Introduction to MV- and Effect-algebras, 2

Effect Algebras and Coordinatization (Extended Version)

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Coordinatization: von Neumann's Continuous Geometry

- In an article in PNAS (US) (1936) "Continuous Geometry" von Neumann says "The purpose of the investigations, the results of which are to be reported briefly in this note, was to complete the elimination of the notion of point (and line,and plane) from geometry."

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- What's left? A (complemented, modular) lattice of subspaces of a space; a dimension function (into [0,1] or $\mathbb{R}$). The subspaces correspond to the principal right ideals of a von-Neumann regular ring.
  Ref.
  `https://en.wikipedia.org/wiki/Continuous_geometry`

What do we want to do?

- Analog of von Neuman's coordinatization program, using inverse semigroups.
- Generalize AF C*-algebra techniques to develop a theory of AF inverse monoids connecting up MV and effect algebras.

- Theorem (Coordinatization Theorem, L-S)

  Let $\mathcal{A}$ be a denumerable MV algebra. Then there exists a boolean coordinatizing AF inverse monoid S s.t. $Ideals(S) = S/\mathcal{J} \cong \mathcal{A}$.

  Here $\mathcal{J}$ is the standard relation: $a\mathcal{J}b$ iff $SaS = SbS$

Some Mundici Examples (1991):

Denumerable MV Algebra	AF C*-correspondent
$\{0,1\}$	$\mathbb{C}$
Chain $\mathcal{M}_n$	$Mat_n(\mathbb{C})$
Finite	Finite Dimensional
Dyadic Rationals	CAR algebra of a Fermi gas
$\mathbb{Q} \cap [0,1]$	Glimm's universal UHF algebra
Chang Algebra	Behncke-Leptin algebra
Real algebraic numbers in [0,1]	Blackadar algebra B.
Generated by an irrational $\rho \in [0,1]$	Effros-Shen Algebra $\mathfrak{F}_p$
Finite Product of Post MV-algebras	Continuous Trace
Free on $\aleph_0$ generators	Universal AF C*-algebra $\mathfrak{M}$
Free on one generator	Farey AF C*-algebra $\mathfrak{M}_1$. Mundici (1988), Boca (2008)

Some Mundici Examples (1991): Coordinatizations(L-S)

+ MSc. Thesis of Wei Lu

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Effect Algebras (of Quantum Effects)

Foulis & Bennet (1994): an abstraction of algebraic structure of self-adjoint operators with spectrum in [0,1] (*quantum effects*).

An Effect Algebra is a *partial* algebra $\langle E; 0, 1, \widetilde{\oplus} \rangle$ satisfying: $\forall a, b, c \in E$ (Using Kleene directed equality $\succeq$)

$$\left.\begin{array}{l} 1.\ a \mathbin{\widetilde{\oplus}} b \succeq b \mathbin{\widetilde{\oplus}} a. \\ 2.\ \text{If } a \mathbin{\widetilde{\oplus}} b \downarrow \text{ then } (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \succeq a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \\ 3.\ 0 \mathbin{\widetilde{\oplus}} a \downarrow \text{ and } 0 \mathbin{\widetilde{\oplus}} a = a \end{array}\right\} \text{PCM}$$

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2. If $a \mathbin{\widetilde{\oplus}} b \downarrow$ then $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \succeq a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c)$
3. $0 \mathbin{\widetilde{\oplus}} a \downarrow$ and $0 \mathbin{\widetilde{\oplus}} a = a$
4. $\forall_{a \in E}\ \exists!_{a' \in E}$ such that $a \mathbin{\widetilde{\oplus}} a' = 1$.
5. $a \mathbin{\widetilde{\oplus}} 1 \downarrow$ implies $a = 0$.

(Items 1–3: PCM; items 4–5: Orthocomplemented)

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Eastern European School: Dvurecenskij, Jenca, Pulmannova, . . .

Nijmegen: Bart Jacobs and his school (Effectus Theory)

Some Basic Examples of Effect Algebras, I

- **Boolean Algebras**: Let $\mathcal{B} = (B, \wedge, \vee, \overline{(\ )}, 0, 1)$ be a Boolean algebra. For $x, y \in B$, define $x' = \overline{x}$ and

$$x \mathbin{\widetilde{\oplus}} y = \begin{cases} x \vee y & \text{if } x \wedge y = 0 \\ \uparrow & \text{else} \end{cases}$$

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- G. Boole [1854] in "An Investigation of the Laws of Thought":
. . . the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common.

Effect Algebras: alternative axiomatizations

An *Effect Algebra* is a partial algebra $\langle E; 0, 1, \widetilde{\oplus} \rangle$ satisfying: $\forall a, b, c \in E$ (Using Kleene directed equality $\succeq$) Various axiomatizations, e.g.:

1. $a \widetilde{\oplus} b \succeq b \widetilde{\oplus} a$.
2. If $a \widetilde{\oplus} b \downarrow$ then $(a \widetilde{\oplus} b) \widetilde{\oplus} c \succeq a \widetilde{\oplus} (b \widetilde{\oplus} c)$ } PCSemigroup

   ~~$0 \widetilde{\oplus} a \downarrow$ and $0 \widetilde{\oplus} a = a$~~

3. $\forall_{a \in E}\ \exists!_{a' \in E}$ such that $a \widetilde{\oplus} a' = 1$.
4. $a \widetilde{\oplus} 1 \downarrow$ iff $a = 0$. } Orthocomplemented

Some Examples of Effect Algebras

- **Interval Effect Algebras:** Let (G, G^+, u) be an ordered abelian group with order unit u. Consider

$$G^+[0,u] = \{a \in G \mid 0 \leqslant a \leqslant u\}.$$

For $a, b \in G^+[0,u]$, set $a \mathbin{\widetilde{\oplus}} b := a + b$ if $a + b \leqslant u$; otherwise undefined. Also set $a' := u - a$. e.g. [0,1] as a partial algebra.

- E.g.: **Standard Effect Algebra** $\mathcal{E}(H)$ of a quantum system.

  $G := \mathcal{B}_{sa}(H)$, (self-adj) bnded linear operators on H,
  G^+ := the positive operators. Let $\mathbb{O}$ = constant zero ,
  $\mathbb{I}$ = identity. $\mathcal{E}(H) := G^+[\mathbb{O}, \mathbb{I}]$.
  - $A \in \mathcal{E}(H)$ represent unsharp measurements
  - Projections $\mathcal{P}(H) \subset \mathcal{E}(H)$ represent sharp measurements

(cf. S. Gudder: Sharp & Unsharp Quantum Effects):
4 kinds of prob./measurement theories

Effect Algebras of Predicates (B. Jacobs, 2012-2015)

Predicates in $\mathcal{C}$: let $\mathcal{C}$ be a category with "good" finite coprods and terminal object 1. Define $Pred_{\mathcal{C}}(X) := \mathcal{C}(X, 1+1)$.

Proposition (Jacobs)

If $\mathcal{C}$ satisfies reasonable p.b. conditions on $+$, $Pred_{\mathcal{C}}(X)$, $X \in \mathcal{C}$, forms an effect algebra. (Such a $\mathcal{C}$ is called an "effectus"). and $Pred : \mathcal{C}^{op} \to Eff$.

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Examples:

- Predicates on Kleisli categories of various distribution monads (e.g. Discrete, Continuous, etc.)
- Predicates on various concrete categories: **Set**, **SemiRing**^{op}, **Ring**^{op}, **DL**^{op}, $(C^*_{PU})^{op}$, ...

Effect Algebras of Predicates (B. Jacobs) II

- **Ring**^{op}

$$\begin{array}{ll} R \xrightarrow{pred} 1+1 & \text{in } \mathbf{Ring}^{op} \\ \hline\hline \mathbb{Z}^2 \longrightarrow R & \text{in } \mathbf{Ring} \\ \hline\hline \text{idempotents in } R & \end{array}$$

(for *commutative* rings, idempotents form BA).

- **DL**^{op}

$$\begin{array}{ll} L \xrightarrow{pred} 1+1 & \text{in } \mathbf{DL}^{op} \\ \hline\hline \mathbf{2} \times \mathbf{2} \longrightarrow L & \text{in } \mathbf{DL} \\ \hline\hline \textit{complementable } x \in L & \end{array}$$

$Pred(L)$ is a boolean sublattice of L.

In general: the effect algebras of predicates correspond to orthogonal or complementable idempotents.

Effect Algebras of Predicates (B. Jacobs) III

Jacobs has yet more general structures which we won't consider:

1. Effect Monoids: these are monoid objects in the category of Effect Algebras ($=$ Effect algebra with multiplication preserving 0, $\oplus$ (in each argument), and $1x = x = x1$.)
2. Effect Modules: Given an effect monoid M and effect algebra E, consider the category of M-actions $\bullet : M \times E \to E$ with usual (M-set laws).

Effect Algebras: Additional Properties

Let E be an effect algebra. Let $a, b, c \in E$. Denote a' by $a^\perp$ or a^*.

1. Partial Order: $a \leqslant b$ iff for some c, $a \mathbin{\widetilde{\oplus}} c = b$.
2. $0 \leqslant a \leqslant 1$, $\forall a \in E$.
3. $a^{\perp\perp} = a$.
4. $0^\perp = 1$ and $1^\perp = 0$.
5. $a \leqslant b$ implies $b^\perp \leqslant a^\perp$
6. (Cancellation) $a \mathbin{\widetilde{\oplus}} c_1 = a \mathbin{\widetilde{\oplus}} c_2$ implies $c_1 = c_2$.
7. (Positivity / conical) $a \mathbin{\widetilde{\oplus}} b = 0$ implies $a = b = 0$

Effect Algebras: Additional Properties II

The analog of "logical subtraction" in MV-algebras.

Define a *partial* operation $b \mathbin{\widetilde{\ominus}} a$ by: $b \mathbin{\widetilde{\ominus}} a = c$ iff $a \mathbin{\widetilde{\oplus}} c = b$. So

$$b \mathbin{\widetilde{\ominus}} a \downarrow \text{ iff } a \leqslant b$$

.
- $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\ominus}} a) = b$
- $a' = a^{\perp} = 1 \mathbin{\widetilde{\ominus}} a$

MV versus MV-Effect Algebras

An *MV-Effect Algebra* is a lattice-ordered effect algebra satisfying

$$(a \vee b) \widetilde{\ominus} a = b \widetilde{\ominus} (a \wedge b)$$

Proposition (Chovanec, Kôka, 1997)

There is a natural 1-1 correspondence between MV-effect algebras and MV-algebras.

Idea: MV-Effect algebras $\longleftrightarrow$ MV-Algebras

$$\langle E, 0, 1, \widetilde{\oplus} \rangle \longmapsto \langle E, 0, 1, \oplus \rangle, \text{ where } x \oplus y = x \widetilde{\oplus} (x' \wedge y)$$

$$\langle E, 0, 1, \widetilde{\oplus} \rangle \longleftarrow \langle E, 0, 1, \oplus \rangle, \text{ where } x \widetilde{\oplus} y = x \oplus y$$

(i.e. restrict to (x, y) s.t. $x \leqslant \neg y$);

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- An effect algebra satisfies RDP (*Riesz Decomposition Property*) iff

$$\begin{array}{ccc} a \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_n & \Rightarrow & \exists a_1, \ldots, a_n \ s.t. \\ a = a_1 \oplus a_2 \oplus \cdots \oplus a_n & \text{with} & a_i \leqslant b_i, i \leqslant n \end{array}$$

Proposition (Bennett & Foulis, 1985)

An effect algebra is an MV-effect algebra iff it is lattice ordered and has RDP.

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Morphisms are different!

$|\,Hom_{\mathbf{MV}}([0,1],[0,1])\,| = 1$, $|\,Hom_{\mathbf{MV}}([0,1]^2,[0,1])\,| = 2$

$|\,Hom_{\mathbf{EA}}([0,1],[0,1])\,| = 1$, $|\,Hom_{\mathbf{EA}}([0,1]^2,[0,1])\,| = 2^{\aleph_0}$

Universal Groups of Effect Algebras: Mundici Anew

- If $(E, +, 0, 1)$ is an effect algebra with RDP, there is a universal monoid $E \hookrightarrow M_E$. This (total) monoid M_E is abelian, cancellative, satisfies a universal property.
- Every cancellative abelian monoid $\mathcal{M}$ has a Grothendieck group $\mathcal{M} \hookrightarrow G_{\mathcal{M}}$ satisfying a universal property (essentially the INT construction yielding $\mathbb{Z}$ from $\mathbb{N}$).

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Theorem (Ravindran,1996)

Let E be an effect algebra with RDP and $E \xrightarrow{\gamma} G_E$ its universal (Groth.) group. Then G_E satisfies:

1. *(i) G_E is partially ordered,*
2. *(ii) $u = \gamma(1)$ is an order unit and (iii) $\gamma : E \cong [0, u]_{G_E}$.*
3. *If E is an MV-algebra, then G_E is an ℓ-group (cf. Mundici).*

Ravindran's Theorem–some details

Essentially an independent approach to Mundici's theorem, via effect algebras. Technique goes back to R. Baer (1949).

Theorem

Let E be an effect algebra satisfying RDP. Then it is an interval effect algebra, with universal group an interpolation group.

Let E^+ be the free (word) semigroup on E. Take the smallest congruence $\sim$ such that the word $(a, b) \sim (a \oplus b)$, whenever $(a \oplus b) \downarrow$. i.e. Take the congruence relation on words generated as: $(a_1, a_2, \cdots a_n) \sim (a_1, a_2, \cdots, a_{k-1}, a_k \oplus a_{k+1}, a_{k+2}, \cdots, a_n)$, whenever $a_k \oplus a_{k+1} \downarrow$. Then $E^+/\sim$ is a positive abelian monoid (get commutativity for free!) with RDP. Its Grothendieck Group is its universal group. If E satisfies RDP, this is the universal group $\gamma : E \to G_E$ of the effect algebra, which is a po-group with $u = \gamma(1)$ an order unit. If E is MV, then $[0, u]$ is lattice and G_E is an $\ell-$group.

Matrix algebras and AF C*-algebras: Mundici II

(*Notes on Real and Complex C*-algebras* by K. R. Goodearl.)

- A finite dimensional C*-algebra is one isomorphic (as a *-algebra) to a direct sum of matrix algebras over $\mathbb{C}$: $\cong M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C})$.
- The ordered list $(m(1), \cdots, m(k))$ is an invariant.
- Many categories arise, with many notions of map!

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- The ordered list $(m(1), \cdots, m(k))$ is an invariant.
- Many categories arise, with many notions of map!
- (Bratteli, 1972) An *AF C*-algebra* (approximately finite C*-algebra) is a countable colimit

$$\varinjlim \left(\mathcal{A}_1 \xrightarrow{\alpha_1} \mathcal{A}_2 \xrightarrow{\alpha_2} \mathcal{A}_3 \xrightarrow{\alpha_3} \cdots \right)$$

of finite-dimensional C*-algebras and *-algebra maps.

Bratteli showed AF C*-algebras have a *standard form*:

Matricial C*-algebras: standard maps

$$\begin{aligned}\mathcal{A} &:= M_{m(1)}(\mathbb{C}) \oplus \cdots \oplus M_{m(k)}(\mathbb{C}) \text{ and} \\ \mathcal{B} &:= M_{n(1)}(\mathbb{C}) \oplus \cdots \oplus M_{n(l)}(\mathbb{C}) \ .\end{aligned}$$

- Define *-algebra maps $\mathcal{A} \to M_{n(i)}(\mathbb{C})$

$$(A_1, \cdots, A_k) \mapsto DIAG_{n(i)}(\overbrace{A_1, \cdots, A_1}^{s_{i1}}, \overbrace{A_2, \cdots, A_2}^{s_{i2}}, \cdots, \overbrace{A_k, \cdots, A_k}^{s_{ik}})$$

  determined by $s_{ik} \in \mathbb{N}$ where $\boxed{s_{i1} m(1) + \cdots + s_{ik} m(k) = n(i)}$.

- A *standard ∗-map* $\mathcal{A} \to \mathcal{B}$ is an l-tuple of such DIAGs:

$$(A_1, \cdots, A_k) \mapsto (DIAG_{n(1)}(\cdots), \ldots, DIAG_{n(l)}(\cdots))$$

  determined by $l \times k$ matrix (s_{ij}) s.t. $\sum_{j=1}^{k}(s_{ij} m(j)) = n(i)$,

Bratteli's Theorem

Theorem (Bratteli)

Any AF C-algebra is isomorphic (as a C*-algebra) to a colimit of a system of matricial C*-algebras and standard maps.*

Bratteli introduced an important graphical language to handle the difficult combinatorics: Bratteli Diagrams.

Bratteli's Diagrams: a combinatorial structure

A Bratteli diagram as an infinite directed multigraph $B = (V, E)$, where $V = \cup_{i=0}^{\infty} V(i)$ and $E = \cup_{i=0}^{\infty} E(i)$.

- Assume $V(0)$ has one vertex, the *root*.
- Edges are only defined from $V(i)$ to $V(i+1)$.

$$\begin{array}{lc} V(i) & \overbrace{m(1) \quad m(2) \quad \cdots \quad m(k)}^{(\mathbb{Z}^k,(m(1),\cdots,m(k)))} \\ V(i+1) & n(1) \quad n(2) \quad \cdots \quad n(l) \end{array}$$

  Draw s_{ij}-many edges between $m(j)$ to $n(i)$. (Of course, for adjacent levels, the s_{ij} must satisfy the combinatorial conditions.)

- Vertices now assigned $\ell\mathbf{AB}_u$ groups $(\mathbb{Z}^k, u)$.

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Colimits along standard maps induces colimits of associated $\mathbb{Z}^k$, called dimension groups.

K_0: Grothendieck group functors

A very general construction:

- K_0: **Ring**$\rightarrow$ **Ab** and K_0 : **AF** $\rightarrow$ **Po-Ab**$_u$
- Roughly: turn the isomorphism classes (of idempotents) in the Karoubi Envelope into an abelian cancellative monoid and then by INT into an abelian group.
- Tricky for AF C*-algebras: technicalities of self-adjoint idempotents (= projections)

AF C*-algebras & Mundici's Theorem II

Approx. finite (AF) C*-algebras classified in deep work of G. Elliott (studied further by Effros, Handelman, Goodearl, et. al).

Theorem (Mundici)

Let $\ell\mathbf{AF}_u$ = *category of AF-algebras, st* $K_0(\mathcal{A})$ *is lattice-ordered with order unit. Let* $\mathcal{MV}_\omega$ = countable *MV-algebras.*

We can extend $\Gamma : \ell\mathcal{G}_u \cong \mathcal{MV}$ *to a functor* $\hat{\Gamma} : \ell\mathbf{AF}_u \to \mathcal{MV}_\omega$,

$$\hat{\Gamma}(\mathcal{A}) := \Gamma(K_0(\mathcal{A}), [1_\mathcal{A}]) = [\,0, [1_\mathcal{A}]\,]_{K_0(\mathcal{A})}$$

(i) $\mathcal{A} \cong \mathcal{B}$ *iff* $\hat{\Gamma}(\mathcal{A}) \cong \hat{\Gamma}(\mathcal{B})$

(ii) $\hat{\Gamma}$ *is full.*

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Inverse Semigroups and Monoids

Definition (Inverse Semigroups)

Semigroups (resp. monoids) satisfying: "Every element x has a unique pseudo-inverse x^{-1}."

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Fundamental Examples

- $\mathcal{I}_X = \mathbf{PBij}(X)$, Symmetric Inverse Monoid. These are partial bijections on the set X, i.e. partial functions $f : X \rightharpoonup X$ which are bijections $dom(f) \to ran(f)$.
  - For each subset $A \subseteq X$, there are partial identity functions $1_A \in \mathcal{I}_X$. These are the idempotents.
  - $f^{-1} \circ f = 1_{dom(f)}$ and $f \circ f^{-1} = 1_{ran(f)}$, partial identities on X .
- **Semisimple:** $=$ Finite Cartesian Products $\mathcal{I}_{X_1} \times \cdots \times \mathcal{I}_{X_n}$.

Examples: Inverse Semigroups & Inv. Monoids

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4. *Tiling semigroups* associated with tilings of $\mathbb{R}^n$.

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2. Symmetric Inverse Monoids $\mathcal{I}_X = \mathbf{PBij}(X)$.
3. *Pseudogroups* (arising in differential geometry): inverse semigroups of partial homeomorphisms between open subsets of a topological space (Veblen-Whitehead, Ehresmann).
4. *Tiling semigroups* associated with tilings of $\mathbb{R}^n$.
5. Connections with topological groupoids (vast area).

Inverse Monoids: Basic Definitions

Let S be an inverse monoid with zero element 0. Let $E(S)$ be the set of idempotents of S.

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- (Compatibility) For $a, b \in S$, define $a \sim b$ iff $a^{-1}b$ & $ab^{-1} \in E(S)$. This is *necessary* for $a \vee b$ to exist.
- (Orthogonality) $a \perp b$ iff $a^{-1}b = 0 = ab^{-1}$.

Non-Commutative Stone Duality

Boolean Inverse monoids arise in various recent areas of noncommutative Stone Duality.

Theorem (Lawson, 2009,2011)

The category of Boolean inverse $\wedge$-semigroups is dual to the category of Hausdorff Boolean groupoids.

Theorem (Kudryavtseva,Lawson 2012)

The category of Boolean inverse semigroups is dual to the category of Boolean groupoids.

Green's Relations

Let S be an inverse monoid. Define:

1. $\mathcal{J}$ on S: $a\mathcal{J}b$ iff $SaS = SbS$ (i.e. equality of principal ideals).
2. $\mathcal{D}$ on $E(S)$: $e\mathcal{D}f$ iff $\exists_{a \in S}(e = dom(a), f = ran(a), e \xrightarrow{a} f)$

Consider $E(S)/\mathcal{D}$, S boolean. For idempotents $e, f \in E(S)$, define $[e] \mathbin{\widetilde{\oplus}} [f]$ as follows: *if* we can find orthogonal idempotents $e' \in [e], f' \in [f]$, let $[e] \mathbin{\widetilde{\oplus}} [f] := [e' \vee f']$. Otherwise, undefined.

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Let S be a factorizable Boolean inverse monoid. Then:

- *$\mathcal{D}$ preserves complementation and $(E(S)/\mathcal{D}, \widetilde{\oplus}, [0], [1])$ is an effect algebra w/ RDP.*

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Call these Foulis Monoids.

Coordinatizing MV Algebras: Main Theorem

- For Foulis monoids S as in the Proposition, $\mathcal{D} = \mathcal{J}$.
- Can identify $E(S)/\mathcal{D}$ with the poset of principal ideals $S/\mathcal{J}$.
- We say S *satisfies the lattice condition* if $S/\mathcal{J}$ is a lattice. It is then in fact an MV-algebra (by Bennet & Foulis).

Theorem (Coordinatization Theorem for MV Algebras: L& S)

For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid S satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$, as MV algebras.

Some Mundici Examples (1991): Coordinatizations

+ MSc. Thesis of Wei Lu

Denumerable MV Algebra	AF C*-correspondent
$\{0,1\}$	$\mathbb{C}$
Chain $\mathcal{M}_n$	$Mat_n(\mathbb{C})$
Finite	Finite Dimensional
Dyadic Rationals	CAR algebra of a Fermi gas
$\mathbb{Q} \cap [0,1]$	Glimm's universal UHF algebra
Chang Algebra	Behncke-Leptin algebra
Real algebraic numbers in [0,1]	Blackadar algebra B.
Generated by an irrational $\rho \in [0,1]$	Effros-Shen Algebra $\mathfrak{F}_p$
Finite Product of Post MV-algebras	Continuous Trace
Free on $\aleph_0$ generators	Universal AF C*-algebra $\mathfrak{M}$
Free on one generator	Farey AF C*-algebra $\mathfrak{M}_1$. Mundici (1988), Boca (2008)

Towards AF inverse monoids

Methodology: redo Bratteli theory, using rook (or boolean) matrices

- A *rook matrix* in $Mat_n(\{0,1\})$ is one where every row and column have at most one 1. Let $R_n :=$ rook matrices.
- There's bijection $\mathcal{I}_n \xrightarrow{\cong} R_n$: $f \mapsto M(f)$, where $M(f)_{ij} = 1$ iff $i = f(j)$.

Up to isomorphism, it's possible to redo the entire theory of Bratteli diagrams using rook matrices and $\mathcal{I}_n$'s instead of $\mathbb{Z}$'s.

Rook Matrices

1. Given $A \in R_m, B \in R_n$, define $A \oplus B := Diag(A, B) \in R_{m+n}$.
2. Define $sA = A \oplus \cdots \oplus A$ (s times). Ditto $\bigoplus_{i=1}^{n} s_i A_i$.
3. Interested in *letter isos*: those wrt a chosen total order on $\mathbf{n}$.
4. *Standard morphisms* $R_{m(1)} \times \cdots \times R_{m(k)} \xrightarrow{\sigma} R_n$ given by $(A_1, \cdots, A_k) \mapsto s_1 A_1 \oplus \cdots \oplus s_k A_k$ for some $s_i \in \mathbb{N}$. More generally, $R_{m(1)} \times \cdots \times R_{m(k)} \xrightarrow{\sigma} R_{n(1)} \times \cdots \times R_{n(l)}$ arises via a matrix (s_{ij}) of coefficients in $\mathbb{N}$ + combinatorial condn.

Lemma (Standard Map Lemma: Rough Version)

Every morphism $\mathcal{I}_{m(1)} \times \cdots \times \mathcal{I}_{m(k)} \xrightarrow{\theta} \mathcal{I}_{n(1)} \times \cdots \times \mathcal{I}_{n(l)}$ *factors as* $\beta^{-1}\sigma\alpha$ *for some standard map* σ *and letter isos.*

Bratteli Diagrams, AF Inverse Monoids and colimits of $\mathcal{I}_n$s

Recall $B = (V, E)$ a Bratteli diagram.

$$\begin{array}{lcccc} V(i) & m(1) & m(2) & \cdots & m(k) \\ V(i+1) & n(1) & n(2) & \cdots & n(l) \end{array}$$

Draw s_{ij}-many edges between $m(j)$ to $n(i)$.

Now associate
$$\begin{array}{ccc} V(0) & \leftrightarrow & S_0 = \mathcal{I}_1 \cong \{0,1\} \\ \vdots & \vdots & \vdots \\ V(i) & \leftrightarrow & S_i = \mathcal{I}_{m(1)} \times \cdots \times \mathcal{I}_{m(k)} \end{array}$$

Monomorphisms $\sigma_i : S_i \to S_{i+1}$ are induced by standard maps.

Combinatorial Conditions are true

An AF Inverse Monoid $I(B) := colim(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$, for Bratteli diagram B.

AF Inverse Monoids and colimits of $\mathcal{I}_n$s

Lemma

(1) Colimits of ω-chains $(S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \cdots)$ of boolean inverse $\wedge$-monoids with monos inherit all the nice features of the factors. In particular, the groups of units are direct limits of groups of units of the S_i.
(2) Given any ω-sequence of semisimple inverse monoids and injective morphisms, the colim(S_i) is isomorphic to $I(B)$, for some Bratteli diagram B.

Theorem

AF inverse monoids are Dedekind finite Boolean inverse monoids in which $\mathcal{D}$ preserves complementation. Their groups of units are direct limits of finite direct products of finite symmetric groups.

The General Coordinatization Theorem

Theorem (Coordinatization Theorem for MV Algebras: L& S)

For each countable MV algebra $\mathcal{A}$, there is a Foulis monoid S satisfying the lattice condition such that $S/\mathcal{J} \cong \mathcal{A}$.

Proof sketch: We know from Mundici every MV algebra $\mathcal{A}$ is isomorphic to an MV-algebra $[0, u]_G$, an interval effect algebra for some order unit u in a countable ℓ-group G. It turns out that G is a countable dimension group. Thus there is a Bratteli diagram B yielding G. Take then $I(B)$, the AF inverse monoid of B. It turns out that $I(B)/\mathcal{J}$ is isomorphic to $[0, u]$ as an MV effect-algebra, and the latter will be a lattice, thus a Foulis monoid. So, we have coordinatized $\mathcal{A}$.

Example 1: Coordinatizing Finite MV-Algebras

Let $\mathcal{I}_n = \mathcal{I}_X$ be the inverse monoid of partial bijections on n letters, $|X| = n$. One can show that all the $\mathcal{I}_n$'s are Foulis monoids. The idempotents in this monoid are partial identities 1_A, where $A \subseteq X$. Two idempotents $1_A \mathcal{D} 1_B$ iff $|A| = |B|$. Indeed we get a bijection $\mathcal{I}_n/\mathcal{J} \xrightarrow{\cong} \mathbf{n+1}$, where $\mathbf{n+1} = \{0, 1, \cdots, n\}$. This induces an order isomorphism, where $\mathbf{n+1}$ is given its usual order, and lattice structure via *max, min*.

The effect algebra structure of $\mathcal{I}_n/\mathcal{J}$ becomes: let $r, s \in \mathbf{n+1}$. $r \mathbin{\widetilde{\oplus}} s$ is defined and equals $r + s$ iff $r + s \leqslant n$. The orthocomplement $r' = n - r$. The associated MV algebra: $r \oplus s = r + min(r', s)$, which equals $r + s$ if $r + s \leqslant n$ and $r \oplus s$ equals n if $r + s > n$.

We get an iso $\mathcal{I}_n/\mathcal{J} \cong \mathcal{M}_n$, the Łukasiewicz chain. But every *finite* MV algebra is a product of such chains, which are then coordinatized by a product of $\mathcal{I}_n$'s.

Example 2: Coordinatizing Dyadic Rationals–Cantor Space

Cuntz (1977) studied C*-algebras of isometries (of a sep. Hilbert space). Also arose in wavelet theory & formal language theory (Nivat, Perrot). We'll describe C_n the nth Cuntz inverse monoid.

Cantor Space A^ω, A finite. For C_n, pick $|A| = n$. For C_2, pick $A = \{a, b\}$. Given the usual topology, we have:

1. Clopen subsets have the form XA^ω, where $X \subseteq A^*$ are *Prefix codes* : finite subsets s.t. $x \precsim y$ (y prefix of x) $\Rightarrow x = y$.
2. Representation of clopen subsets by prefix codes is not unique. E.g. $aA^\omega = (aa + ab)A^\omega$.
3. We can make prefixes X in clopens uniquely representable: define *weight* by $w(X) = \sum_{x \in X} |x|$. **Theorem:** *Every* clopen is generated by a unique prefix code X of minimal weight.

Cuntz and n-adic AF-Inverse Monoids

Definition (The Cuntz inverse monoid, Lawson (2007))

$C_n \subseteq \mathcal{I}_{A^\omega}$ consists of those partial bijections on prefix sets of same cardinality: $(x_1 + \cdots x_r)A^\omega \longrightarrow (y_1 + \cdots y_r)A^\omega$ such that $x_i u \mapsto y_i u$, for any right infinite string u.

Proposition (Lawson (2007))

C_n is a Boolean inverse $\wedge$-monoid, whose set of idempotents $E(C_n)$ is the unique countable atomless B.A. Its group of units is the Thompson group V_n.

Definition (n-adic inverse monoid $Ad_n \subseteq C_n$)

Ad_n = those partial bijections in C_n of the form $x_i \mapsto y_i$, where $|x_i| = |y_i|$, $i \leqslant r$. Ad_2 = the dyadic inverse monoid.

Cuntz and Dyadic AF-Inverse Monoids

Theorem

The MV-algebra of dyadic rationals is co-ordinatized by Ad_2.

The proof uses Bernoulli measures on Cantor spaces.

Proposition (Characterizing Ad_2 as an AF monoid)

The dyadic inverse monoid is isomorphic to the direct limit of the sequence of symmetric inverse monoids (partial bijections)

$$\mathcal{I}_2 \to \mathcal{I}_4 \to \mathcal{I}_8 \to \cdots$$

called the CAR inverse monoid. The group of units is a colimit of symmetric groups: $Sym(1) \to Sym(2) \to \cdots Sym(2^r) \to \cdots$.

Cuntz and Dyadic AF-Inverse Monoids: Invariant Measures

General theory of measures on Cantor Space is recent research (Akin, Handelman, ...). Look at simple *Bernoulli Measures*.

Definition

Let S be a Boolean inverse monoid. An invariant measure *is a function $\mu : E(S) \to [0,1]$ satisfying: (i) $\mu(1) = 1$,*
(ii) $\forall s \in S(\mu(s^{-1}s) = \mu(ss^{-1}))$,
(iii) If $e, f \in E(S), e \perp f$ then $\mu(e \vee f) = \mu(e) + \mu(f)$.

A good invariant measure μ is an invariant measure such that:
$\mu(e) \leqslant \mu(f) \;\Rightarrow \exists e'[e' \leqslant f \wedge \mu(e) = \mu(e')]$

Example If $|A| = n$ and $a \in A$, let $\mu(a) = \frac{1}{n}$. If $x \in A^*$, let $\mu(x) = \frac{1}{n^{|x|}}$. For prefix set X, let $\mu(X) = \sum_{x \in X} \mu(x)$.
(If $n = 2$, μ is called *Bernoulli measure*.)

Bernoulli Measures

A general property:

Lemma

If S is a boolean inverse monoid with a good invariant measure μ that reflects the $\mathcal{D}$ relation (i.e. $\mu(e) = \mu(f) \Rightarrow e\mathcal{D}f$) then S is (i) Dedekind finite, (ii) $\mathcal{D}$ preserves complementation, and (iii) $S/\mathcal{J}$ is linearly ordered.

Lemma

Ad_2 has a good invariant measure that reflects the $\mathcal{D}$ relation. Hence $Ad_2/\mathcal{J}$ is linearly ordered.

The main coordinatization theorem in this example then follows:

M. Lawson , P. Scott, AF Inverse Monoids and the structure of Countable MV Algebras, *J. Pure and Applied Algebra* 221 (2017), pp. 45–74.

Coordinatizing $\mathbb{Q} \cap [0, 1]$: thesis of Wei Lu

Definition (Omnidivisional sequence)

A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is omnidivisional if it satisfies the following properties.

- *For all i, $n_i \mid n_{i+1}$.*
- *For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.*

Example

The sequence $\{n!\}_{n=1}^{\infty}$.

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Example

The sequence $\{n!\}_{n=1}^{\infty}$.

Theorem (Coordinatization of the rationals)

Let $D = \{n_i\}_{n=1}^{\infty}$ be an omnidivisional sequence. Then, the directed colimit of the sequence

$$Q\colon \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \ldots,$$

coordinatizes $\mathbb{Q} \cap [0, 1]$.

Open Questions

- Coordinatize all the countable examples from Mundici.
- Recent work of Fred Wehrung shows that *all* MV algebras can be coordinatized. But in the countable case, he does not obtain AF inverse monoids. Can full coordinatization be done using our methods, in the general case?
- Constructing Boolean Inverse semigroups from lattice-ordered abelian groups (à la Wehrung). E.g. Wehrung has recent representation theorems: for every such G, G^+ arises as $(E(S)/\mathcal{D}, \oplus)$ in a universal way, for some Boolean Inverse S. Lawson and I are working on new approaches to Wehrung's representation theorems.
- (Speculative!) Elliott's original construction of what he calls the "local semigroup" of an AF C*-algebra was the main construction that our original paper parallels. In fact, Elliott constructs an effect algebra. Are effect algebras a finer class of invariants for certain of these C*-algebras?