Coordinatizing MV-Algebras

(In honour of Prakash's 60th birthday)
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(Joint work with Mark Lawson, Heriot-Watt)

Précis

- Mundici (1986) surprisingly connected up MV-algebras (arising from many-valued logics) with G. Elliott's program for classification of AF C*-algebras via countable dimension groups.
- In 1990's, the algebraic theory of quantum effects in physics led to *Effect Algebras* developed by Bennett & Foulis and the Eastern European school: Jencova, Pulmannova, et. al.

  Theorem [J & P, 2008] There are three categorical equivalences: unital AF C*-algebras $\cong$ countable dimension effect algebras $\cong$ countable dimension groups.

What do we want to do?

- Find a setting that encompasses both frameworks, based on Inverse Semigroup Theory.

- Connect this work up with recent works on noncommutative Stone-Duality, étale groupoids, pseudogroups, tilings, formal language theory, etc. (Lawson, Lenz, Resende, et. al.)

- Generalize AF C*-algebra techniques (Bratteli diagrams) to develop a theory of AF inverse monoids (e.g. the dyadic or CAR Inverse Monoid) and connect it up with effect algebras.

- cf. von Neumann's coordinatization of projective geometry

Theorem (Coordinatization Theorem, L-S)

Let $\mathcal{A}$ be a denumerable MV algebra. Then there exists a boolean inverse "coordinatizing" monoid S s.t. $Ideals(S) = S/\mathcal{J} \cong \mathcal{A}$.

Here $\mathcal{J}$ is the standard relation: $a\mathcal{J}b$ iff $SaS = SbS$

Some Mundici Examples (1991)

Denumerable MV Algebra	AF C*-correspondent
$\{0,1\}$	$\mathbb{C}$
Chain $\mathcal{M}_n$	$Mat_n(\mathbb{C})$
Finite	Finite Dimensional
Dyadic Rationals	CAR algebra of a Fermi gas
$\mathbb{Q} \cap [0,1]$	Glimm's universal UHF algebra
Real algebraic numbers in [0,1]	Blackadar algebra B.
Generated by an irrational $\rho \in [0,1]$	Effros-Shen Algebra $\mathfrak{F}_p$
Finite Product of Post MV-algebras	Continuous Trace
Free on $\aleph_0$ generators	Universal AF C*-algebra $\mathfrak{M}$
Free on one generator	Farey AF C*-algebra $\mathfrak{M}_1$. Mundici (1988)

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Łukasiewicz many-valued logics

Łukasiewicz (1878-1956) introduced many-valued logics in the 1920's. What are they? In brief:

- "Fuzzy" logics $\mathcal{L}$ with truth values in [0,1] (also related ones with truth values in $\mathbb{Q} \cap [0,1]$ or $\mathbb{Q}_{\mathsf{Dyad}} \cap [0,1]$).
- Finite Łukasiewicz logics $\mathcal{L}_n$, with truth values in $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$.

Łukasiewicz Logics and their Algebras

- Studied by Polish logicians in 1920's, including Lesniewski, Tarski (in parallel with Post (1921) in U.S.)
- 1940's & early 1950's: Rosenbloom, Rosser, McNaughton.
- Mid-1950's: major advances by CC. Chang: MV-algebras, Chang Completeness Thm, lattice ordered abelian groups.
- From mid-1980's: large body of work by D. Mundici, et.al.
  - MV-Algebras & AF C*-algebras.
  - Connections to works of Elliott, Effros, Handleman: dimension groups and Grothendieck's K_0 functor.
  - States & probability distributions.
- Sheaf Representation of MV-Algebras: Dubuc/Poveda (2010)
- Łukasiewicz μ-calculus, Matteo Mio and Alex Simpson (2013)
- Morita Equivalence of MV-algebras (Caramello, 2014)

What are MV Algebras?

MV algebras are structures $\mathcal{M} = \langle M, \oplus, \neg, 0 \rangle$ satisfying:

- $\langle M, \oplus, 0 \rangle$ is a commutative monoid.
- $\neg$ is an involution: $\neg\neg x = x$, for all $x \in M$.
- $1 := \neg 0$ is absorbing: $x \oplus 1 = 1$, for all $x \in M$.
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Writing $x \multimap y := \neg x \oplus y$, we can rewrite the last equation:

- $$(x \multimap y) \multimap y = (y \multimap x) \multimap x$$

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Notation: $x \otimes y := \neg(\neg x \oplus \neg y)$

$x \leqslant y$ iff for some z, $x \oplus z = y$ iff $x \multimap y = 1$

Facts: (i) $\leqslant$ is a partial order.

(ii) $\otimes$ is left adjoint to $\multimap$, i.e.

$x \otimes y \leqslant z$ iff $x \leqslant (y \multimap z)$

Further MV Lattice Structure

Further Facts:

(i) Let $x \ominus y := x \otimes \neg y$. Then

$$x \leqslant y \ \text{ iff } \ x \ominus y = 0 \ \text{ iff } \ y = x \oplus (y \ominus x)$$

(ii) $\ominus$ is left adjoint to $\oplus$, i.e. $x \ominus z \leqslant y$ iff $x \leqslant y \oplus z$

Lattice Structure ("Additives")

The order on an MV algebra determines a distributive lattice structure with 0, 1:

$$x \vee y := (x \otimes \neg y) \oplus y = (x \ominus y) \oplus y$$
$$x \wedge y := \neg(\neg x \vee \neg y)$$

Fundamental Example of an MV Algebra: $[0,1]$

For $x, y \in [0,1]$, define:

1. $\neg x = 1 - x$
2. $x \oplus y = min(1, x + y)$
3. $x \otimes y = max(0, x + y - 1)$

Other models: similarly consider the same operations on:

- $\mathbb{Q} \cap [0,1]$ and $\mathbb{Q}_{\text{dyad}} \cap [0,1]$.
- Finite MV algebras $\mathcal{M}_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ (subalgebras of [0,1]). Note: $\mathcal{M}_2 = \{0,1\}$.

Fact (Barr)

$([0,1], \otimes, \oplus, 1, 0, \neg)$ *also forms a *-autonomous poset.*

Moreover, it has products ($\wedge$) and thus coproducts ($\vee$).

Example 2: Lattice-Ordered Abelian Groups

- Let $\langle G, +, -, 0, \leqslant \rangle$ be a partially ordered abelian group, i.e. an abelian group with translation invariant partial order.
- If G is lattice-ordered, call G an *ℓ-group*, G^+ its positive cone.
- If G is an ℓ-group and $t \in G$, then $t + (\ )$ preserves $\vee$ and $\wedge$.
- If G is an ℓ-group, an *order unit* $u \in G$ is an *Archimedian element*: $\forall g \in G$, $\exists n \in \mathbb{N}^+$ s.t. $g \leqslant nu$.
- If G is an ℓ-group with order unit u, define the G-interval

$$[0, u]_G = \{g \in G \mid 0 \leqslant g \leqslant u\} \quad \text{(just a poset)}$$

G-interval MV algebras

Example: $\Gamma(G, u) = ([0, u]_G, \oplus, \otimes, *, 0, 1)$ is an MV algebra, via:

$$\begin{aligned} x \oplus y &:= u \wedge (x + y) \\ x^* &:= u - x \\ x \otimes y &:= (x^* \oplus y^*)^* \end{aligned}$$

$0 := 0_G$ and $1 := u$

All previous examples are special cases

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Let $\mathcal{MV}$ = the category of MV-algebras and MV-morphisms. Let $\ell\mathcal{G}_u$ be the category of ℓ-groups and order-unit preserving homs.

Theorem (Mundici, 1986)

Γ *induces an equivalence of categories* $\ell\mathcal{G}_u \cong \mathcal{MV}$

$\therefore$ For every MV algebra A, there exists an ℓ-group G with order unit u, unique up to isomorphism, s.t. $A \cong \Gamma(G,u)$, and $|G| \leqslant max(\aleph_0, |A|)$.

Completeness Theorems for Łukasiewicz logic

Theorem (Chang Completeness, 1955-58)

1. *Every MV algebra is a subdirect product of MV Chains.*
2. *An MV equation holds in* $[0,1]$ *iff it holds in all MV algebras.*

Corollary (Existence of Free MV-Algebras)

The free MV algebra $\mathcal{F}_\kappa$ *on* κ *free generators is the smallest MV-algebra of functions* $[0,1]^\kappa \to [0,1]$ *containing all projections (as generators) and closed under the pointwise operations.*

Theorem (McNaughton, 1950: earlier than Chang!)

The free MV algebra $\mathcal{F}_n$ *is exactly the algebra of* McNaughton Functions*: continuous, piecewise linear polynomial functions (in n vbls, with integer coefficients):* $[0,1]^n \to [0,1]$.

Matrix algebras and AF C*-algebras:

(*Notes on Real and Complex C*-algebras* by K. R. Goodearl.)

- A finite dimensional C*-algebra is one isomorphic (as a *-algebra) to a direct sum of matrix algebras over $\mathbb{C}$: $\cong M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C})$.
- The ordered list $(m(1), \cdots, m(k))$ is an invariant and determines $\mathcal{A}$. *We write* $\mathcal{A} \cong (m(1), \cdots, m(k))$.
- Many categories arise, with many notions of map!

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-  Many categories arise, with many notions of map!
- (Bratteli, 1972) An *AF C*-algebra* (approximately finite C*-algebra) is a countable colimit

$$\varinjlim \left(\mathcal{A}_1 \xrightarrow{\alpha_1} \mathcal{A}_2 \xrightarrow{\alpha_2} \mathcal{A}_3 \xrightarrow{\alpha_3} \cdots\right)$$

of finite-dimensional C*-algebras and *-algebra maps.

Bratteli showed AF C*-algebras have a *standard form*:

Matrix C*-algebras: standard maps

Consider matrix C*-algebras $\mathcal{A} = M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C})$.

- Define *-algebra maps $\mathcal{A} \to M_{n(i)}(\mathbb{C})$ mapping to a block diagonal $n(i) \times n(i)$-matrix:

$$(A_1, \cdots, A_k) \mapsto DIAG_{n(i)}(\overbrace{A_1, \cdots, A_1}^{s_{i1}}, \overbrace{A_2, \cdots, A_2}^{s_{i2}}, \cdots, \overbrace{A_k, \cdots, A_k}^{s_{ik}})$$

  determined by $s_{ik} \in \mathbb{N}$ where $\boxed{s_{i1}m(1) + \cdots + s_{ik}m(k) = n(i)}$.

- Let $\mathcal{A} = (m(1), \cdots, m(k))$, $\mathcal{B} = (n(1), \cdots, n(l))$ be algebras.

  A *standard $*$-map* $\mathcal{A} \to \mathcal{B}$ is an l-tuple of such DIAGs:

$$(A_1, \cdots, A_k) \mapsto (DIAG_{n(1)}(\cdots), \ldots, DIAG_{n(l)}(\cdots))$$

  determined by $l \times k$ matrix (s_{ij}) s.t. $\sum_{j=1}^{k}(s_{ij}m(j)) = n(i)$, $1 \leqslant i \leqslant l$. The s_{ij} are sometimes called *partial multiplicities*.

Bratteli's Theorem

Theorem (Bratteli)

Any AF C-algebra is isomorphic (as a C*-algebra) to a colimit of a system of matricial C*-algebras and standard maps.*

Bratteli introduced an important graphical language to handle the difficult combinatorics: Bratteli Diagrams.

Bratteli's Diagrams: a combinatorial structure

A Bratteli diagram as an infinite directed multigraph $B = (V, E)$, where $V = \cup_{i=0}^{\infty} V(i)$ and $E = \cup_{i=0}^{\infty} E(i)$.

- Assume $V(0)$ has one vertex, the *root*.
- Edges are only defined from $V(i)$ to $V(i+1)$. Vertices have weights on them.

$$\begin{array}{lcccc} V(i) & m(1) & m(2) & \cdots & m(k) \\ \\ V(i+1) & n(1) & n(2) & \cdots & n(l) \end{array}$$

Draw s_{ij}-many edges between $m(j)$ to $n(i)$. (Of course, for adjacent levels, the s_{ij} must satisfy the combinatorial conditions.)

Bratteli associated groups to labels $m(i)$; the diagrams generate so-called dimension groups. We shall associate certain inverse semigroups.

K_0: Grothendieck group functor **Ring**$\rightarrow$ **Ab**

- A general functorial construction $K_0(-)$. Gives a pre- or p.o.-abelian group $K_0(\mathcal{A})$ for classes of structures $\mathcal{A}$.
- Roughly, we construct a commutative monoid on isomorphism classes of idempotents in a category of idempotents (a kind of Karoubi envelope/$\mathcal{A}$). E.g. say idempotents $e \sim f$ iff there exists maps $x : e \rightarrow f$ and $y : f \rightarrow e$ in Karoubi($\mathcal{A}$) such that $xy = e$ and $yx = f$. But how to add classes $[e] + [f] =$?
- E.g. $\mathcal{A} =$ ring. Move to matrix ring over $\mathcal{A}$. Define "gen. idempotents" $E(\mathcal{A}) = \cup_{n=1}^{\infty}\{\text{idempotents in } M_n(\mathcal{A})\}$. If $e \in M_k$, $f \in M_n$, then $e \oplus f = Diag(e, f) \in M_{k+n}$ is an idempotent. $E(\mathcal{A})/\sim$ is commutative monoid. Want cancellative monoid (why?). Use *stably equiv. idempotents*: $e \approx f$ iff $e \oplus g \sim f \oplus g$ for some $g \in E(\mathcal{A})$. Get cancellative abelian monoid. Apply now the formal INT construction (like building $\mathbb{Z}$ from $\mathbb{N}$). Get functor K_0 : **Rings** $\rightarrow$ **Ab**.

K_0: Grothendieck group for C*-algebras A

- Suppose A is a *-algebra. Now use self-adjoint idempotents (= projections): $e = e^* = e^2$. (Note if $e \neq 0$, $||e|| = ||e^*|| = ||e||^2$, so $||e|| = 1$).
- For projections $e, f \in A$, $e \overset{*}{\sim} f$ if for some $w \in A$, $f \xrightarrow{w} e$ in Karoubi(A), $w^*w = f, ww^* = e$. Note: $e \overset{*}{\sim} f$ implies $e \sim f$.
- For C*-algebras A, again use matrices, using $\overset{*}{\sim}$, $\overset{*}{\approx}$, and $P(A) = \cup_{n=1}^{\infty}\{\text{projections in } M_n(\mathcal{A})\}$. Next facts are increasingly hard to prove: see Goodearl's text:
- Prop: $K_0 :$ C*-alg $\to$ **Preord-Ab**$_u$ is a functor preserving colimits.
- Prop: If start with AF C*-algebra, $K_0 :$ **AF** $\to$ **Po-Ab**$_u$.
- Prop: If A is an AF C*-algebra, then $K_0(A)$ is a countable dimension group with an order unit.

AF C*-algebras & Mundici's Work

Approx. finite (AF) C*-algebras classified in deep work of G. Elliott (studied further by Effros, Handelman, Goodearl, et. al).

Theorem (Mundici)

Let $\ell\mathbf{AF}_u$ = *category of AF-algebras, st* $K_0(\mathcal{A})$ *is lattice-ordered with order unit. Let* $\mathcal{MV}_\omega$ = countable *MV-algebras.*

We can extend $\Gamma : \ell\mathcal{G}_u \cong \mathcal{MV}$ *to a functor* $\hat{\Gamma} : \ell\mathbf{AF}_u \to \mathcal{MV}_\omega$,

$$\hat{\Gamma}(\mathcal{A}) := \Gamma(K_0(\mathcal{A}), [1_\mathcal{A}])$$

(i) $\mathcal{A} \cong \mathcal{B}$ *iff* $\hat{\Gamma}(\mathcal{A}) \cong \hat{\Gamma}(\mathcal{B})$

(ii) $\hat{\Gamma}$ *is full.*

Effect Algebras (of Quantum Effects)

Introduced by Foulis & Bennet (1994) as an abstraction of the algebraic structure of self-adjoint operators with spectrum in [0,1] (called *quantum effects*). (See recent work of Bart Jacobs)

An *Effect Algebra* is a *partial* algebra $\langle E; 0, 1, \widetilde{\oplus} \rangle$ satisfying: $\forall a, b, c \in E$ (Using Kleene directed equality $\succeq$)

1. $a \,\widetilde{\oplus}\, b \succeq b \,\widetilde{\oplus}\, a$.

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2. If $a \,\widetilde{\oplus}\, b \downarrow$ then $(a \,\widetilde{\oplus}\, b) \,\widetilde{\oplus}\, c \succeq a \,\widetilde{\oplus}\, (b \,\widetilde{\oplus}\, c)$

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3. $0 \mathbin{\widetilde{\oplus}} a \downarrow$ and $0 \mathbin{\widetilde{\oplus}} a = a$
4. $\forall_{a \in E} \ \exists!_{a' \in E}$ such that $a \mathbin{\widetilde{\oplus}} a' = 1$.

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$$\left.\begin{array}{l} 1.\ a \widetilde{\oplus} b \succeq b \widetilde{\oplus} a. \\ 2.\ \text{If } a \widetilde{\oplus} b \downarrow \text{ then } (a \widetilde{\oplus} b) \widetilde{\oplus} c \succeq a \widetilde{\oplus} (b \widetilde{\oplus} c) \\ 3.\ 0 \widetilde{\oplus} a \downarrow \text{ and } 0 \widetilde{\oplus} a = a \end{array}\right\} \text{PCM}$$

$$\left.\begin{array}{l} 4.\ \forall_{a \in E}\ \exists!_{a' \in E} \text{ such that } a \widetilde{\oplus} a' = 1. \\ 5.\ a \widetilde{\oplus} 1 \downarrow \text{ implies } a = 0. \end{array}\right\} \text{Orthocomplemented}$$

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An *Effect Algebra* is a partial algebra $\langle E; 0, 1, \widetilde{\oplus} \rangle$ satisfying: $\forall a, b, c \in E$ (Using Kleene directed equality $\succeq$) Various axiomatizations, e.g.:

$$\left.\begin{array}{l} \text{1. } a \mathbin{\widetilde{\oplus}} b \succeq b \mathbin{\widetilde{\oplus}} a. \\ \text{2. If } a \mathbin{\widetilde{\oplus}} b \downarrow \text{ then } (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \succeq a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \end{array}\right\} \text{PCSemigroup}$$

$\cancel{0 \mathbin{\widetilde{\oplus}} a \downarrow \text{ and } 0 \mathbin{\widetilde{\oplus}} a = a}$

$$\left.\begin{array}{l} \text{3. } \forall_{a \in E}\ \exists!_{a' \in E} \text{ such that } a \mathbin{\widetilde{\oplus}} a' = 1. \\ \text{4. } a \mathbin{\widetilde{\oplus}} 1 \downarrow \ \text{ iff } \ a = 0. \end{array}\right\} \text{Orthocomplemented}$$

Some Examples of Effect Algebras

- **Interval Effect Algebras:** Let (G, G^+, u) be an ordered abelian group with order unit u. Consider

$$G^+[0, u] = \{a \in G \mid 0 \leqslant a \leqslant u\}.$$

For $a, b \in G^+[0, u]$, set $a \,\widetilde{\oplus}\, b := a + b$ if $a + b \leqslant u$; otherwise undefined. Also set $a' := u - a$.

- E.g.: **Standard Effect Algebra** $\mathcal{E}(H)$ of quantum system.

  $G := \mathcal{B}_{sa}(H)$, (self-adj) bnded linear operators on H,
  G^+ := the positive operators. Let $\mathbb{O}$ = constant zero ,
  $\mathbb{I}$ = identity. $\mathcal{E}(H) := G^+[\mathbb{O}, \mathbb{I}]$.
  - $A \in \mathcal{E}(H)$ represent unsharp measurements
  - Projections $\mathcal{P}(H) \subset \mathcal{E}(H)$ represent sharp measurements

(cf. S. Gudder: Sharp & Unsharp Quantum Effects):
4 kinds of prob./measurement theories

Effect Algebras of Predicates (B. Jacobs, 2012)

- **Predicates in $\mathcal{C}$**: let $\mathcal{C}$ be a category with "good" coprods. Define $Pred_{\mathcal{C}}(X) := \mathcal{C}(X, 1+1\}$.

Proposition (Jacobs)

If $\mathcal{C}$ has coproducts satisfying some reasonable p.b. conditions, $Pred_{\mathcal{C}}(X)$, $X \in \mathcal{C}$, forms an effect algebra

Examples: (Discrete Distribution Monads, $\mathbf{SRel}_{fin}$, etc.)

$\mathcal{D}(X) = \{m \in [0,1]^X \mid m$ has finite support $\& \sum_x m(x) = 1\}$

$\cong$ formal sums $\{\sum_{i=1}^n r_i x_i \mid r_i \in [0,1] \;\&\; x_i \in X \;\&\; \sum_i r_i = 1\}$

Effect Algebras: Additional Properties

Let E be an effect algebra. Let $a, b, c \in E$. Denote a' by $a^{\perp}$ or a^{*}.

1. Partial Order: $a \leqslant b$ iff for some c, $a \mathbin{\widetilde{\oplus}} c = b$.
2. $0 \leqslant a \leqslant 1$, $\forall a \in E$.
3. $a^{\perp\perp} = a$.
4. (Cancellation) $a \mathbin{\widetilde{\oplus}} c_1 = a \mathbin{\widetilde{\oplus}} c_2$ implies $c_1 = c_2$.
5. (Positivity) $a \mathbin{\widetilde{\oplus}} b = 0$ implies $a = b = 0$
6. $0^{\perp} = 1$ and $1^{\perp} = 0$.
7. $a \leqslant b$ implies $b^{\perp} \leqslant a^{\perp}$

Define a *partial* operation $b \mathbin{\widetilde{\ominus}} a$ by: $b \mathbin{\widetilde{\ominus}} a = c$ iff $a \mathbin{\widetilde{\oplus}} c = b$. So

$$b \mathbin{\widetilde{\ominus}} a \downarrow \text{ iff } a \leqslant b$$

- $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\ominus}} a) = b$
- $a' = a^{\perp} = 1 \mathbin{\widetilde{\ominus}} a$

MV versus MV-Effect Algebras

An *MV-Effect Algebra* is a lattice-ordered effect algebra satisfying

$$(a \vee b) \mathbin{\widetilde{\ominus}} a = b \mathbin{\widetilde{\ominus}} (a \wedge b)$$

Proposition (Chovanec, Kôka, 1997)

There is a natural 1-1 correspondence between MV-effect algebras and MV-algebras.

Idea: MV-Effect algebras $\longleftrightarrow$ MV-Algebras

$$\langle E, 0, 1, \widetilde{\oplus} \rangle \longmapsto \langle E, 0, 1, \oplus \rangle, \text{ where } x \oplus y = x \mathbin{\widetilde{\oplus}} (x' \wedge y)$$

$$\langle E, 0, 1, \widetilde{\oplus} \rangle \longleftarrow \langle E, 0, 1, \oplus \rangle, \text{ where } x \mathbin{\widetilde{\oplus}} y = x \oplus y$$

(i.e. restrict to (x, y) s.t. $x \leqslant \neg y$);

Equivalences of MV- and MV-Effect Algebras

Various facts (mostly due to Bennett & Foulis (1995))

- For lattice-ordered effect algebras E,
  E is MV $\Leftrightarrow \forall a, b \in E, a \wedge b = 0 \Rightarrow a \oplus b \downarrow$.
- An effect algebra satisfies RDP (*Riesz Decomposition Property*) iff

$$\begin{array}{ll} a \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_n & \Rightarrow \\ a = a_1 \oplus a_2 \oplus \cdots \oplus a_n & \text{with} \quad a_i \leqslant b_i, i \leqslant n \end{array}$$

Proposition (B& F)

An effect algebra is an MV-effect algebra iff it is lattice ordered and has RDP.

Universal Groups

Let E be an effect algebra, K an abelian group. A map $E \to K$ is a *K-valued measure* if $a \oplus b \downarrow$ then $\varphi(a \oplus b) = \varphi(a) + \varphi(b)$.

Theorem (Bennet-Foulis, 1997)

If E is an interval effect algebra, $\exists!$ p.o. abelian group G with positive generating cone G^+, and order unit $u \in G^+$ so that

- *$E \cong G^+[0, u]$ and $G^+[0, u]$ generates G^+*
- *Every K-valued measure $E \xrightarrow{\varphi} K$ extends uniquely to a group hom $G \xrightarrow{\varphi^*} K$. G is called the universal group of E*

Theorem (Ravindran,1996)

Let E be an effect algebra with RDP. Then

- *The universal group $E \xrightarrow{\gamma} G_E$ satisfies: (i) it's partially ordered, (ii) $u = \gamma(1)$ is an order unit and (iii) $E \cong G_E[0, u]$.*
- *If E is an MV-algebra, then G_E is an ℓ-group.*

Inverse Semigroups and Monoids

Definition (Inverse Semigroups)

Semigroups (resp. monoids) satisfying: "Every element x has a unique pseudo-inverse y."

- $\forall x \exists ! y (xyx = x \;\&\; yxy = y)$

Fact (Preston-Wagner): Equivalent axiomatization, (i) & (ii):

(i) Existence of pseudo-inverses: $\forall x \exists y (xyx = x \;\&\; yxy = y)$

(ii) Idempotents commute:
$\forall x, y\ [(x^2 = x \;\&\; y^2 = y) \;\Rightarrow xy = yx\ ]$.

We denote the unique pseudo-inverse of x by x^{-1}. So the equations of an inverse semigroup/monoid are:

$$xx^{-1}x = x \;\;\&\;\; x^{-1}xx^{-1} = x^{-1}$$

Ref: M.V. Lawson *Inverse semigroups: the theory of partial symmetries*, World Scientific Publishing Co., 1998.

Examples: Inverse Semigroups & Inv. Monoids

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4. *Tiling semigroups* associated with tilings of $\mathbb{R}^n$.
5. Connections with topological groupoids (vast area of modern mathematics).

Inverse Monoids: Basic Definitions

Let S be an inverse monoid with zero element 0. Let $E(S)$ be the set of idempotents of S.

- For $a, b \in S$, define $a \leqslant b$ iff $a = be$, for some $e \in E(S)$.
- $E(S)$ is always a $\wedge$-semi-lattice.
- S is $\wedge$-*inverse monoid* if $a \wedge b$ exists, $\forall a, b \in S$.
- $\leqslant$ on S is compatible with multiplication.
- Note $a \leqslant b$ implies $a^{-1} \leqslant b^{-1}$!
- For $a \in S$, define $dom(a) := a^{-1}a, ran(a) := aa^{-1} \in E(S)$, so $dom(a) \xrightarrow{a} ran(a)$.
- (Compatibility) For $a, b \in S$, define $a \sim b$ iff $a^{-1}b$ & $ab^{-1} \in E(S)$. This is *necessary* for $a \vee b$ to exist.
- (Orthogonality) $a \perp b$ iff $a^{-1}b = 0 = ab^{-1}$.
- S is *boolean* if: (i) $E(S)$ is a boolean algebra, (ii) compatible elements have joins, (iii) multiplication distributes over $\vee$'s.

Non-Commutative Stone Duality

Boolean Inverse monoids arise in various recent areas of noncommutative Stone Duality.

Theorem (Lawson, 2009,2011)

The category of Boolean inverse $\wedge$-semigroups is dual to the category of Hausdorff Boolean groupoids.

Theorem (Kudryavtseva,Lawson 2012)

The category of Boolean inverse semigroups is dual to the category of Boolean groupoids.

Green's Relations

Let S be an inverse monoid. Define:

1. $\mathcal{J}$ on S: $a\mathcal{J}b$ iff $SaS = SbS$ (i.e. equality of principal ideals).
2. $\mathcal{D}$ on $E(S)$: $e\mathcal{D}f$ iff $\exists_{a\in S}(e = dom(a), f = ran(a), e \xrightarrow{a} f)$
3. For the classes of inverse semigroups we study, $\mathcal{J} = \mathcal{D}$.
4. S is *completely semisimple* if $e\mathcal{D}f \leqslant e$ implies $e = f$.

Consider $E(S)/\mathcal{D}$, S boolean. For idempotents $e, f \in E(S)$, define $[e] \mathbin{\widetilde{\oplus}} [f]$ as follows: *if* we can find orthogonal idempotents $e' \in [e], f' \in [f]$, let $[e] \mathbin{\widetilde{\oplus}} [f] := [e' \vee f']$. Otherwise, undefined.

Proposition

Let S be a Boolean inverse monoid.

- $(E(S)/\mathcal{D}, \widetilde{\oplus}, [0], [1])$ *is a well-defined PCM satisfying (RDP).*
- *If $\mathcal{D}$ preserves complementation and S is completely semi-simple then $(E(S)/\mathcal{D}, \widetilde{\oplus}, [0], [1])$ is an effect algebra w/ RDP.*

Coordinatizing MV Algebras: Main Theorem

- Consider such completely semisimple Boolean inverse monoids S where $\mathcal{D}$ preserves complementation. Call them *Foulis monoids*.
- For Foulis monoids S as in the Proposition, $\mathcal{D} = \mathcal{J}$.
- We can identify $E(S)/\mathcal{D}$ with the poset of principal ideals $S/\mathcal{J}$.
- We say S *satisfies the lattice condition* if $S/\mathcal{J}$ is a lattice. It is then in fact an MV-algebra (by Bennet & Foulis).

Theorem (Coordinatization Theorem for MV Algebras: L& S)

For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid S satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$.

Bratteli Diagrams & AF inverse monoids: Rook Matrices

- A *rook matrix* in $Mat_n(\{0,1\})$ is one where every row and column have at most one 1. Let $R_n :=$ rook matrices.
- There's bijection $\mathcal{I}_n \xrightarrow{\cong} R_n$: $f \mapsto M(f)$, where $M(f)_{ij} = 1$ iff $i = f(j)$.

Up to isomorphism, it's possible to redo the entire theory of Bratteli diagrams for rook matrices. We get:

Bratteli Diagrams of Inverse Monoids and colimits of $\mathcal{I}_n$s

Recall $B = (V, E)$ a Bratteli diagram, where $V = \cup_{i=0}^{\infty} V(i)$ and $E = \cup_{i=0}^{\infty} E(i)$. We assume $V(0)$ has one vertex, the *root*. Edges are only defined from $V(i)$ to $V(i+1)$. Vertices have weights.

$$\begin{array}{lcccc} V(i) & m(1) & m(2) & \cdots & m(k) \\ \\ V(i+1) & n(1) & n(2) & \cdots & n(l) \end{array}$$

Draw s_{ij}-many edges between $m(j)$ to $n(i)$.

$$\text{Associate} \quad \begin{array}{ccc} V(0) & \leftrightarrow & S_0 = \mathcal{I}_1 \cong \{0, 1\} \\ \vdots & \vdots & \vdots \\ V(i) & \leftrightarrow & S_i = \mathcal{I}_{m(1)} \times \cdots \times \mathcal{I}_{m(k)} \end{array}$$

Monomorphisms $\sigma_i : S_i \to S_{i+1}$ are induced by standard maps. $\boxed{\text{Combinatorial Conditions are true}}$

An AF Inverse Monoid $I(B) := colim(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$, for Bratteli diagram B.

AF Inverse Monoids and colimits of $\mathcal{I}_n$s

Lemma

(1) Colimits of ω-chains $(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$ of boolean inverse $\wedge$-monoids with monos inherit all the nice features of the factors. In particular, the groups of units are direct limits of groups of units of the S_i.
(2) Given any ω-sequence of semisimple inverse monoids and injective morphisms, the colim(S_i) is isomorphic to $I(B)$, for some Bratteli diagram B.

Theorem

AF inverse monoids are completely semisimple Boolean inverse monoids in which $\mathcal{D}$ preserves complementation. Their groups of units are direct limits of finite direct products of finite symmetric groups.

Example 1: Coordinatizing Finite MV-Algebras

Let $\mathcal{I}_n = \mathcal{I}_X$ be the inverse monoid of partial bijections on n letters, $|X| = n$. One can show that all the $\mathcal{I}_n$'s are Foulis monoids. The idempotents in this monoid are partial identities 1_A, where $A \subseteq X$. Two idempotents $1_A \mathcal{D} 1_B$ iff $|A| = |B|$. Indeed we get a bijection $\mathcal{I}_n/\mathcal{J} \xrightarrow{\cong} \mathbf{n{+}1}$, where $\mathbf{n{+}1} = \{0, 1, \cdots, n\}$. This induces an order isomorphism, where $\mathbf{n{+}1}$ is given its usual order, and lattice structure via *max, min*.

The effect algebra structure of $\mathcal{I}_n/\mathcal{J}$ becomes: let $r, s \in \mathbf{n{+}1}$. $r \mathbin{\widetilde{\oplus}} s$ is defined and equals $r + s$ iff $r + s \leqslant n$. The orthocomplement $r' = n - r$. The associated MV algebra: $r \oplus s = r + min(r', s)$, which equals $r + s$ if $r + s \leqslant n$ and $r \oplus s$ equals n if $r + s > n$.

We get an iso $\mathcal{I}_n/\mathcal{J} \cong \mathcal{M}_n$, the Łukasiewicz chain. But every *finite* MV algebra is a product of such chains, which are then coordinatized by a product of $\mathcal{I}_n$'s.

Example 2: Coordinatizing Dyadic Rationals–Cantor Space

Cuntz (1977) studied C*-algebras of isometries (of a sep. Hilbert space). They have also arisen in wavelet theory. Associated formal inverse monoids also arose in formal language theory (Nivat, Perrot). We'll describe C_n the nth Cuntz inverse monoid.

Cantor Space A^ω, A finite. For C_n, pick $|A| = n$. For C_2, pick $A = \{a, b\}$. Given the usual topology, we have:

1. Clopen subsets have the form XA^ω, where $X \subseteq A^*$ are *Prefix codes* : finite subsets s.t. $x \precsim y$ (y prefix of x) $\Rightarrow x = y$.
2. Representation of clopen subsets by prefix codes is not unique. E.g. $aA^\omega = (aa + ab)A^\omega$.
3. We can make prefixes X in clopens uniquely representable: define *weight* by $w(X) = \sum_{x \in X} |x|$. Every clopen is generated by unique prefix codes X of minimal weight.

Cuntz and n-adic AF-Inverse Monoids

Definition (The Cuntz inverse monoid, Lawson (2007))

$C_n \subseteq \mathcal{I}_{A^\omega}$ consists of those partial bijections on prefix sets of same cardinality: $(x_1 + \cdots x_r)A^\omega \longrightarrow (y_1 + \cdots y_r)A^\omega$ such that $x_i u \mapsto y_i u$, for any right infinite string u.

Proposition (Lawson (2007))

C_n is a Boolean inverse $\wedge$-monoid, whose set of idempotents $E(C_n)$ is the unique countable atomless B.A. Its group of units is the Thompson group V_n.

Definition (n-adic inverse monoid $Ad_n \subseteq C_n$)

$Ad_n =$ those partial bijections in C_n of the form $x_i \mapsto y_i$, where $|x_i| = |y_i|$, $i \leqslant r$. $Ad_2 =$ the dyadic inverse monoid.

Cuntz and Dyadic AF-Inverse Monoids

Theorem

The MV-algebra of dyadic rationals is co-ordinatized by Ad_2.

The proof takes a small detour through aspects of Bernoulli measures on prefix sets.

Proposition (Characterizing Ad_2)

The dyadic inverse monoid is isomorphic to the direct limit of the sequence of symmetric inverse monoids (partial bijections)

$$\mathcal{I}_2 \to \mathcal{I}_4 \to \mathcal{I}_8 \to \cdots$$

called the CAR inverse monoid. The group of units is a colimit of symmetric groups: $Sym(1) \to Sym(2) \to \cdots Sym(2^r) \to \cdots$.

The General Coordinatization Theorem

Theorem (Coordinatization Theorem for MV Algebras: L& S)

For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid S satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$.

Proof sketch: We know from Mundici every MV algebra $\mathcal{A}$ is isomorphic to an MV-algebra $[0, u]$, which is an interval effect algebra for some order unit u in a countable ℓ-group G. It turns out that G is a countable dimension group. Thus there is a Bratteli diagram B yielding G. Take then $I(B)$, the AF inverse monoid of B. It turns out that $I(B)/\mathcal{J}$ is isomorphic to $[0, u]$ as an MV effect-algebra, and the latter will be a lattice, thus a Foulis monoid. So, we have coordinatized $\mathcal{A}$.

Open Questions

- Coordinatize all the countable examples from Mundici.
- Can we coordinatize larger cardinality MV-algebras (related to (hard!) open problems in operator algebras).