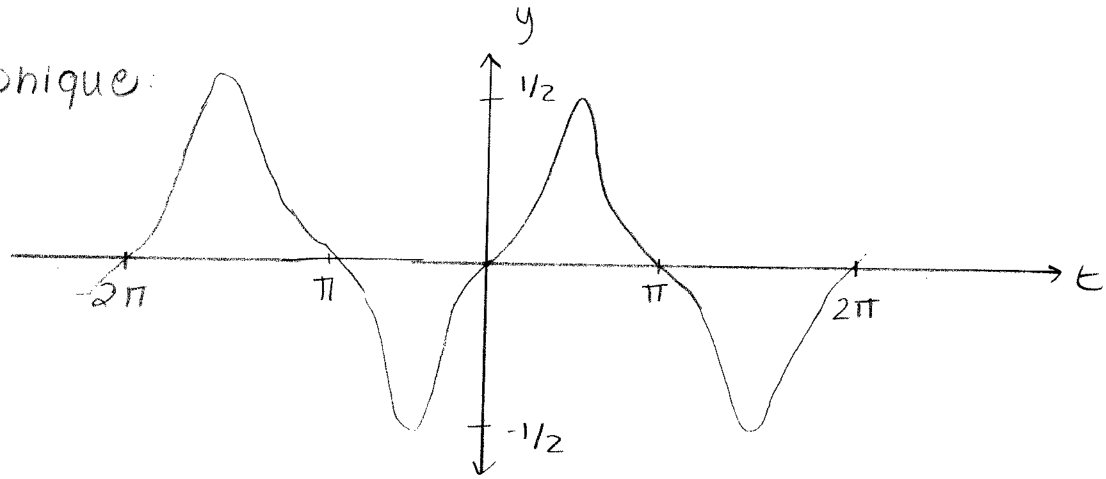
$D = -B, \quad 9B - B = 3 \qquad 8B = 3 \quad B = 3/8 \quad D = -3/8$

$$Y(S) = \frac{3}{8(S^2+1)} - \frac{3}{8(S^2+9)}$$

$$\mathcal{L}^{-1}(Y s) = y(t) = \frac{3\sin t}{8} - \frac{\sin 3t}{8}$$

Réponse: $y(t) = \dfrac{3\sin t - \sin 3t}{8}$

Graphique:



5.45 $y'' + 5y' + 6y = 3e^{-2t}$ $\quad y(0) = 0 \quad y'(0) = 1$

$$L(y'') + \mathcal{L}(5y') + \mathcal{L}(6y) = \mathcal{L}(3e^{-2t})$$

$$S^2Y(S) - s\cancel{y(0)}^{0} - y'(0) + 5sY(S) - \cancel{y(0)}^{0} + 6Y(S) = \frac{3}{S+2}$$

$$Y(S)(S^2 + 5S + 6) = \frac{3}{S+2} + 1$$

$$Y(S) = \frac{3}{(S+2)(S^2+5S+6)} + \frac{1}{(S^2+5S+6)}$$

$$Y(S) = \frac{3}{(S+2)(S+2)(S+3)} + \frac{1}{(S+2)(S+3)}$$

$$Y(S) = \frac{3}{(S+2)(S+2)(S+3)} + \frac{S+2}{(S+2)(S+2)(S+3)}$$

$$Y(S) = \frac{S+5}{(S+2)(S+2)(S+3)}$$

$$\frac{A}{(S+2)} + \frac{B}{(S+2)^2} + \frac{C}{(S+3)} = \frac{S+5}{(S+2)^2(S+3)}$$

$A(S+2)(S+3) + B(S+3) + C(S+2)(S+2) = S+5$
$A(S^2+5S+6) + B(S+3) + C(S^2+4S+4) = S+5$

$S^2 \cdot \quad A+C=0 \qquad A=-C$
$S^1 \quad 5A+B+4C=1 \qquad B=1-4C+5C = 1+C.$
$S^0 \cdot \quad 6A+3B+4C=5 \qquad -6C+3(1+C)+4C=5$
$-6C+3+3C+4C=5$

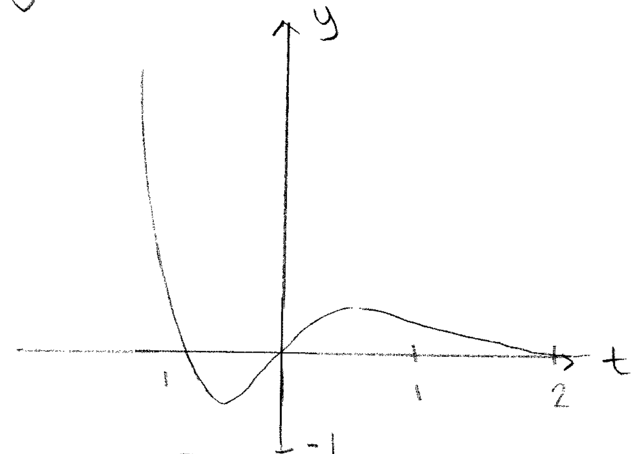
$A=-2 \quad B=3 \quad C=2$

$Y(S) = \frac{-2}{(S+2)} + \frac{3}{(S+2)^2} + \frac{2}{S+3}$

$\mathcal{L}^{-1}(Y(s)) = y(t) = -2e^{-2t} + 3te^{-2t} + 2e^{-3t}$

Réponse : $y(t) = -2e^{-2t} + 3te^{-2t} + 2e^{-3t}.$

Graphique :



5.49. $y'' - 5y' + 6y = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}, \quad y(0)=0, \; y'(0)=1.$

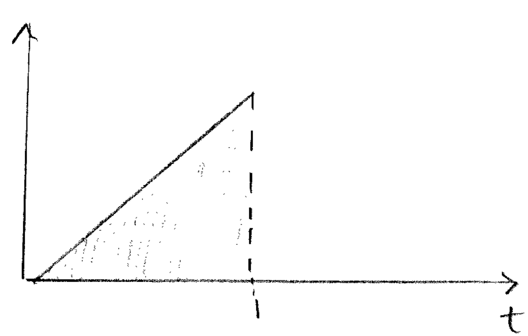
$L(y'') + L(-5y') + L(6y) = ①$

$= S^2 Y(s) - Sy(0) - y'(0) - 5S Y(s) + y(0) + 6Y(s)$
$0 \qquad\qquad 0$

$Y(S)(S^2-5S+6) - 1 = ?$

① → Heaviside

$f(t) = \begin{Bmatrix} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{Bmatrix}.$



$f(t) = t - u(t-1)t + \cancel{u(t-1) \times 0}$

$$f(t) = t - u(t-1)t$$

$$\mathcal{L}(f t) = \mathcal{L}(t) + \frac{d}{ds}\{\mathcal{L}(u(t-1)\}$$

$$= \frac{1}{s} - \frac{d}{ds}\left(\frac{e^{-s}}{s}\right) = \frac{1}{s} + \left(\frac{-e^{-s}s - e^{-s}}{s^2}\right)$$

$$= \frac{1}{s} - e^{-s}\left(\frac{s}{s^2} + \frac{1}{s}\right) + = \frac{1}{s} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = ①$$

alors...

$$Y(s)(s^2 - 5s + 6) - 1 = \frac{1}{s} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s^2(s^2 - 5s + 6)} - \frac{e^{-s}}{s^2(s^2 - 5s + 6)} - \frac{e^{-s}}{s(s^2 - 5s + 6)} + \frac{1}{s^2 - 5s + 6}$$

$$= \underbrace{\frac{1}{s^2(s-2)(s-3)}}_{1} - \underbrace{\frac{e^{-s}}{s^2(s-2)(s-3)}}_{2} - \underbrace{\frac{e^{-s}}{s(s-2)(s-3)}}_{3} + \underbrace{\frac{1}{(s-2)(s-3)}}_{4}$$

Résolution par fractions simples (4 parties)

$$1 \rightarrow \frac{As + B}{s^2} + \frac{C}{(s-2)} + \frac{D}{(s-3)} = \frac{1}{s^2(s-2)(s-3)}$$

$$(s-2)(s-3)(As + B) + C(s^2)(s-3) + D(s^2)(s-2) = 1$$

$$(As + B)(s^2 - 5s + 6) + C(s^3 - 3s^2) + D(s^3 - 2s^2) = 1$$

$$As^3 - A5s^2 + 6As + Bs^2 - 5Bs + 6B + Cs^3 - C3s^2$$
$$+ Ds^3 - 2Ds^2 = 1$$

s^3: $A + C + D = 0$ $\qquad$ s^1: $6A - 5B = 0$

s^2: $-5A + B - 3C - 2D = 0$ $\qquad$ s^0: $6B = 1$

$B = 1/6 \quad A = 5/36 \quad C = -1/4 \quad D = \frac{1}{9}$

alors,

$$1 \rightarrow \frac{5s + 6}{36s^2} - \frac{1}{4(s-2)} + \frac{1}{9(s-3)}$$

$$2 \rightarrow -e^{-s}\left(\frac{5s + 6}{36s^2} - \frac{1}{4(s-2)} + \frac{1}{9(s-3)}\right)$$

$3 \rightarrow -e^{-s}\left[\dfrac{1}{S(S-2)(S-3)}\right] = \dfrac{A}{S} + \dfrac{B}{S-2} + \dfrac{C}{S-3}$

$A(S-2)(S-3) + B(S)(S-3) + C(S)(S-2) = 1$

$A(S^2-5S+6) + B(S^2-3S) + C(S^2-2S) = 1$

$S^2: \; A+B+C=0. \qquad S^0 \; 6A = 1$

$S^1: \; -5A-3B-2C=0$

$A = \dfrac{1}{6} \quad B = -\dfrac{1}{2} \quad C = \dfrac{1}{3}$

$3 \rightarrow -e^{-s}\left[\dfrac{1}{6S} - \dfrac{1}{2(S-2)} + \dfrac{1}{3(S-3)}\right]$

$4 \rightarrow \dfrac{1}{(S-2)(S-3)} = \dfrac{A}{S-2} + \dfrac{B}{S-3} \rightarrow A(S-3) + B(S-2) =$

$S^1: \; A+B=0. \qquad S^0 \; -3A-2B = 1$

$A = -1 \qquad B = 1$

[illegible] $+ \dfrac{1}{S-3}$

Finalement

$Y(S) = \dfrac{1}{S-2)} + \dfrac{1}{S-3)} + \dfrac{5S+6}{36S^2} - \dfrac{1}{4(S-2)} + \dfrac{1}{9(S-3)}$

$e^{-s}\left[\dfrac{5S+6}{36S^2} - \dfrac{1}{4(S-2)} + \dfrac{1}{9(S-3)} + \dfrac{1}{6S} - \dfrac{1}{2(S-2)} + \dfrac{1}{3(S-3)}\right]$

$= \dfrac{1}{S} - e^{-s}\left(\dfrac{1}{S^2} + \dfrac{1}{S}\right)$

$\mathcal{L}^{-1}(Y(s)) = y(t) =$

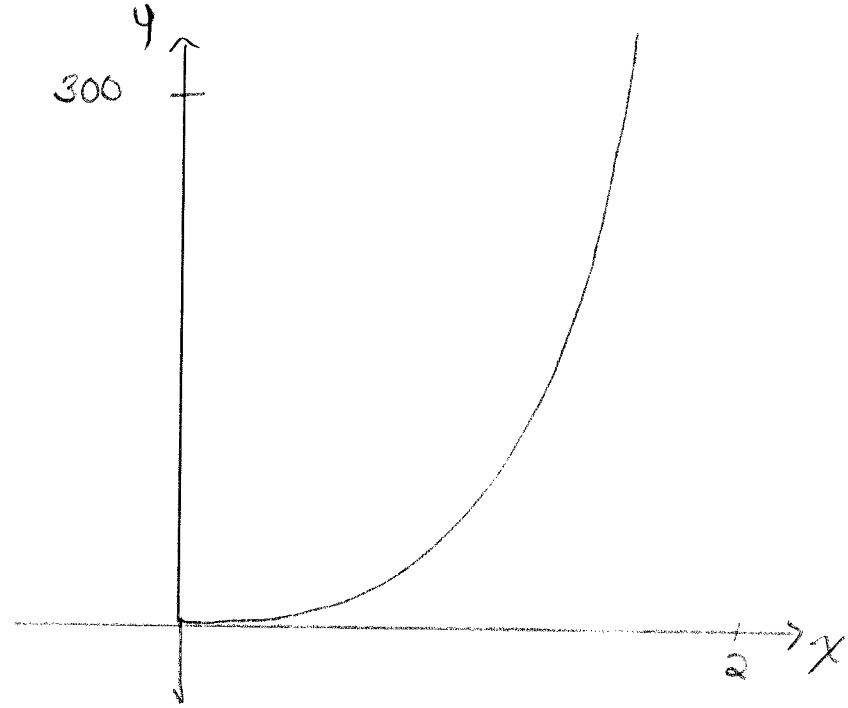
Réponse $y(t) = -\dfrac{5}{4}e^{2t} + \dfrac{10}{9}e^{3t} + \dfrac{5}{36} + \dfrac{t}{6}$

$-u(t-1)\left[\dfrac{11}{36} + \dfrac{1}{6}(t-1) - \dfrac{3}{4}e^{2t-2} + \dfrac{4}{9}e^{3t-3}\right]$

pas mu grec
mais u latin

✓

Graphique:



✱9.9. Méthode des trapèzes. $\int_1^3 \ln x\,dx$.

Intervalle

$a = 1 \quad b = 3$.

$f(x) = \ln x \qquad f'(x) = \frac{1}{x} \qquad f''(x) = \frac{-1}{x^2}$

Nous calculons $f''(x)$ selon $a = 1$ et $b = 3$.

$f''(1) = -1$

$f''(3) = -1/9$.

$$M = \left| \underset{\text{si } 1 \leq x \leq 3}{\text{maximum}} f''(x) \right| = 1 \qquad \left(|-1| > \left|-\frac{1}{9}\right| \right)$$

Trouver h et n

$$\left| \frac{(b-a)h^3}{12} f''(\xi) \right| \leq \frac{2h^2}{12} M =$$

$$\left| \frac{2h^3}{12} f''(\xi) \right| \leq \frac{2h^2}{12}(1) \Rightarrow \frac{h^3}{6} \leq 10^{-3}$$

$h^3 \leq 0{,}006 \qquad h = 0{,}0775$

Trouver n $\quad nh = (b-a)$

$n(0{,}0775) = 2 \qquad n = 25{,}80645$

Trouver h. $\quad h = \frac{2}{26} = 0{,}076923 \qquad \approx 26$ ✓

Réponse: $h = 0{,}076923$ $n = 26$

✱9.10. méthode des pts. milieux. $\int_0^2 \frac{1}{x+4}\,dx$.

$f'(x) = -(x+4)^{-2} = \frac{-1}{(x+4)^2}$

$f''(x) = 2(x+4)^{-3} = \frac{2}{(x+4)^3}$

$a = 0 \quad b = 2$

$f''(a) = \frac{+1}{32} \qquad f''(b) = \frac{1}{108}. \qquad$ maximum: $f''(0) = \boxed{\frac{+1}{32}}$

$$|\varepsilon| < \frac{(b-a)h^2}{24} f''(\xi).$$

D6.9

$$\frac{(2-0)}{24} \cdot h^2 \cdot \frac{1}{32} > 10^{-5}.$$

$$h^2 = \frac{32\ 24 \cdot 10^{-5}}{2} \qquad h = \sqrt{\frac{32 \cdot 24\ 10^{-5}}{2}} = 0{,}06196773$$

5 décimales:

0,06197

$$h = \frac{b-a}{n} = \frac{2-0}{n} \qquad n = \frac{2}{0{,}06197} = 32{,}27368$$

≈ 33

$$h = \frac{2}{33} = 0{,}06061$$

Réponses. $n = 33$, $h = 0{,}06061$