Detecting singularities with continuous wavelet transforms and application to leak detection

Rémi Vaillancourt (University of Ottawa, Canada)

Department of Mathematics, Osaka University

remi@uottawa.ca

www.site.uottawa.ca/~remi

26th of November to 9th of December 2003

Overview

- Pipe system diagnosis and leak detection
- Fourier analysis
  - Global regularity and decay
  - Windowed Fourier transform
  - Uncertainty principle
- Wavelet analysis
  - Continuous wavelet transform
  - Lipschitz regularity
  - Wavelet vanishing moments
  - Wavelet transform modulus maxima
  - Detection of singularities
- Application to leak detection

REFERENCES

References

[1] M. Ferrante and B. Brunone, *Pipe system diagnosis and leak detection by unsteady-state tests. 1. Harmonic analysis*, Advances in Water Resources, **26** (2003) 95–105.

[2] M. Ferrante and B. Brunone, *Pipe system diagnosis and leak detection by unsteady-state tests. 2. Wavelet analysis*, Advances in Water Resources, **26** (2003) 107–116.

[3] S. Mallat, *A wavelet tour of signal processing*, 2nd ed., Academic Press, San Diego CA, 1999, Chapter VI Wavelet Zoom.

Pipe system diagnosis and leak detection

Single pipe system diagnosis and leak detection by unsteady-leak tests.

- Harmonic analysis
  - The governing equation for transient flow in pressurized pipes are solved directly in the frequency domain by means of the impulse response method.
  - Information about arrival time of pressure waves are lost.
- Wavelet analysis
  - Retains information coming from the time domain analysis.
  - Detects local singularities in the pressure history due to a leak

Basic equation in the time domain

The simplified one-dimensional momentum and continuity equations for unsteady-state pipe flow:

$$\frac{\partial h}{\partial x} + L\,\frac{\partial q}{\partial t} + R' = 0, \qquad \frac{\partial q}{\partial x} + C\,\frac{\partial q}{\partial t} = 0,$$

where h is the piezometric head; q is the flow rate; x is the spatial coordinate; and t is the time.

The inertance, L, the capacitance, C, and the resistance per unit length, R', are given by

$$L = \frac{1}{gA}, \qquad C = \frac{gA}{a^2}, \qquad R' = \frac{fq^2}{2gDA^2},$$

where g is the gravitational acceleration; A is the pipe cross-sectional area; a is the pressure wave speed; and f is the Darcy–Weisbach friction factor.

Pressure signal

- In a single pipe system, when a valve is suddenly closed, an abrupt increase in pressure occurs and a pressure wave propagates upstream along the pipe.
- This positive pressure wave is reflected back by the reservoir, bringing on a negative pressure wave.
- Diameter changes, junctions, or leak give rise to a partial reflection of the incoming pressure wave.
- Through correctly interpreting the pressure time-history at the measurement section (called *pressure signal*), it is possible to extract the information carried by the reflected waves on discontinuity characteristics, such as leak location and size.

Fourier and inverse Fourier transforms

The Fourier transform of $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt.$$

It measures "how much" oscillation there is in f at frequency ω.
The inverse Fourier transform of $\hat{f} \in L^1(\mathbb{R})$ or $\hat{f} \in L^2(\mathbb{R})$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$

Example 1. The Fourier transform of the indicator function $f = 1_{[-1,1]}$:

$$\hat{f}(\omega) = \int_{-1}^{1} e^{-i\omega t}\, dt = \frac{2\sin\omega}{\omega}.$$

Global regularity and decay

If $\hat{f} \in L^1(\mathbb{R})$, then f is continuous and bounded:

$$|f(t)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{i\omega t} \hat{f}(\omega)| \, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, dw < +\omega.$$

Proposition 1. A function f is bounded and p times continuously differentiable with bounded derivatives if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|(1 + |\omega|^p) \, dw < +\infty.$$

The global regularity of a signal f depends on the decay of $|\hat{f}(\omega)|$ when the frequency ω increases.

Example 2. Since $\hat{f}(\omega) = \dfrac{2 \sin \omega}{\omega}$ is not in $L^1(\mathbb{R})$, its inverse Fourier transform, $f = 1_{[-1,1]}$, is discontinuous.

Windowed Fourier transform

A real symmetric window $g(t) = g(-t)$ translated by u and modulated by the frequency ξ:

$$g_{u,\xi}(t) = e^{i\xi t} g(t-u).$$

Normalized: $\|g\|_2 = 1$ so that $\|g_{u,\xi}\|_2 = 1$ for any $(u, \xi) \in \mathbb{R}^2$.
Windowed Fourier transform of $f \in L^2\mathbb{R})$:

$$S(u,\xi) = \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{\infty} f(t) g(t-u)\, e^{-i\xi t}\, dt.$$

Uncertainty principle

The state of a one-dimensional particle is described by a wave function $f \in L^2(\mathbb{R})$. The average location and average momentum of this particle:

$$u = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} t|f(t)|^2 \, dt, \qquad \xi = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} \omega|\hat{f}(\omega)|^2 \, d\omega.$$

The variances around u and ξ:

$$\sigma_t^2 = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (t-u)^2|f(t)|^2 \, dt, \qquad \sigma_\omega^2 = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} (\omega-\xi)^2|\hat{f}(\omega)|^2 \, d\omega.$$

Theorem 1 (Heisenberg Uncertainty). The temporal variance and the frequency variance of $f \in L^2(\mathbb{R})$ satisfy

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}.$$

For Heisenberg rectangle for windowed Fourier transform, see Figure 1.1.

Continuous wavelet transform

Wavelet function $\psi \in L^2(\mathbb{R})$ with zero average and norm 1:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0, \qquad \|\psi\|_2 = 1.$$

Scaling by s and translating by u:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\, \psi\left(\frac{t-u}{s}\right), \qquad \widehat{\psi}_{u,s}(\omega) = e^{-i\omega u}\sqrt{s}\, \widehat{\psi}(s\omega).$$

Wavelet transform of $f \in L^2(\mathbb{R})$:

$$Wf(u,s) = \langle f, \psi_{u,s}\rangle = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}}\, \psi^*\left(\frac{t-u}{s}\right) dt.$$

As a convolution:

$$Wf(u,s) = f * \bar{\psi}_s(u), \qquad \bar{\psi}_s(t) = \frac{1}{\sqrt{s}}\, \psi^*\left(\frac{-t}{s}\right).$$

For wavelet Heisenberg rectangle, see Figure 1.2.

Mexican hat wavelet

Normalized Mexican hat wavelet, equal to the second derivative of a Gaussian:

$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3\sigma}}\left(\frac{t^2}{\sigma^2} - 1\right)\exp\left(\frac{-t^2}{2\sigma^2}\right), \qquad \|\psi\|_2 = 1.$$

For $\sigma = 1$, Figure 4.6 plots $-\psi$ and its Fourier transform

$$\hat{\psi}(\omega) = \frac{-\sqrt{8}\sigma^{5/2}\pi^{1/4}}{\sqrt{3}}\,\omega^2 \exp\left(\frac{-\sigma^2\omega^2}{2}\right).$$

Figure 4.7 shows the wavelet transform of a signal that is piecewise regular on the left and almost everywhere singular on the right.

Inverse continuous wavelet transform

Theorem 2 (Calderón, Grossmann, Morlet). Let $\psi \in L^2(\mathbb{R})$ be a real function such that

$$C_\psi = \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega < +\infty. \qquad \left(\text{Note:} \quad \widehat{\psi}(0) = \int_{-\infty}^\infty \psi(x)\, dx = 0.\right)$$

Then, any $f \in L^2(\mathbb{R})$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty Wf(u,s)\, \frac{1}{\sqrt{s}}\, \psi\left(\frac{t-u}{s}\right) du\, \frac{ds}{s^2},$$

and

$$\int_{-\infty}^\infty |f(t)|^2\, dt = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty |Wf(u,s)|^2\, du\, \frac{ds}{s^2}.$$

Lipschitz (or Hölder) regularity

Definition 1.

- A function f is pointwise Lipschitz $\alpha \geq 0$ at v it there exists $K > 0$ and a polynomial p_v of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in \mathbb{R}, \quad |f(t) - p_v(t)| \leq K|t - v|^\alpha.$$

- f is uniformly Lipschitz $\alpha \geq 0$ over $[a, b]$ if it satisfies the above inequality for all $v \in [a, b]$ with a constant K independent of v.
- The Lipschitz regularity of f at v or over $[a, b]$ is the sup of the α such that f is Lipschitz α.

Theorem 4. A function f is bounded and uniformly Lipschitz α if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|(1 + |\omega|^\alpha)\, dw < +\infty. \quad \left(\text{Note:} \quad f'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} i\omega \hat{f}(\omega)\, d\omega.\right)$$

Wavelet vanishing moments

To measure the local regularity of a signal, it is not so important to use a wavelet with narrow frequency support, but vanishing moments are crucial.

Definition 2. The function ψ has n vanishing moments if

$$\int_{-\infty}^{\infty} t^k \psi(t)\, dt = 0 \quad \text{for} \quad 0 \le k < n.$$

It has fast decay if, for any $m \in \mathbb{R}$, there exist C_m such that

$$\forall t \in \mathbb{R}, \quad |\psi(t)| \le \frac{C_m}{1 + |t|^m}.$$

Theorem 5. A wavelet ψ with a fast decay has n vanishing moments if and only if there exists θ with a fast decay such that

$$\psi(t) = (-1)^n \frac{d^n \theta(t)}{dt^n}.$$

Thus

$$Wf(u,s) = s^n \frac{d^n}{dt^n} (f * \bar{\theta}_s)(u), \quad \text{where} \quad \bar{\theta}_s(t) = s^{-1/2}\theta(-t/s).$$

Moreover, ψ has no more than n vanishing moments if and only if $\int_{-\infty}^{\infty} \theta(t)\, dt \neq 0$.

Regularity measurements with wavelets

- The decay of the wavelet transform amplitude across scales is related to the uniform and pointwise Lipschitz regularity of the signal.
- Measuring this asymptotic decay is equivalent to zooming into signal structures with a scale that goes to zero.
- We suppose that the wavelet ψ has n vanishing moments and is C^n with derivatives that have fast decay:

$$\forall t \in \mathbb{R}, \quad |\psi^{(k)}(t)| \leq \frac{C_m}{1+|t|^m}, \qquad 0 \leq k \leq n.$$

Theorem 6. If $f \in L^2(\mathbb{R})$ is uniformly Lipschitz $\alpha \leq n$ over $[a,b]$, then there exists $A > 0$ such that

$$\forall (u,s) \in [a,b] \times \mathbb{R}^+, \quad |Wf(u,s)| \leq A s^{\alpha+1/2}. \tag{1}$$

Conversely, suppose that f is bounded and that $Wf(u,s)$ satisfies the above inequality for an $\alpha < n$ that is not an integer. Then f is uniformly Lipschitz α on $[a+\epsilon, b-\epsilon]$ for any ϵ.

Pointwise Lipschitz regularity

A difficult subject that has been made simpler by Jaffard. Remember that the wavelet ψ has n vanishing moments and n derivatives having a fast decay.

Theorem 7 (Jaffard). If $f \in L^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ at v, then there exists $A > 0$ such that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+, \quad |Wf(u,s)| \leq A s^{\alpha+1/2} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right). \tag{2}$$

Conversely, if $\alpha < n$ is not an integer and there exist A and $\alpha' < \alpha$ such that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+, \quad |Wf(u,s)| \leq A s^{\alpha+1/2} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right), \tag{3}$$

then f is Lipschitz α at v.

Cone of influence

- To interpret the necessary and sufficient conditions (2)–(3) in Jaffard's Theorem 7, we suppose that ψ has compact support equal to $[-C, C]$.
- The *cone of influence* of v in the scale-space plane is the set of points (u, s) in the support of

$$\psi_{u,s}(t) = s^{-1/2}\psi\left(\frac{t-u}{s}\right).$$

- Thus the cone of influence of v, shown in Figure 6.2, is

$$|u - v| \leq Cs.$$

- Conditions (2)–(3) can be written as (1) of Theorem 6:

$$|Wf(u, s)| \leq A' s^{\alpha+1/2}.$$

- In Figure 4.7, the high amplitude wavelet coefficients are in the cone of influence of each singularity.

Oscillating singularities

- Surprisingly, conditions (2)–(3) in Jaffard's Theorem 7 imposes a condition on the wavelet transform outside the cone of influence of v.
- To control the oscillations of f that might generate singularities at v it is necesary to impose the decay condition

$$|Wf(u,s)| \leq A' s^{\alpha-\alpha'+1/2} |u-v|^{\alpha}$$

for u outside the cone of influence, that is, for

$$|u-v| > Cs.$$

- For the highly oscillatory function

$$f(t) = \sin\frac{1}{t}$$

high amplitude coefficients are along a parabola below the cone of influence of $t=0$.

See Figure 6.3.

Wavelet transform modulus maxima

- Theorems 6 and 7 prove that the local Lipschitz regularity of f at v depends on the decay at fine scales of $|Wf(u,s)|$ in the neighborhood of v.
- Measuring this decay in the scale-space plane (u,s) is not necessary.
- The decay of $|Wf(u,s)|$ can indeed be controlled from its local maxima values.

Definition 3. The term *modulus maxima* describes any point (u_0, s_0) such that $|Wf(u,s_0)|$ ($\neq$ const) is locally maximum at $u = u_0$. A *maxima line* is any connected curve $s(u)$ in the time-scale plane (u,s) along which all points are modulus maxima.

See Figure 6.5.

Definition 3 implies that

$$\frac{\partial Wf(u_0,s_0)}{\partial u} = 0.$$

Detection of singularities

Singularities are detected by finding the abscissa where the wavelet modulus maxima converge at fine scales. To better understand the properties of these maxima the wavelet transform is written as a multiscale differential operator from Theorem 5:

$$Wf(u,s) = s^n \frac{d^n}{dt^n}(f * \bar{\theta}_s)(u), \quad \text{where} \quad \bar{\theta}_s(t) = s^{-1/2}\theta(-t/s).$$

- If the wavelet has only one vanishing moment, wavelet modulus maxima are the maxima of f' smoothed by $\bar{\theta}_s$ (see Fig. 6.4.) thus locating discontinuities and edges in images.
- If the wavelet has two vanishing moment, the modulus maxima correspond to high curvature (see Fig. 6.4).

Some cases in the detection of singularities

A list of possibile cases:

- Oscillating singularities, previously mentioned
- No modulus maxima at fine scale $\Rightarrow f$ regular
- Maxima propagation traced with a Gaussian wavelet
- Isolated singularities
- Smoothed singularities
- Noisy signal in leak detection
- Multiscale edge detection (not covered in this talk)
- Multifractals (not covered in this talk)

No modulus maxima at fine scale $\Rightarrow f$ regular

Under the hypotheses of Theorem 5 the following theorem proves that if $Wf(u,s)$ has no modulus maxima at fine scale, then f is locally regular.

Theorem 8 (Hwang, Mallat). Suppose that ψ is C^n with a compact support and $\psi = (-1)^n \theta^{(n)}$ with $\int_{-\infty}^{\infty} \theta(t)\,dt \neq 0$. Let $f \in L^1[a,b]$. If there exists $s_0 > 0$ such that $|Wf(u,s)|$ has no local maximum for $u \in [a,b]$ and $s < s_0$, then f is uniformly Lipschitz n on $[a-\epsilon, b-\epsilon]$ for any $\epsilon > 0$.

Maxima propagation

For arbitray $\psi = (-1)^n \theta^{(n)}$, there is no guarantee that a modulus maximum located at (u_0, s_0) belongs to a maxima line that propagates towards finer scales. But this is never the case if θ is a Gaussian.

Theorem 9 (Hummel, Poggio, Yuille). Let $\psi = (-1)^n \theta^{(n)}$ where θ is a Gaussian. For any $f \in L^2(\mathbb{R})$, the modulus maxima of $Wf(u, s)$ belong to connected curves that are never interrupted when the scale decreases.

Idea of proof. The wavelet transform $Wf(u, s)$ is written as the solution of the heat diffusion equation, where s is proportional to the diffusion time. The maximum principle applied to the heat diffusion equation proves that maxima may not disappear when s decreases.

- Derivatives of Gaussians are most often used to guarantee that all maxima lines propagate up to the finest scales.
- Chaining together maxima into maxima lines is also a procedure for removing spurious modulus maxima created by numerical errors in regions where the wavelet transform is close to zero.

Isolated singularities

- A sequence of local maxima may converge to an abscissa v of f even though f is regular at v. See Figure 6.5: maxima line converging to $v = 0.23$.
- The Lipschitz regularity is calculated from the decay of the modulus maxima amplitude.
- By Theorem 6, f is uniformly Lipschitz α in a neighborhood of v if and only if, in the cone $|u - v| \le Cs$,

  $$|Wf(u,s)| \le A\, s^{\alpha+1/2}$$

  which is equivalent to

  $$\log_2 |Wf(u,s)| \le \log_2 A + \left(\alpha + \frac{1}{2}\right) \log_2 s.$$

- The Lipschits regularity at v is thus the maximum slope of $\log_2 |Wf(u,s)|$ as a function of $\log_2 s$.

Smoothed singularities

In the neighborhoot of a sharp transition at v, we suppose that

$$f(t) = f_0 * g_\sigma(t), \qquad g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-t^2}{2\sigma^2}\right),$$

where g_σ is a Gaussian of variance σ^2.

Theorem 10 Let $\psi = (-1)^n \theta^{(n)}$ with $\theta(t) = \lambda \exp(-t^2/(2\beta^2))$. If $f = f_0 * g_\sigma$ and f_0 is uniformly Lipschitz α on $[v-h, v+h]$ then there exists A such that

$$\forall (u,s) \in [v-h, v+h] \times \mathbb{R}^+, \quad |Wf(u,s)| \le A s^{\alpha+1/2} \left(1 + \frac{\alpha^2}{\beta^2 s^2}\right)^{-(n-\alpha)/2}.$$

- At large scales $s \gg \sigma/\beta$, the Gaussian averaging is not felt by the wavelet transform which decays like $s^{\alpha+1/2}$.
- For $s \le \sigma/\beta$, the variation of f at v is not sharp because of the Gaussian averaging. At fine scales, the wavelet transform decays like $s^{n+1/2}$ because f is C^∞. See Fig. 6.6.

Leak detection in laboratory single pipe system

- An upstream air vessel in which the pressure can be held nearly constant.
- Polyethylene pipe, 350.5 m in length of nominal diameter DN110 and wall thickness 8.1 mm, in concentric circles of minimum radius equal to 1.5 m, almost horizontal except for the last short part.
- The hand operated ball valve at the end section discharges into a free surface thank.
- To simulate the leak, a device with an orifice at its wall has been used at distance $l_2 = 128.3$ m from the end section of the pipe.
- Test no. 1: intact pipe with leak area 0 cm^2.
- Test no. 2: pipe with leak area 0.77 cm^2.
- Test no. 3: pipe with leak area 1.99 cm^2.