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MAT 2784A
10.10.15

DEVOIR #4

3.20 $(1-x^2)y'' - 2xy' = 0$ $\qquad y_1(x) = 1 \quad y_1'(x) = 0 \quad y_1''(x) = 0$

On pose $y_2(x) = u(x)\,y_1(x)$

$$y_2'(x) = u'y_1 + uy_1'$$
$$y_2''(x) = u''y_1 + u'y_1' + u'y_1' + uy_1'' = u''y_1 + 2u'y_1' + y_1''u$$

$$(1-x^2)(u''y_1 + 2u'y_1' + y_1''u) - 2x(u'y_1 + uy_1') = 0$$
$$u''y_1 + 2u'y_1' + y_1''u - x^2u''y_1 - 2x^2u'y_1' - x^2y_1''u - 2xu'y_1 - 2xuy_1' = 0$$
$$\underbrace{Ly_2}_{0} = u\underbrace{Ly_1}_{0} + u'(2y_1' - 2x^2y_1' - 2xy_1) + u''(y_1 - x^2y_1)$$
$$0 = u'(2y_1' - 2x^2y_1' - 2xy_1) + u''(y_1 - x^2y_1)$$
$$0 = u'(0 - 0 - 2x) + u''(1 - x^2)$$
$$0 = u'(-2x) + u''(1-x^2)$$

Soit $v = u'$ et $v' = u''$

$$(-2x)v = -(1-x^2)\frac{dv}{dx}$$
$$2xv = (1-x^2)\frac{dv}{dx}$$
$$\int \frac{2x\,dx}{(1-x^2)} = \int \frac{dv}{v}$$
$$-\ln(1-x^2) = \ln v$$
$$e^{-\ln(1-x^2)} = v$$
$$\frac{1}{1-x^2} = v = u'$$
$$u = \int u' = \int v = \int\left(\frac{1}{1-x^2}\right) =$$

PAR FRACTIONS SIMPLES :

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$$
$$= \frac{A}{1-x} + \frac{B}{1+x}$$
$$\Rightarrow 1 = (1+x)A + (1-x)B$$
$$= (A-B)x + A + B$$
$$A - B = 0$$
$$A + B = 1$$
$$\Rightarrow A = B = 1/2$$
$$u = \frac{1}{2}\int \frac{1}{1-x}\,dx + \frac{1}{2}\int \frac{1}{1+x}\,dx$$
$$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|$$

$$\boxed{y_2(x) = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|}$$

Le vendredi 15 Octobre 2010
MAT 2784 A

Devoir 4

#3.20. $(1-x^2)y'' - 2xy' = 0$. Si $y_1(x) = 1$ $y_1'(x) = 0$
on cherche $y_2(x)$? $y_1''(x) = 0$
On pose: $y_2(x) = u(x)y_1(x)$

$y_2'(x) = u'(x)y_1(x) + u(x)y_1'(x)$
$y_2''(x) = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)$
$y_2'' = u''y_1 + 2u'y_1' + uy_1''$

$(1-x^2)(u''y_1 + 2u'y_1' + uy_1'') - 2x(u'y_1 + uy_1') = 0$
$u''y_1 + 2u'y_1' + uy_1'' - x^2u''y_1 - 2x^2u'y_1' - x^2uy_1''$
$- 2xu'y_1 - 2xuy_1' = 0$
$u(y_1'' - x^2y_1'' - 2xy_1') + u'(2y_1' - 2x^2y_1' - 2xy_1)$
$+ u''(y_1 - x^2y_1) = 0$
$u(0-0-0) + u'(0-0-2x) + u''(1-x^2)$
$u'(-2x) + u''(1-x^2) = 0$
Si $v = u'$ et $v' = u''$ $\Rightarrow (-2x)v + v'(1-x^2) = 0$
$-2xv + \frac{dv}{dx}(1-x^2) = 0$ $\int \frac{dv}{v} = \int \frac{2x}{(1-x^2)}dx$
$\ln|v| = -\ln(1-x^2)$
$v = e^{\ln(1-x^2)^{-1}}$ $v = 1/(1-x^2) = u'(x)$
$u(x) = \int u'(x) = \int \frac{1}{(1-x^2)}dx = \cancel{-\frac{\ln(1-x^2)}{2x}}$ v. solution

$\boxed{\text{Réponse: } y_2(x) = \cancel{-\frac{\ln(1-x^2)}{2x}}}$ $\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$ -1 FRACTION PARTIELLE

#3.23. $y'' + y' = 3x^2$ $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$
$y'' + y' = 0$
$e^{\lambda x}(\lambda^2 + \lambda) = 0$ $\lambda(\lambda + 1) = 0$ $\lambda_1 = 0$ $\lambda_2 = -1$
$\lambda_1 \neq \lambda_2$
$y_1(x) = e^{\lambda_1 x}$ $y_2(x) = e^{\lambda_2 x}$

$y_1(x) = 1 \quad y_2(x) = e^{-x}$
$y_h = c_1 + c_2 e^{-x}$

$x\,y_p = ax^3 + bx^2 + cx$
$y_p' = 3ax^2 + 2bx + c$
$y_p'' = 6ax + 2b$

$6ax + 2b + 3ax^2 + 2bx + c = 3x^2$
$a = 1 \qquad (6a + 2b)x = 0$
$b = -3 \qquad (2b + c) = 0$
$c = 6$
$y_p(x) = x^3 - 3x^2 + 6x \qquad y(x) = y_p + y_h$
$y(x) = c_1 + c_2 e^{-x} + x^3 - 3x^2 + 6x$

$\boxed{\text{Reponse: la solution générale est: } y(x) = c_1 + c_2 e^{-x} + x^3 - 3x^2 + 6x}$ ✓

#3.29. $Ly := y'' + y = 3x^2 - 4\sin x \qquad y(0) = 0 \qquad y'(0) = 1$

$Ly = y'' + y = 0. \quad y = e^{dx} \quad y' = de^{dx}, y'' = d^2 e^{dx}$
$e^{dx}(d^2 + 1) = 0.$
$d^2 = -1 \qquad d_{1,2} = \pm i$
$y_1(x) = \cos x \quad y_2(x) = \sin x$
$y_h(x) = A\cos x + B\sin x$
$y_p(x) = ax^2 + bx + c + dx\cos x$
$y_p'(x) = 2ax + b + d\cos x - dx\sin x$
$y_p''(x) = 2a - d\sin x - d\sin x - dx\cos x$
$= 2a - 2d\sin x - dx\cos x$
$Ly_p = y_p'' + y_p = r(x)$

$2a - 2d\sin x - \cancel{dx\cos x} + ax^2 + bx + c + \cancel{dx\cos x} = 3x^2 - 4\sin x$
$2a - 2d\sin x + ax^2 + bx + c = 3x^2 - 4\sin x$
$a = 3 \quad b = 0 \quad (2a + c) = 0 \quad c = -6.$
$-2d\sin x = -4\sin x \quad d = 2.$
$y_p = 3x^2 + 2x\cos x - 6$

$y_h + y_p =$

$$\boxed{\text{Solution générale: } y(x) = A\cos x + B\sin x + 3x^2 - 6 + 2x\cos x}$$

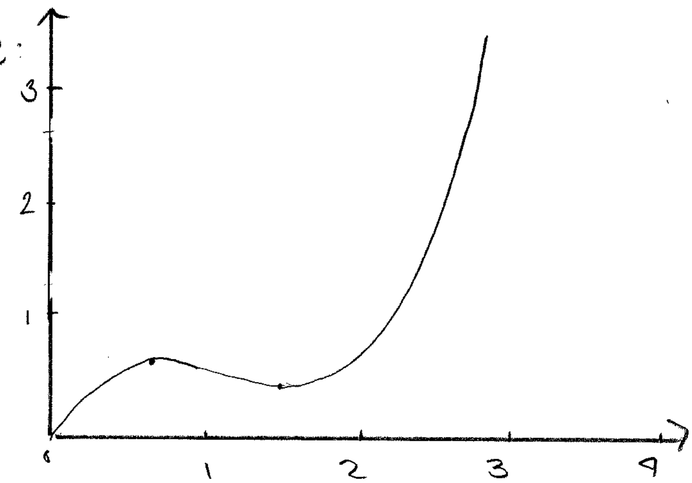
$y'(x) = -A\sin x + B\cos x + 6x + 2\cos x - 2x\sin x$

$0 = A - 6 \quad A = 6$

$1 = B + 2 \quad B = -1$

$$\boxed{\begin{array}{l}\text{Solution unique:}\\ y(x) = 6\cos x - \sin x + 3x^2 + 2x\cos x - 6.\end{array}}$$

Graphique:



✓

#3.30. $y'' + y = \dfrac{1}{\sin x} \qquad y = e^{dx} \quad y' = de^{dx} \quad y'' = d^2 e^{dx}$

$d^2 + 1 = 0 \quad d_{1,2} = \pm i \quad y_1 = \cos x \quad y_2 = \sin x$

$y_h = A\cos x + B\sin x$

$y_p = c_1 \cos x + c_2 \sin x$

méthode variation des paramètres.

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sin x \end{bmatrix}$$

MATRICE ORTHOGONALE $A^{-1} = A^T$

$c_1' \cos x + c_2' \sin x = 0 \qquad c_1' = -c_2' \dfrac{\sin x}{\cos x}$

$-c_1' \sin x + c_2' \cos x = 1/\sin x$

$c_2' \dfrac{\sin^2 x}{\cos x} + c_2' \cos x = \dfrac{1}{\sin x} \qquad c_2' = \dfrac{\cos x}{\sin x} \quad c_1' = -1$

$c_1 = \int -c_1' = -\int dx = -x$

$$C_2 = \int C_2' = \int \frac{\cos x}{\sin x}\, dx = \ln|\sin x|$$

$$\boxed{\begin{array}{l}\text{Reponse:}\\ \text{Solution générale: } y(x) = C_1\cos x + C_2\sin x - x\cos x + \ln|\sin x|\sin x\end{array}}$$

#3.36. $y'' + y = \tan x \qquad y(0) = 1 \qquad y'(0) = 0$

$Ly = 0 = y'' + y$

$\lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i$

$y_h = A\cos x + B\sin x$

$y_p = C_1\cos x + C_2\sin x$

Méthode des paramètres:

MATRICE ORTHOGONALE: $A^{-1} = A^T$

$$\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}\begin{bmatrix}C_1'\\ C_2'\end{bmatrix} = \begin{bmatrix}0\\ \tan x\end{bmatrix}$$

$C_1'\cos x + C_2'\sin x = 0 \qquad C_1' = -\dfrac{C_2'\sin x}{\cos x}$

$-C_1'\sin x + C_2'\cos x = \tan x$

$+\dfrac{C_2'\sin^2 x}{\cos x} + C_2'\cos x = \tan x$

$\dfrac{C_2'\sin^2 x + C_2'\cos^2 x}{\cos x} = \tan x \rightarrow C_2' = \tan x \cdot \cos x = \sin x$

$C_1' = \dfrac{-\sin^2 x}{\cos x} = \dfrac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$

$C_1 = \int C_1' = \int \cos x\, dx - \int \sec x\, dx = \sin x - \ln|\sec x + \tan x|$

$C_2 = \int C_2' = \int \sin x\, dx = -\cos x$

$y_p = \cos x(\sin x - \ln|\sec x + \tan x|) - \cancel{\sin x\cos x}$

$y_p = \cos x \ln|\sec x + \tan x|$

$y_p + y_h$:

Solution générale:

$y(x) = A\cos x + B\sin x + \cos\ln|\sec x + \tan x|$

$$y'(x) = -A\sin x + B\cos x - \frac{(\sec x\tan x + \sec^2 x)}{\sec x + \tan x}\cos x + (\ln|\sec x + \tan x|)\sin x$$

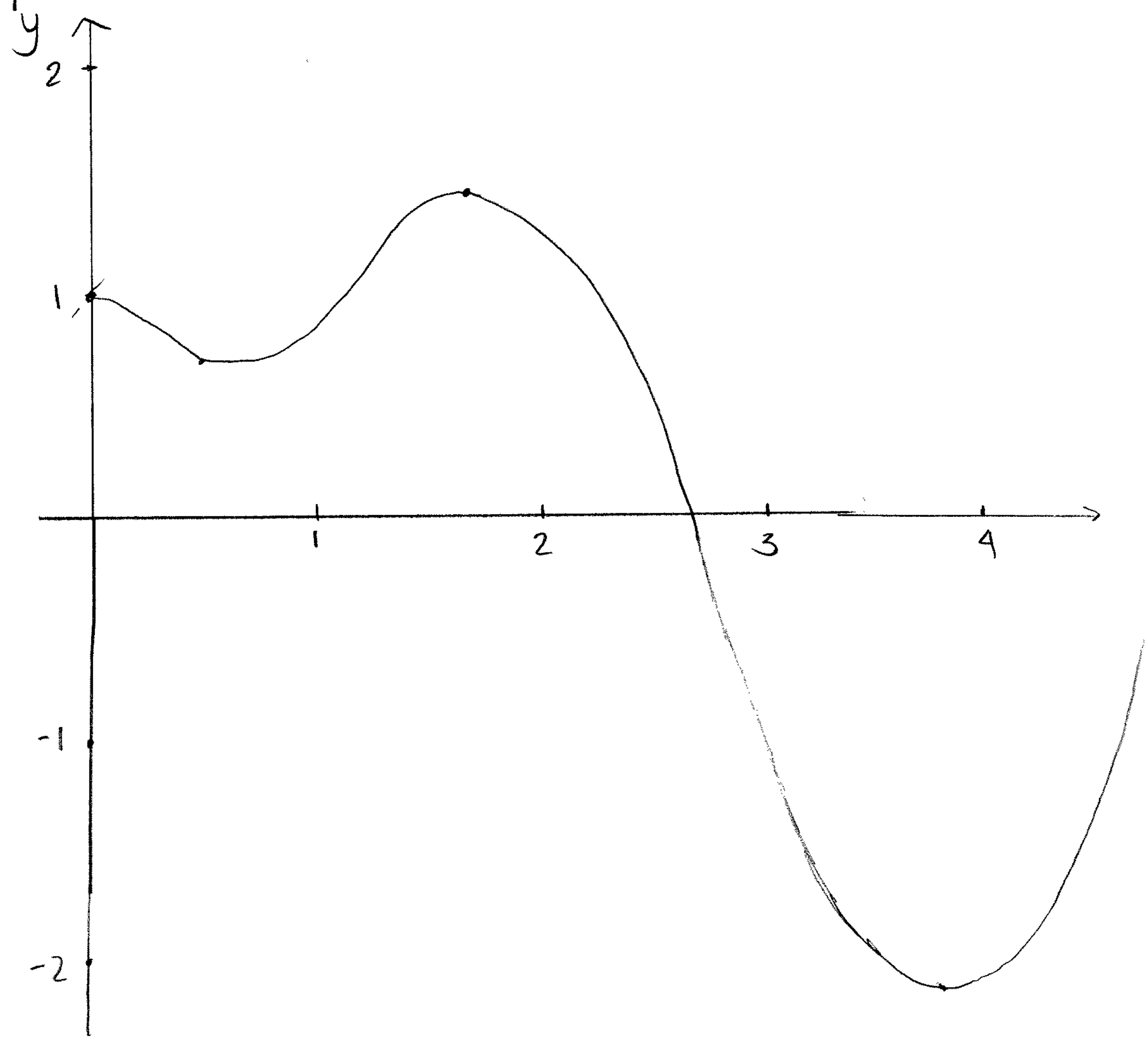
On connait : $y(0) = 1 \quad y'(0) = 0$.

$y(0) = 1 = A + 0 \qquad A = 1$

$y'(0) = 0 = B - 1 + 0 \qquad B = 1$

$$\boxed{\begin{array}{l}\text{Solution unique:}\\ y(x) = \cos x + \sin x - \cos x \ln|\sec x + \tan x|\end{array}}$$

Graphique :



✓

*3.38 $2x^2y'' + xy' - 3y = x^{-2}$ $y(1)=0$ $y'(1)=2$

on pose $y = x^m$ $xy' = mx^m$ $x^2y'' = m(m-1)x^m$

$2m^2 - m - 3 = 0$ $m_1 = 3/2$ $m_2 = -1$

CAS: $m_1 \neq m_2$ $y_h(x) = c_1x^{-1} + c_2x^{3/2}$

$y_p(x) = c_1(x)x^{-1} + c_2(x)x^{3/2}$

$\frac{1}{2x^2}r(x) = \frac{x^{-2}}{2x^2} = \frac{1}{2}x^{-4}$

$$\begin{bmatrix} x^{-1} & x^{3/2} \\ -x^{-2} & \frac{3}{2}x^{1/2} \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2\,x^{-4} \end{bmatrix}$$

$$\begin{bmatrix} x^{-1} & x^{3/2} \\ 0 & \frac{5}{2}x^{1/2} \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (1/2)\,x^{-4} \end{bmatrix}$$

$c_1'x^{-1} + c_2'x^{3/2} = 0$

$\frac{5}{2}x^{1/2}c_2' = \frac{1}{2}x^{-4}$ $c_2' = \frac{x^{-9/2}}{5}$

$c_1'(x) = x^{3/2}\left(\frac{x^{-9/2}}{5}\right)/(x^{-1})$ $c_1'(x) = -\frac{1}{5}x^{-2}$

$c_1 = \int c_1' = \int -1/5\, x^{-2} = \frac{1}{5}x^{-1}$

$c_2 = \int c_2' = \int \frac{x^{-9/2}}{5}$ $c_2 = -\frac{2}{35}x^{-7/2}$

$y_p = (1/5)x^{-1}\cdot x^{-1} - (2/35)x^{-7/2}\cdot x^{3/2} = 1/5x^{2} - 2/35x^{2} = \frac{1}{7}x^{-2}$

$y(x) = Ax^{-1} + Bx^{3/2} + \frac{1}{7}x^{-2}$

$y'(x) = -Ax^{-2} + \frac{3}{2}Bx^{1/2} - \frac{2}{7}x^{-3}$ } nous connaissons $y(1)=0$ $y'(1)=2$

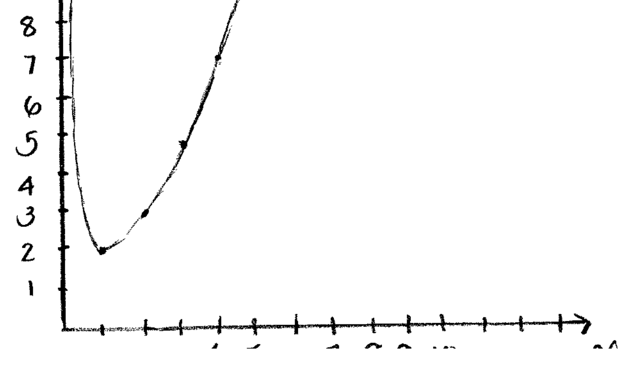
$y(1) = 0 = A + B + \frac{1}{7}$

$B = -A - 1/7$ $B = (2 + \frac{2}{7} + A)\cdot\frac{2}{3}$ $A = -1$

$y'(1) = 2 = -A + (3/2)B - (2/7)$ $B = 6/7$

$$\boxed{\begin{array}{l} \text{Solutions: générale: } Ax^{-1} + Bx^{3/2} + 1/7\,x^{-2} \\ \text{unique: } x^{-1} + \frac{6}{7}x^{3/2} + \frac{1}{7}x^{-2} \end{array}}$$

Graphique:



#8.5.

i	x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	3.2	22.0			
			8,400		
1	2.7	17.8		2.856	
			2.118		−0.528
2	1.0	14.2		2.011	
			6.342		−4.273 ✓
3	4.8	38.3		−10.382	
			−41,413		
4	5.6	5.17			

$$\frac{f_2 - f_1}{x_2 - x_1} = \frac{14.2 - 17.8}{1 - 2.7} = 2.118$$

$$\frac{f_3 - f_2}{x_3 - x_2} = 6.342 \qquad \frac{f_4 - f_3}{x_4 - x_3} = -41.4125$$

$$\frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{6.342 - 2.118}{4.8 - 2.7} = 2.011$$

$$\frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2} = \frac{-41.413 - 6.342}{5.6 - 1} = -10.382$$

$$\frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1} = \frac{-10.382 - 2.011}{5.6 - 2.7} = -4.273$$

$$P_3(x) = f_0 + f[x_0 x_1](x_1 - x_0) + f[x_0, x_1 x_2](x - x_0)(x - x_1) + f[x_0 x_1 x_2 x_3](x - x_0)(x - x_1)(x - x_2)$$

$$\boxed{P_3(x) = 22.0 + 8.4(x - 3.2) + 2.856(x - 3.2)(x - 2.7) - 0.528(x - 3.2)(x - 2.7)(x - 1)}$$

\# 8.7 Résoudre avec polynômes de Newton aux différences divisées

Évaluer polynôme en $x = 0{,}45$

$f(x) = \ln(x+1) \quad x_0 = 0 \quad x_1 = 0{,}6 \quad x_2 = 0{,}9$

$f_0 = \ln(0+1) = 0$

$f_1 = \ln(1{,}6) = 0{,}470003629$

$f_2 = \ln(1{,}9) = 0{,}641853886$

$f(0{,}45) = 0{,}371563556$

- Degré 1: $p_1(x) = a_0 + a_1(x - x_0)$

$a_0 = f_0 = 0$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = \frac{0{,}470003629 - 0}{0{,}6 - 0} = 0{,}783339381$$

$$\boxed{\begin{aligned} p_1(x) &= 0 + 0{,}783(x) \\ p_1(0{,}45) &= 0{,}783(0{,}45) = 0{,}352502721 \end{aligned}}$$

- Degré 2: $p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$a_0 = 0 \quad a_1 = 0{,}783$

$$a_2 = \frac{f_2 - f_1 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$a_2 = \frac{0{,}641853886 - 0 - 0{,}783(0{,}9)}{(0{,}9 - 0)(0{,}9 - 0{,}6)} = -0{,}23276$$

$$\boxed{\begin{aligned} p_2(x) &= 0 + 0{,}783(x-0) + -0{,}23276(x-0)(x-0{,}6) \\ &= 0{,}783x - 0{,}23276(x)(x-0{,}6) \\ p_2(0{,}45) &= 0{,}36806 \end{aligned}}$$

Erreur: Degré 1: $\dfrac{|0{,}352502721 - 0{,}371563556|}{0{,}371563556} = 5{,}13\%$

Degré 2: $\dfrac{|0{,}36806 - 0{,}371563556|}{0{,}371563556} = 0{,}94\%$