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ON THE INSTABILITY OF THE BDF FORMULAS*

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Dedicated to Peter Henrici on the occasion of his 60th birthday

Abstract. We give a short proof of the instability of the BDF formulas for $k \geqq 7$.
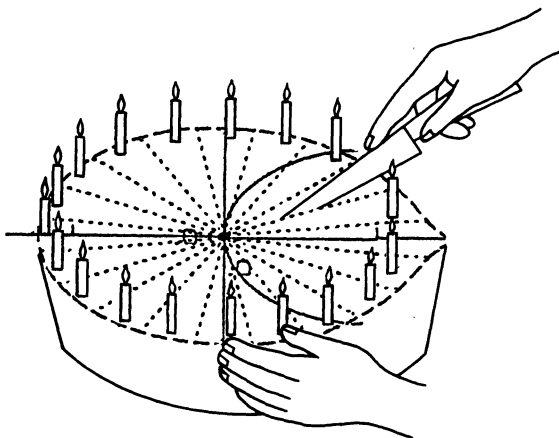
The backward difference formulas

$$
(1) \qquad \sum_{j=1}^{k} \frac{1}{j} \nabla^{j} y_{n+1} = h f_{n+1}
$$

are the most widely used methods for the solution of (stiff) differential equations and became particularly prominent with the books of Henrici (1962) and Gear (1971). For 30 years it has been "known" from numerical calculations that they are unstable for $k \geqq 7$ (Mitchell and Craggs (1953)), but a rigorous mathematical proof of this fact has only been given as late as 1971 in a fifty-page report of Cryer (see Cryer (1971), (1972), also Creedon and Miller (1975)).

THEOREM. *The k-step* BDF *formula* (1) *is stable for* $k \leqq 6$, *and unstable for* $k \geqq 7$. *Proof.* As the ρ-polynomial of (1) is given by

$$
\rho(\zeta) = \sum_{j=1}^{k} \frac{1}{j} \zeta^{k-j}(\zeta - 1)^{j},
$$

the transformation $\zeta = 1/(1-z)$ yields

$$
(2) \qquad p(z) = (1-z)^{k} \rho(\zeta) = \sum_{j=1}^{k} \frac{z^{j}}{j}.
$$

Hence, by definition of stability, formula (1) is stable iff all roots of (2) are outside the disk $\{z\,;\,|z-1|\le 1\}$, with simple roots allowed on the boundary.

It is known (see Jentzsch (1918, p. 219)) that each point on the circle of convergence of a Taylor series is a limit point of the zeros of the truncated series (2); therefore instability is necessary for k tending to infinity. However, for k fixed, we need a more refined estimation.

The roots of (2) are displayed in Fig. 1 for different values of k, and the validity of the theorem can be observed for $k \le 11$. For k larger, we write

$$(3) \qquad p(z) = \int_0^z \sum_{j=1}^k \zeta^{j-1} d\zeta = \int_0^z \frac{1-\zeta^k}{1-\zeta} d\zeta = \int_0^r (1 - e^{ik\theta} s^k) \phi(s)\, ds$$

with

$$\zeta = s e^{i\theta}, \quad z = r e^{i\theta}, \quad \phi(s) = \frac{e^{i\theta}}{1 - s e^{i\theta}}.$$
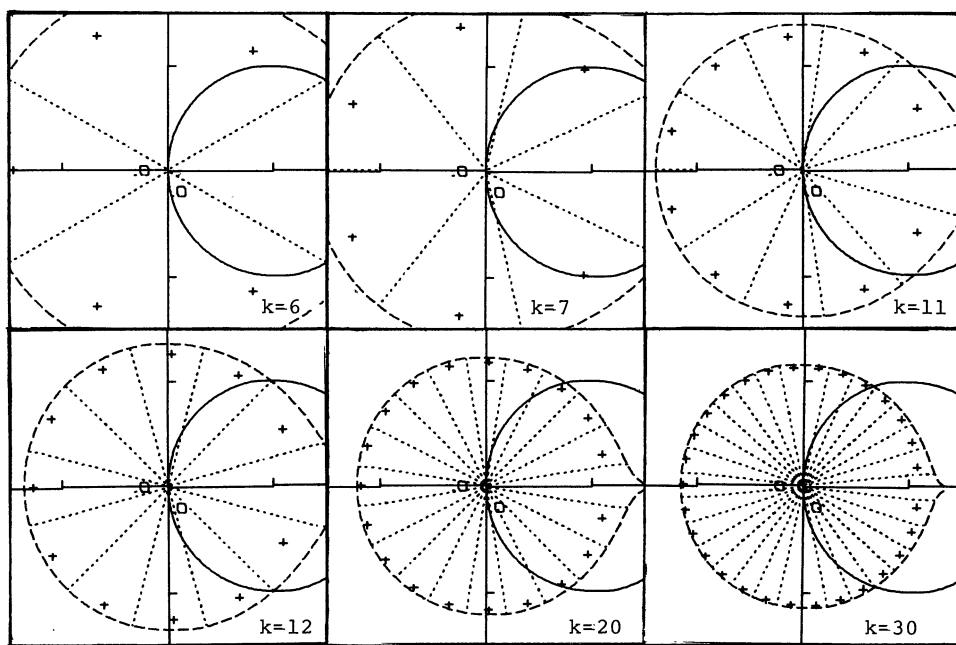

FIG. 1. *Henrici's birthday cake* $(+\cdots$ *zeros of* $p(z))$.

1. We cut the complex plane into k sectors,

$$S_j = \left\{ z\,;\, \frac{2\pi(j-\frac{1}{2})}{k} < \arg(z) < \frac{2\pi(j+\frac{1}{2})}{k},\, \delta < |z| < R \right\},$$

$$j = 0, 1, \cdots, k-1, \delta \text{ small}, R \text{ large}.$$

On the rays limiting S_j we have $e^{ik\theta} = -1$, so that from (3)

$$p(z) = \int_0^r (1 + s^k) \phi(s)\, ds$$

with a *positive* weight function. Therefore the argument of $p(z)$ lies always in the angle which is spanned by $\phi(s)$, i.e., the angle between $e^{i\theta}$ and $e^{i\pi} = -1$. So no

revolution of $\arg(p(z))$ is possible on these rays, and, due to the one revolution of $\arg(z^k)$ at infinity between $\theta = 2\pi(j - \frac{1}{2})/k$ and $\theta = 2\pi(j + \frac{1}{2})/k$, the principle of the argument (e.g. Henrici (1974, p. 278)) implies (see Fig. 2):

   In each sector S_j ($j = 1, \cdots, k - 1$, with the exception of $j = 0$) lies exactly one root of $p(z)$.
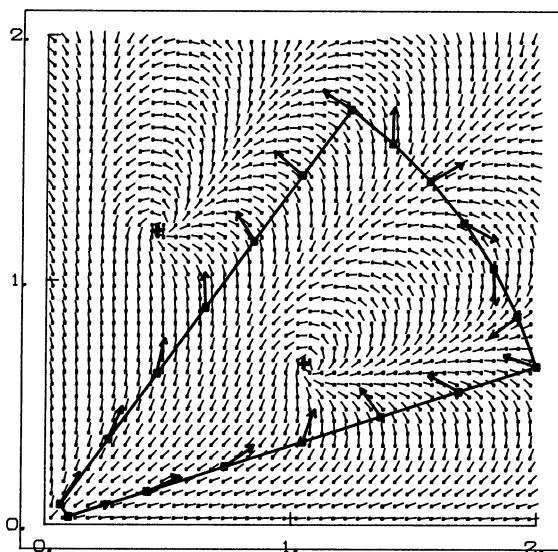

FIG. 2. *The first sector S_1 for $k = 10$ with indication of $\arg(p(z))$.*

2. In order to complete the proof, we will have to bound the zeros of $p(z)$ from above. We observe that in (3) the term s^k becomes large for $s > 1$. We therefore partition (3) into two integrals $p(z) = I_1 - I_2$, where

$$I_1 = \int_0^r \phi(s)\, ds - \int_0^1 e^{ik\theta} s^k \phi(s)\, ds, \qquad I_2 = e^{i\theta k} \int_1^r s^k \phi(s)\, ds.$$

Since $|\phi(s)| \le B(\theta)$, where

$$(4) \qquad\qquad B(\theta) = \begin{cases} \dfrac{1}{\sin\theta}, & 0 < \theta \le \dfrac{\pi}{2}, \\[2mm] 1, & \dfrac{\pi}{2} \le \theta \le \pi, \end{cases}$$

we obtain

$$(5) \qquad\qquad |I_1| \le \left(r + \frac{1}{k+1}\right) B(\theta) < rB(\theta)\frac{k+2}{k+1} \qquad (r > 1).$$

Secondly, since s^k is positive,

$$I_2 = e^{i\theta k}\phi \int_1^r s^k\, ds, \quad \text{where } \phi \in \operatorname{conv}\{\phi(s); 1 \le s \le r\}.$$

Since by some elementary geometry we have $|\phi| \ge 1/2r$ for $r \ge 1$, and get

$$(6) \qquad\qquad |I_2| \ge \frac{r^{k+1} - 1}{k+1}\frac{1}{2r} > r\frac{r^{k-1} - 1}{2k+2} \qquad (r > 1).$$

From (5) and (6) we obtain that

$$(7) \qquad r \geqq R(\theta) = ((2k+4)B(\theta)+1)^{1/(k-1)}$$

implies $|I_2| > |I_1|$, so that $p(z)$ cannot be zero. The curve $R(\theta)$ is also plotted in Fig. 1 and cuts from the sectors S_j, what we call "Henrici's birthday cake," each slice of which (with $j \neq 0$) must contain a zero of $p(z)$. A simple analysis shows that for $k = 12$ the cake, for S_1, is small enough to assure the presence of zeros of $p(z)$ inside the disk $\{z \, ; \, |z-1| \leqq 1\}$. As $R(\theta)$, for fixed θ, as well as $R(\pi/k)$ are monotonically decreasing in k, the same is true for all $k \geqq 12$.

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