Towards a Typed Geometry of Interaction

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Abstract. Girard's Geometry of Interaction (GoI) develops a mathematical framework for modelling the dynamics of cut-elimination. We introduce a typed version of GoI, called Multiobject GoI (MGoI) for multiplicative linear logic without units in categories which include previous (untyped) GoI models, as well as models not possible in the original untyped version. The development of MGoI depends on a new theory of partial traces and trace classes, as well as an abstract notion of orthogonality (related to work of Hyland and Schalk) We develop Girard's original theory of types, data and algorithms in our setting, and show his execution formula to be an invariant of Cut Elimination. We prove Soundness and Completeness Theorems for the MGoI interpretation in partially traced categories with an orthogonality.

1 Introduction

Geometry of Interaction (GoI) is a novel interpretation of linear logic, introduced by Girard in a fundamental series of papers beginning in the late 80's [12, 11, 13] and continued recently in [14]. One striking feature of this work is that it provides a mathematical framework for modelling cut-elimination (normalization) as a dynamical process of information flow, independent of logical syntax. To these ends, Girard introduces methods from functional analysis and operator algebras to model proofs and their dynamical behaviour. At the same time, these methods allow GoI to provide new foundational insights into the theory of algorithms.

Girard's original framework, based on C*-algebras, was studied in detail in several works of Danos and Regnier (for example in [8]) and by Malacaria and Regnier [26]. The GoI program itself has been applied to the analysis of optimal reduction by Gonthier, Abadi, and Lévy [9], to complexity theory [6], to game semantics and token machines [5, 24], etc.

Let us briefly recall some aspects of Girard's original GoI. Traditional denotational semantics models normalization of proofs (or lambda terms) by static equalities: if Π, Π' are proofs and if Π reduces to Π' by cut-elimination, then in any appropriate model, $[\![\Pi]\!] = [\![\Pi']\!]$. Instead, in his GoI program, Girard considers proofs (or algorithms) as operators, pictured as I/O boxes: a proof of a sequent $\vdash \Gamma$ is interpreted as a box with input and output wires labelled by Γ. The formulas or types in Γ form the I/O-*interface* of the proof. Girard works in an *untyped* setting, so in fact the labels of the wires range over a space U satisfying various domain equations (see below). Now consider a proof Π of a sequent $\vdash [\Delta], \Gamma$, where Δ is a list of all the cut-formulas used. Girard associates to such a proof a pair of operators (u, σ), where u is a hermitian of norm at most 1, and σ is a partial symmetry representing the cuts Δ. The dynamics of cut-elimination may now be captured in a solution of a system of *feedback* equations, summarized in an operator $EX(u, \sigma)$ (the *Execution Formula*). We remark that our general categorical framework (based on partial traces) permits a structured approach to solving such feedback equations and deriving properties of the Execution formula. Finally, it can be shown ([12, 17]) that for denotations of proofs ($u = [\![\Pi]\!]$) of appropriate types in System F, $EX([\![\Pi]\!], \sigma)$ is an invariant of cut-elimination.

Categorical foundations of GoI were initiated in the 90's in lectures by M. Hyland and by S. Abramsky. An early categorical framework was given in [4]. Recent work has stressed the role of Joyal-Street-Verity's *traced monoidal categories* [23] (with additional structure). For example, Abramsky's *GoI situations* [1, 15, 3] provide a basic algebraic foundation for GoI for multiplicative, exponential linear logic (MELL). Recently, we used a special kind of GoI situation (with traced unique decomposition categories) to axiomatize the details of Girard's original GoI 1 paper [17].

In our previous papers, we emphasized several important aspects of Girard's seminal work (at least in GoI 1 and 2).

1. The original Girard framework is essentially *untyped*: there is a reflexive object U in the underlying model (with various retractions and/or domain isomorphisms, e.g. $U \otimes U \lhd U$).
2. Cut-elimination is interpreted by *feedback*, naturally represented in traced monoidal categories. The execution formula, defined via trace, provides an invariant for cut-elimination.
3. Girard introduced an *orthogonality* operation $\perp$ on endomaps of U together with the notion of *types* (as sets of endomaps equal to their biorthogonal).
4. There are notions of *data* and *algorithm* encoded into this dynamical setting, with fundamental theorems connecting types, algorithms, and the convergence of the execution formula.

Points (1) and (2) above were already emphasized in the Abramsky program, as well as in the work of Danos and Regnier [1, 3, 17, 8]. Orthogonalities have been studied abstractly by Hyland and Schalk [21]. The points (1)–(4) are critical to our view of GoI in [17, 18] and to the technical developments in this paper.

Alas, Girard's original GoI is not without its own share of syntactical bureaucracy: there are domain isomorphisms (of the reflexive object U) and an associated $*$-algebra of codings and uncodings. On the one hand, this means the original GoI interpretation of proofs is essentially untyped (i.e. categorically, proofs are interpreted in the monoid $Hom(U, U)$, using the above-mentioned algebra) (see [3, 17, 18]). On the other hand, this led Danos and Regnier ([8]) to study this algebra in detail in certain concrete models, leading to their extensive analysis of reduction paths in untyped lambda calculus.

Our aim in this paper is to move away from "uni-object GoI" to a typed version. This permits us to both generalize GoI and axiomatize its essential features. For example, by removing reflexive objects U, we also unlock the possibilities of generalizing Girard-style GoI to more general tensor categories including cases where the tensor is "product-like" in addition to "sum-like". We shall illustrate both of these styles in the examples below.

The contributions of this paper can be summarized as follows:

- We introduce an axiomatization for partially traced symmetric monoidal categories and provide examples based on $\mathbf{Vec_{fd}}$, finite dimensional vector spaces, and **CMet**, complete metric spaces. Our axiomatization is different from that in [2], although related in spirit.
- We introduce an abstract orthogonality [21], appropriate for GoI, on our models.
- We introduce a multiobject version of Girard's GoI semantics (MGoI) in partially traced models with orthogonality. This includes Girard's notions of *types, datum, algorithm* and the *execution formula*. We give an MGoI interpretation for the multiplicative fragment of linear logic without units (MLL) and show that the execution formula is an invariant of cut-elimination (see Section 5 below). Recall that Girard's original GoI (as presented in [3]) requires a reflexive object $U \neq \{\mathbf{0}\}$, with a retraction $U \oplus U \lhd U$, which is impossible in $\mathbf{Vec_{fd}}$.
- We prove a soundness and completeness theorem for our MGoI interpretation of MLL in arbitrary partially traced categories with an orthogonality relation. As an application, we can also prove a completeness result for untyped GoI semantics of MLL (see our [17]) in a traced UDC based GoI Situation; the latter result will appear in the journal version of this paper.

It is worth remarking that GoI does not work well with units. They are not part of the original interpretation ([12]), and fail to satisfy the properties demanded by the main theorems. In [18] we show that the "natural" category of types and associated morphisms in certain GoI-situations fails to have tensor and par units act correctly. We suspect the same is true for the MGoI case introduced here.

The rest of the paper is organized as follows. In Section 2 we introduce partially traced symmetric monoidal categories and discuss some examples. In Section 3 we introduce the abstract orthogonality relation in a partially traced symmetric monoidal category and discuss how it relates to the work in [21]. In Section 4 we introduce our new semantics, MGoI, and give an interpretation for MLL. Section 5 discusses the execution formula and the soundness theorem, while in Section 6 we prove a completeness theorem for the MGoI interpretation of MLL in a partially traced category with an orthogonality relation. Finally, Section 7 contains some thoughts about possible future directions, projects and links to related work in the literature.

Note: The full proofs of the results here will appear in the journal version of this paper, available on our websites.

2 Trace Class

The notion of categorical trace was introduced by Joyal, Street and Verity in an influential paper [23]. The motivation for their work arose in algebraic topology and knot theory, although the authors were aware that such traces also have many applications in Computer Science, where they include such notions as feedback, fixedpoints, iteration theories, etc. For references and history, see [1, 3, 17].

In this paper we go one step further and look at partial traces. The idea of generalizing the abstract trace of [23] to the partial setting is not new. For example, partial traces were already studied in work of Abramsky, Blute, and Panangaden [2], in unpublished lecture notes of Gordon Plotkin [27], work of A. Jeffrey [22] (discussed below) and others. The guiding example in [2] is the relationship between trace class operators on a Hilbert space and Hilbert-Schmidt operators. This allows the authors to establish a close correspondence between trace and nuclear ideals in a tensor $*$-category. Plotkin's work develops a theory of Conway ideals on biproduct categories, and an associated categorical trace theory. Unfortunately none of these extant theories is appropriate for Girard's GoI. So we present an axiomatization for partial traces suitable for our purposes.

Recall, following Joyal, Street, and Verity [23], a (parametric) trace in a symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ is a family of maps $Tr^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \to \mathbb{C}(X, Y)$, satisfying various well-known naturality equations. A *partial* (parametric) trace requires instead that each $Tr^U_{X,Y}$ be a partial map (with domain denoted $\mathbb{T}^U_{X,Y}$) and satisfy various closure conditions.

Definition 1 (Trace Class). Let $(\mathbb{C}, \otimes, I, s)$ be a symmetric monoidal category. A *(parametric) trace class* in $\mathbb{C}$ is a choice of a family of subsets, for each object U of $\mathbb{C}$, of the form

$$\mathbb{T}^U_{X,Y} \subseteq \mathbb{C}(X \otimes U, Y \otimes U) \text{ for all objects } X, Y \text{ of } \mathbb{C}$$

together with a family of functions, called a *(parametric) partial trace*, of the form

$$Tr^U_{X,Y} : \mathbb{T}^U_{X,Y} \to \mathbb{C}(X, Y)$$

subject to the following axioms. Here the parameters are X and Y and a morphism $f \in \mathbb{T}^U_{X,Y}$, by abuse of terminology, is said to be *trace class.*

- **Naturality** in X and Y: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \to X$ and $h : Y \to Y'$,

$$(h \otimes 1_U) f (g \otimes 1_U) \in \mathbb{T}^U_{X',Y'}$$

and $$Tr^U_{X',Y'}((h \otimes 1_U) f (g \otimes 1_U)) = h\, Tr^U_{X,Y}(f)\, g$$

– **Dinaturality** in U: For any $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$,

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \;\text{ iff }\; f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y},$$

$$\text{and} \qquad Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g)).$$

– **Vanishing I**: $\mathbb{T}^I_{X,Y} = \mathbb{C}(X \otimes I, Y \otimes I)$ and for $f \in \mathbb{T}^I_{X,Y}$

$$Tr^I_{X,Y}(f) = \rho_Y f \rho_X^{-1}.$$

Here $\rho_A : A \times I \to A$ is the right unit isomorphism of the monoidal category.

– **Vanishing II**: For any $g : X \otimes U \otimes V \to Y \otimes U \otimes V$, if $g \in \mathbb{T}^V_{X \otimes U, Y \otimes U}$, then

$$g \in \mathbb{T}^{U \otimes V}_{X,Y} \;\text{ iff }\; Tr^V_{X \otimes U, Y \otimes U}(g) \in \mathbb{T}^U_{X,Y},$$

$$\text{and} \qquad Tr^{U \otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \otimes U, Y \otimes U}(g)).$$

– **Superposing**: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : W \to Z$,

$$g \otimes f \in \mathbb{T}^U_{W \otimes X, Z \otimes Y},$$

$$\text{and} \qquad Tr^U_{W \otimes X, Z \otimes Y}(g \otimes f) = g \otimes Tr^U_{X,Y}(f).$$

– **Yanking**: $s_{UU} \in \mathbb{T}^U_{U,U}$, and $Tr^U_{U,U}(s_{U,U}) = 1_U$.

A symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ with such a trace class is called a *partially traced category*, or a *category with a trace class*. If we let X and Y be I (the unit of the tensor), we get a family of operations $Tr^U_{I,I} : \mathbb{T}^U_{I,I} \to \mathbb{C}(I, I)$ defining what we call a *non-parametric trace*.

Remark 1. An early definition of a partial parametric trace is due to Abramsky, Blute and Panangaden in [2]. Our definition is different but related to theirs. First, we have used the Yanking axiom in Joyal, Street and Verity [23], whereas in [2] they use a conditional version of the so-called “generalized yanking”; that is, for $f : X \to U$ and $g : U \to Y$, $Tr^U_{X,Y}(s_{U,Y}(f \otimes g)) = gf$ *whenever* $s_{U,Y}(f \otimes g)$ is of trace class. It was shown in [15] that for traced monoidal categories the two axioms of yanking and generalized yanking are equivalent in the presence of all the other axioms. This equivalence remains valid for the partially traced categories introduced here. In our theory s_{UU} is traceable for all U; on the other hand, many examples in [2] do not have this property. Our Vanishing II axiom differs from and is weaker than the one proposed in [2]: it is a “conditional” equivalence. More importantly, we do not require one of the ideal axioms in [2]. Namely, we do **not** ask that for $f \in \mathbb{T}^U_{X,Y}$ and any $h : U \to U$, $(1_Y \otimes h)f$ and $f(1_X \otimes h)$ be in $\mathbb{T}^U_{X,Y}$. Indeed in the next section we prove that the categories $(\mathbf{Vec_{fd}}, \oplus)$ of finite dimensional vector spaces, and $(\mathbf{CMet}, \times)$ of complete metric spaces are partially traced. It can be shown that in both categories the above ideal axiom and Vanishing II of [2] fail and hence they are not traced in the sense of ABP. In defense of not enforcing this ideal axiom, we observe that it is not required for any of the trace axioms. Any partially traced category in the sense of ABP for which the yanking axiom holds will be partially traced according to our definition. Finally, we observe that the nonparametric version of our partial trace is also different from the one in [2].

Other notions of categorical partial trace have been examined by Alan Jeffrey [22] and also by various category theorists. For example, Jeffrey cuts down the domain of the trace operator to admissible (traceable) objects U which form a full subcategory of the original category. This is not possible for us: our trace classes do not form subcategories. For example, in keeping with functional analysis on infinite dimensional spaces, the ABP theory of traced ideals [2], and with Girard's papers on GoI, we do not wish the identity map to be traced; nor are our trace classes necessarily closed under all possible compositions.

One is obliged to say that there are many different approaches to partial categorical traces and ideals; ours is geared to Girard's GoI. We should also note that our examples will not be partially traced categories according to Jeffrey's definition. It is not possible to capture our traceability conditions on morphisms using his approach, as they cannot be characterized as object properties.

2.1 Examples of Partial Traces

(a) Finite Dimensional Vector Spaces

The category $\mathbf{Vec_{fd}}$ of finite dimensional vector spaces and linear transformations is a symmetric monoidal, indeed an additive, category (see [25]), with monoidal product taken to be $\oplus$, the direct sum (biproduct). Hence, given $f : \oplus_I X_i \to \oplus_J Y_j$ with $|I| = n$ and $|J| = m$, we can write f as an $m \times n$ matrix $f = [f_{ij}]$ of its components, where $f_{ij} : X_j \to Y_i$ (notice the switch in the indices i and j).

We give a trace class structure on the category $(\mathbf{Vec_{fd}}, \oplus, \mathbf{0})$ as follows. We shall say an $f : X \oplus U \to Y \oplus U$ is trace class iff $(I - f_{22})$ is invertible, where I is the identity matrix, and I and f_{22} have size $dim(U)$. In that case, we write

$$Tr^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21} \qquad (1)$$

This definition is motivated by a generalization of the fact that for a matrix A, $(I - A)^{-1} = \sum_i A^i$, whenever the infinite sum converges. Clearly this sum converges when the matrix norm of A is strictly less than 1, or when A is nilpotent, but in both cases the general idea is the desire to have $(I - A)$ invertible. If the infinite sum for $(I - f_{22})^{-1}$ exists, the above formula for $Tr^U_{X,Y}(f)$ becomes the usual "particle-style" trace in [1, 3, 17]. One advantage of formula (1) is that it does not *a priori* assume the convergence of the sum, nor even that $(I - f_{22})^{-1}$ be computable by iterative methods.

Proposition 1. $(\mathbf{Vec_{fd}}, \oplus, \mathbf{0})$ *is partially traced, with trace class as above.*

The proof of Proposition 1 uses the following standard facts from linear algebra:

Lemma 1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a partitioned matrix with blocks* A $(m \times m)$, B $(m \times n), C$ $(n \times m)$ *and* D $(n \times n)$. *If* D *is invertible, then* M *is invertible iff* $A - BD^{-1}C$ *(the* Schur Complement of D*) is invertible.*

Lemma 2. *Given* A $(m \times n)$ *and* B $(n \times m)$, $(I_m - AB)$ *is invertible iff* $(I_n - BA)$ *is invertible. Moreover* $(I_m - AB)^{-1}A = A(I_n - BA)^{-1}$.

(b) Other Finite Dimensional Examples

Proposition 1 remains valid for the category ($\mathbf{Hilb_{fd}}$,$\oplus$) of finite dimensional Hilbert spaces and bounded linear maps. As discussed in Remark 1, the category ($\mathbf{Vec_{fd}}, \oplus$) is *not* partially traced in the sense of ABP; nor is it traced in the sense of A. Jeffrey, since (for example) the identity is not trace class.

(c) Metric Spaces

Consider the category **CMet** of complete metric spaces with non-expansive maps. Define $f : (M, d_M) \to (N, d_N)$ to be *non-expansive* iff there is a fixed $0 \leq \alpha \leq 1$ such that $d_N(f(x), f(y)) \leq \alpha d_M(x, y)$, for all $x, y \in M$. Note that the tempting collection of complete metric spaces and contractions ($\alpha < 1$) is not a category: there are no identity morphisms! **CMet** has products, namely given (M, d_M) and (N, d_N) we define $(M \times N, d_{M \times N})$ with $d_{M \times N}((m, n), (m', n')) = max\{d_M(m, m'), d_N(n, n')\}$.

We define the trace class structure on **CMet** (where $\otimes = \times$) as follows. We say that a morphism $f : X \times U \to Y \times U$ is in $\mathbb{T}^U_{X,Y}$ iff for every $x \in X$ the induced map $\pi_2 \lambda u.f(x, u) : U \to U$ has a unique fixed point; in other words, iff for every $x \in X$, there is a unique u, and a y, such that $f(x, u) = (y, u)$. Note that in this case y is necessarily unique. Also, note that contractions have unique fixed points, by the Banach fixed point theorem.

Suppose $f \in \mathbb{T}^U_{X,Y}$. We define $Tr^U_{X,Y}(f) : X \to Y$ by $Tr^U_{X,Y}(f)(x) = y$, where $f(x, u) = (y, u)$ for the unique u. Equivalently, $Tr^U_{X,Y}(f)(x) = \pi_1 f(x, u)$ where u is the unique fixed point of $\pi_2 \lambda t.f(x, t)$.

Proposition 2. $(\mathbf{CMet}, \times, \{*\})$ *is a partially traced category with trace class as above.*

Lemma 3. *Let* A *and* B *be complete metric spaces,* $f : A \to B$ *and* $g : B \to A$. *Then,* gf *has a unique fixed point if and only if* fg *does. Moreover, let* $a \in A$ *be the unique fixed point of* $gf : A \to A$ *and* $b \in B$ *be the unique fixed point of* $fg : B \to B$. *Then* $f(a) = b$ *and* $g(b) = a$.

Proposition 2 remains valid for the category ($\mathbf{Sets}, \times$) of sets and mappings. The latter then becomes a partially traced category with the same definition for trace class morphisms as in **CMet**. However, this fails for the category ($\mathbf{Rel}, \times$), of sets and relations: consider the sets $A = \{a\}$, $B = \{b, b'\}$, and let $f = \{(a, b), (a, b')\}$ and $g = \{(b, a), (b', a)\}$.

(d) Total Traces

Of course, all (totally-defined) traces in the usual definition of a traced monoidal category yield a trace class, namely the entire homset is the domain of Tr. In particular, all the examples in our previous work on uni-object GoI [17, 18], for example based on unique decomposition categories, still apply here.

Remark 2. [A Non-Example]
Consider the structure $(\mathbf{CMet}, \times)$. Defining the trace class morphisms as f such that $\pi_2 \lambda u.f(x, u) : U \to U$ is a contraction for every $x \in X$, does not yield a partially traced category: all axioms are true except for dinaturality and Vanishing II.

3 Orthogonality Relations

Girard originally introduced orthogonality relations into linear logic to model formulas (or types) as sets equal to their biorthogonal (e.g. in the phase semantics of the original paper [10] and in GoI 1 [11]). Recently M. Hyland and A. Schalk gave an abstract approach to orthogonality relations in symmetric monoidal closed categories [21]. They also point out that an orthogonality on a traced symmetric monoidal category $\mathbb{C}$ can be obtained by first considering their axioms applied to $Int(\mathbb{C})$, the compact closure of $\mathbb{C}$, and then translating them down to $\mathbb{C}$. Below we give this translation (not explicitly calculated in [21]), using the so-called "GoI construction" $\mathcal{G}(\mathbb{C})$ [1, 15] instead of $Int(\mathbb{C})$. The categories $\mathcal{G}(\mathbb{C})$ and $Int(\mathbb{C})$ are both compact closures of $\mathbb{C}$, and are shown to be isomorphic in [15]. Alas, we do not have the space to give the details of these constructions; however the reader can safely ignore the remarks above and use the definition below independently of its motivation. To understand the detailed constructions behind the definition, the interested reader is referred to the above references.

As we are dealing with partial traces we need to take extra care in stating the axioms below; namely, an axiom involving a trace should be read with the proviso: "whenever all traces exist".

Definition 2. Let $\mathbb{C}$ be a traced symmetric monoidal category. An *orthogonality relation* on $\mathbb{C}$ is a family of relations $\perp_{UV}$ between maps $u : V \to U$ and $x : U \to V$

$$V \xrightarrow{u} U \ \perp_{UV} \ U \xrightarrow{x} V$$

subject to the following axioms:

(i) *Isomorphism*: Let $f : U \otimes V' \to V \otimes U'$ and $\hat{f} : U' \otimes V \to V' \otimes U$ be such that $Tr^{V'}(Tr^{U'}((1 \otimes 1 \otimes s_{U',V'})\alpha^{-1}(f \otimes \hat{f})\alpha))) = s_{U,V}$ and $Tr^{V}(Tr^{U}((1 \otimes 1 \otimes s_{U,V})\alpha^{-1}(\hat{f} \otimes f)\alpha))) = s_{U',V'}$. Here $\alpha = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ with s at appropriate types. Note that this simply means that $f : (U, V) \to (U', V')$ and $\hat{f} : (U', V') \to (U, V)$ are inverses of each other in $\mathcal{G}(\mathbb{C})$.
Then for all $u : V \to U$ and $x : U \to V$,

$$u \perp_{UV} x \ \text{ iff } \ Tr^{U}_{V',U'}(s_{U,U'}(u \otimes 1_{U'})f s_{V',U}) \perp_{U'V'} Tr^{V}_{U',V'}((1_{V'} \otimes x)\hat{f})$$

(ii) *Tensor*: For all $u : V \to U$, $v : V' \to U'$ and $h : U \otimes U' \to V \otimes V'$,

$$u \perp_{UV} Tr^{U'}_{U,V}((1_V \otimes v)h) \text{ and } v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})hs_{U',U})$$
$$\text{implies} \qquad (u \otimes v) \perp_{U\otimes U',V\otimes V'} h$$

(iii) *Implication*: For all $u : V \to U$, $y : U' \to V'$ and $f : U \otimes V' \to V \otimes U'$

$$u \perp_{UV} Tr^{V'}_{U,V}((1_V \otimes y)f) \text{ and } Tr^{V}_{V',U'}(s_{V,U'}f(u \otimes 1_{V'})s_{V',V}) \perp_{U'V'} y$$
$$\text{implies} \qquad f \perp_{V\otimes U',U\otimes V'} (u \otimes y)$$

(iv) *Identity*: For all $u : V \to U$ and $x : U \to V$

$$u \perp_{UV} x \text{ implies } 1_I \perp_{II} Tr^{V}_{I,I}(xu)$$

(v) *Symmetry*: For all $u : V \to U$ and $x : U \to V$

$$u \perp_{UV} x \text{ iff } x \perp_{VU} u$$

Remark 3. (i) It should be noted that for a (partially) traced symmetric monoidal category, the axioms for Tensor and Implication are equivalent in the presence of the other axioms: by dinaturality of trace we have $Tr^{V}_{V',U'}(s_{V,U'}f(u\otimes 1_{V'})s_{V',V}) = Tr^{U}_{V',U'}(s_{U,U'}(u\otimes 1_{U'})fs_{V',U}))$, then use the Symmetry axiom. Thus we shall drop the Implication axiom.

(ii) Our work on GoI reveals that one needs another axiom which we observe as the converse of the Tensor axiom and relaxation of one of the premises. This is related to abstract computation and the notion of datum in GoI. Hence, we shall replace the Tensor axiom by the following Strong Tensor axiom. Our Strong Tensor axiom is similar to, but *not* the same as the Precise Tensor axiom of [21]. The latter requires an additional property on the biconditional.

Strong Tensor: For all $u : V \to U$, $v : V' \to U'$ and $h : U \otimes U' \to V \otimes V'$,

$$v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})hs_{U',U}) \quad \text{iff} \quad (u \otimes v) \perp_{U\otimes U',V\otimes V'} h,$$

whenever the trace exists. It can be shown that in the presence of the Strong Tensor, Isomorphism, and Symmetry axioms, $v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})hs_{U',U})$ implies $u \perp_{UV} Tr^{U'}_{U,V}((1_V \otimes v)h)$, whenever all traces exist.

Definition 3. Let $\mathbb{C}$ be a traced symmetric monoidal category. A *strong orthogonality* relation is defined as in Definition 2 but with the Tensor axiom replaced by the Strong Tensor axiom above, and the Implication axiom dropped.

In the context of GoI, we will be working with strong orthogonality relations on endomorphism sets of objects in the underlying categories. Biorthogonally closed (i.e. $X = X^{\perp\perp}$) subsets of certain endomorphism sets are important as they define *types* (GoI interpretation of formulae.) We have observed that all the orthogonality relations that we work with in this paper can be characterized using trace classes. This suggests the following, which seems to cover many known examples.

Example 1 (Orthogonality as trace class) Let $(\mathbb{C}, \otimes, I, Tr)$ be a partially traced category where $\otimes$ is the monoidal product with unit I, and Tr is the partial trace operator as in Section 2. Let A and B be objects of $\mathbb{C}$. For $f : A \to B$ and $g : B \to A$, we can define an orthogonality relation by declaring $f \perp_{BA} g$ iff $gf \in \mathbb{T}^A_{I,I}$. It turns out [3] that this is a variation of the notion of *Focussed orthogonality* of Hyland and Schalk [21].

Hence, from our previous discussion on traces, we obtain the following examples:

- $\mathbf{Vec_{fd}}$. For $A \in \mathbf{Vec_{fd}}$, $f, g \in End(A)$, define $f \perp g$ iff $I - gf$ is invertible.
- **CMet**. Let $M \in \mathbf{CMet}$. For $f, g \in End(M)$, define $f \perp g$ iff gf has a unique fixed point.

4 Multi-object GoI Interpretation

In this section we introduce the multiobject Geometry of Interaction semantics for MLL in a partially traced symmetric monoidal category $(\mathbb{C}, \otimes, I, Tr, \perp)$ equipped with an orthogonality relation $\perp$ as in the previous section. Here $\otimes$ is the monoidal product with unit I and Tr is the partial trace operator as in Section 2. We do not require that the category $\mathbb{C}$ have a reflexive object, so uni-object GoI semantics (as in [12, 17]) may not be possible to carry out in $\mathbb{C}$.

Interpreting formulae:

Let A be an object of $\mathbb{C}$ and let $f, g \in End(A)$. We say that f is *orthogonal to* g, denoted $f \perp g$, if $(f, g) \in \perp$. Also given $X \subseteq End(A)$ we define

$$X^{\perp} = \{f \in End(A) \,|\, \forall g \in X, f \perp g\}.$$

We now define an operator on the objects of $\mathbb{C}$ as follows: Given an object A, $\mathcal{T}(A) = \{X \subseteq End(A) \,|\, X^{\perp\perp} = X\}$. We shall also need the notion of a denotational interpretation of formulas. We define an interpretation map $[\![-]\!]$ on the formulas of MLL as follows. Given the value of $[\![-]\!]$ on the atomic propositions as objects of $\mathbb{C}$, we extend it to all formulas by:

- $[\![A^{\perp}]\!] = [\![A]\!]$
- $[\![A \invamp B]\!] = [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$.

We then define the MGoI-interpretation for formulas as follows. We use the notation $\theta(A)$ for this interpretation.

- $\theta(\alpha) \in \mathcal{T}([\![\alpha]\!])$, where α is an atomic formula.
- $\theta(\alpha^{\perp}) = \theta(\alpha)^{\perp}$
- $\theta(A \otimes B) = \{a \otimes b \,|\, a \in \theta(A), b \in \theta(B)\}^{\perp\perp}$
- $\theta(A \invamp B) = \{a \otimes b \,|\, a \in \theta(A)^{\perp}, b \in \theta(B)^{\perp}\}^{\perp}$

[3] We thank the anonymous referee for pointing out this connection.

Two easy consequences of the definition are: (i) for any formula A, $(\theta A)^\perp = \theta A^\perp$, and (ii) $\theta(A) \subseteq End(\llbracket A \rrbracket)$.

Interpretation of Proofs:

We define the MGoI interpretation for proofs of MLL without units, similarly to [17]. Every MLL sequent will be of the form $\vdash [\Delta], \Gamma$ where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have already been made in the proof of $\vdash \Gamma$ (see [12, 17]). This device is used to keep track of the cuts in a proof of $\vdash \Gamma$. A proof Π of $\vdash [\Delta], \Gamma$ is represented by a morphism $\llbracket \Pi \rrbracket \in End(\otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Delta \rrbracket)$. With $\Gamma = A_1, \cdots, A_n$, $\otimes \llbracket \Gamma \rrbracket$ stands for $\llbracket A_1 \rrbracket \otimes \cdots \otimes \llbracket A_n \rrbracket$, similarly for Δ. We drop the double brackets wherever there is no danger of confusion. We also define $\sigma = s \otimes \cdots \otimes s$ (m-copies) where s is the symmetry map at different types (omitted for convenience), and $|\Delta| = 2m$. The morphism σ represents the cuts in the proof of $\vdash \Gamma$, i.e. it models Δ. In the case where Δ is empty (that is for a cut-free proof), we define $\sigma : I \to I$ to be 1_I where I is the unit of the monoidal product in $\mathbb{C}$.

Let Π be a proof of $\vdash [\Delta], \Gamma$. We define the MGoI interpretation of Π, denoted by $\llbracket \Pi \rrbracket$, by induction on the length of the proof as follows.

1. Π is an *axiom* $\vdash A, A^\perp$, $\llbracket \Pi \rrbracket := s_{V,V}$ where $\llbracket A \rrbracket = \llbracket A^\perp \rrbracket = V$.
2. Π is obtained using the *cut* rule on Π' and Π'' that is

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{matrix} \quad \begin{matrix}\Pi'' \\ \vdots \\ \vdash [\Delta''], A^\perp, \Gamma''\end{matrix}}{\vdash [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma''}\ cut$$

   Define $\llbracket \Pi \rrbracket = \tau^{-1}(\llbracket \Pi' \rrbracket \otimes \llbracket \Pi'' \rrbracket)\tau$, where τ is the permutation
   $\Gamma' \otimes \Gamma'' \otimes \Delta' \otimes \Delta'' \otimes A \otimes A^\perp \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes A^\perp \otimes \Gamma'' \otimes \Delta''$,

3. Π is obtained using the *exchange* rule on the formulas A_i and A_{i+1} in Γ'. That is Π is of the form

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{matrix}}{\vdash [\Delta], \Gamma}\ exchange$$

   where $\Gamma' = \Gamma_1', A_i, A_{i+1}, \Gamma_2'$ and $\Gamma = \Gamma_1', A_{i+1}, A_i, \Gamma_2'$. Then, $\llbracket \Pi \rrbracket = \tau^{-1} \llbracket \Pi' \rrbracket \tau$, where $\tau = 1_{\Gamma_1'} \otimes s \otimes 1_{\Gamma_2' \otimes \Delta}$.
4. Π is obtained using an application of the *par* rule, that is Π is of the form:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B\end{matrix}}{\vdash [\Delta], \Gamma', A \parr B}\ \parr \quad . \text{ Then } \llbracket \Pi \rrbracket = \llbracket \Pi' \rrbracket$$

5. Π is obtained using an application of the *times* rule, that is Π is of the form:

$$\dfrac{\begin{array}{cc} \Pi' & \Pi'' \\ \vdots & \vdots \\ \vdash [\Delta'], \Gamma', A & \vdash [\Delta''], \Gamma'', B \end{array}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B} \otimes$$

Then $[\![\Pi]\!] = \tau^{-1}([\![\Pi']\!] \otimes [\![\Pi'']\!])\tau$, where τ is the permutation $\Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \Delta' \otimes \Delta'' \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes \Gamma'' \otimes B \otimes \Delta''$. This corresponds exactly to the definition of tensor product in Abramsky's $\mathcal{G}(\mathbb{C})$ (see [1, 15].)

Example 1. (a) Let Π be the following proof:

$$\dfrac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}} \; cut$$

Then the MGoI semantics of this proof is given by

$$[\![\Pi]\!] = \tau^{-1}(s \otimes s)\tau = s_{V \otimes V, V \otimes V}$$

where $\tau = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ and $[\![A]\!] = [\![A^{\perp}]\!] = V$.

(b) Now consider the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B, B^{\perp} \quad \vdash C, C^{\perp}}{\vdash B, C, B^{\perp} \otimes C^{\perp}}}{\vdash B, B^{\perp} \otimes C^{\perp}, C}}{\vdash B^{\perp} \otimes C^{\perp}, B, C}}{\vdash B^{\perp} \otimes C^{\perp}, B \parr C} \; .$$

Its denotation is $s_{V \otimes W, V \otimes W}$, where $[\![B]\!] = [\![B^{\perp}]\!] = V$ and $[\![C]\!] = [\![C^{\perp}]\!] = W$.

Proposition 3. *Let Π be an MLL proof of $\vdash [\Delta], \Gamma$ where $|\Delta| = 2m$ and $|\Gamma| = n$ (counting occurrences of propositional variables). Then $[\![\Pi]\!]$ is a fixed-point free involutive permutation on $n + 2m$ objects of $\mathbb{C}$. That is $[\![\Pi]\!] : V_1 \otimes \cdots \otimes V_{n+2m} \to V_1 \otimes \cdots \otimes V_{n+2m}$ induces a permutation π on $\{1, 2 \cdots, n + 2m\}$ and*

- $\pi^2 = 1$
- *For all $i \in \{1, 2, \cdots, n + 2m\}$, $\pi(i) \neq i$.*
- *For all $i \in \{1, 2, \cdots, n + 2m\}$, $V_i = V_{\pi(i)}$.*

4.1 Dynamics

Dynamics is at the heart of the GoI interpretation as compared to denotational semantics and it is hidden in the cut-elimination process. The mathematical model of cut-elimination is given by the so called *execution formula* defined as follows:

$$EX([\![\Pi]\!], \sigma) = Tr^{\otimes \Delta}_{\otimes \Gamma, \otimes \Gamma}((1 \otimes \sigma)[\![\Pi]\!]) \tag{2}$$

where Π is a proof of the sequent $\vdash [\Delta], \Gamma$, and $\sigma = s \otimes \cdots \otimes s$ (m times) models Δ. Note that $EX([\![\Pi]\!], \sigma)$ is a morphism from $\otimes\Gamma \to \otimes\Gamma$, when it exists. We shall prove below (see Theorem 2) that the execution formula always exists for any MLL proof Π.

Example 2. Consider the proof Π in Example 1 above. Recall also that $\sigma = s$ in this case ($m = 1$). Then $EX([\![\Pi]\!], \sigma) = Tr((1 \otimes s_{V,V}) s_{V \otimes V, V \otimes V}) = s_{V,V}$.

Note that in this case we have obtained the MGoI interpretation of the cut-free proof of $\vdash A, A^{\perp}$, obtained by applying Gentzen's Hauptsatz to the proof Π.

5 Soundness of the Interpretation

In this section we state one of the main results of this paper: the soundness of the MGoI interpretation. We show that if a proof Π is reduced (via cut-elimination) to another proof Π', then $EX([\![\Pi]\!], \sigma) = EX([\![\Pi']\!], \tau)$; that is, $EX([\![\Pi]\!], \sigma)$ is an invariant of reduction. In particular, if Π' is cut-free (i.e. a normal form) we have $EX([\![\Pi]\!], \sigma) = [\![\Pi']\!]$. Intuitively this says that if one thinks of cut-elimination as computation then $[\![\Pi]\!]$ can be thought of as an algorithm. The computation takes place as follows: if $EX([\![\Pi]\!], \sigma)$ exists then it yields a datum (cf. cut-free proof). This intuition will be made precise below (Theorems 2 & 3).

The next fundamental lemma follows directly from our trace axioms:

Lemma 4 (Associativity of cut). *Let Π be a proof of $\vdash [\Gamma, \Delta], \Lambda$ and σ and τ be the morphisms representing the cut-formulas in Γ and Δ respectively. Then*

$$EX([\![\Pi]\!], \sigma \otimes \tau) = EX(EX([\![\Pi]\!], \tau), \sigma) = EX(EX((1 \otimes s)[\![\Pi]\!](1 \otimes s), \sigma), \tau),$$

whenever all traces exist. (This is essentially the Church-Rosser Property).

Definition 4. Let $\Gamma = A_1, \cdots, A_n$ and $V_i = [\![A_i]\!]$.

- A *datum of type $\theta\Gamma$* is a morphism $M : \otimes_i V_i \to \otimes_i V_i$ such that for any $\beta_i \in \theta(A_i^{\perp})$, $\otimes_i \beta_i \perp M$ and $M \centerdot \beta_1 := Tr^{V_1}(s^{-1}_{\otimes_i V_i, V_1}(\beta_1 \otimes 1_{V_2} \otimes \cdots \otimes 1_{V_n}) M s_{\otimes_i V_i, V_1})$ exists. (In Girard's notation [12], $M \centerdot \beta_1$ corresponds to $ex(CUT(\beta_1, M))$.)
- An *algorithm of type $\theta\Gamma$* is a morphism $M : \otimes_i V_i \otimes [\![\Delta]\!] \to \otimes_i V_i \otimes [\![\Delta]\!]$ for some $\Delta = B_1, B_2, \cdots, B_{2m}$ with m a nonnegative integer and $B_{i+1} = B_i^{\perp}$ for $i = 1 \cdots, 2m-1$, such that if $\sigma : \otimes_{j=1}^{2m} [\![B_j]\!] \to \otimes_{j=1}^{2m} [\![B_j]\!]$ is $\otimes_{j=1}^{2m-1} s_{[\![B_j]\!], [\![B_{j+1}]\!]}$, $EX(M, \sigma)$ exists and is a datum of type $\theta\Gamma$.
(Here σ is defined to be 1_I for $m = 0$.)

Lemma 5. *Let $\Gamma = A_2, \cdots, A_n$, $V_i = [\![A_i]\!]$, and $M : \otimes_i V_i \to \otimes_i V_i$, for $i = 1, \cdots, n$. Then, M is a datum of type $\theta(A_1, \Gamma)$ iff for every $a_1 \in \theta(A_1^{\perp})$, $M \centerdot a_1$ (defined as above) exists and is in $\theta(\Gamma)$.*

Theorem 2 (Proofs as algorithms). *Let Π be an MLL proof of a sequent $\vdash [\Delta], \Gamma$. Then $[\![\Pi]\!]$ is an algorithm of type $\theta\Gamma$.*

Corollary 1 (Existence of Dynamics). *Let Π be an MLL proof of a sequent $\vdash [\Delta], \Gamma$. Then $Ex([\![\Pi]\!], \sigma)$ exists.*

Theorem 3 (EX is an invariant). *Let Π be an MLL proof of a sequent $\vdash [\Delta], \Gamma$. Then,*

- *If Π reduces to Π' by any sequence of cut-eliminations, then $EX([\![\Pi]\!], \sigma) = EX([\![\Pi']\!], \tau)$. So $EX([\![\Pi]\!], \sigma)$ is an invariant of reduction.*
- *In particular, if Π' is any cut-free proof obtained from Π by cut-elimination, then $EX([\![\Pi]\!], \sigma) = [\![\Pi']\!]$.*

6 Completeness

In this section we give a completeness theorem for MLL in a partially traced category equipped with an orthogonality relation, under MGoI semantics. Recall from Proposition 3 that the denotation of a proof $[\![\Pi]\!]$ induces a fixed-point free involutive permutation. We now seek a converse.

Theorem 4 (Completeness). *Let M be a fixed-point free involutive permutation from $V_1 \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_n$ (induced by a permutation μ on $\{1, 2, \cdots, n\}$) where $n > 0$ is an even integer, $V_i = [\![A_i]\!]$, and $V_i = V_{\mu(i)}$ for all $i = 1, \cdots, n$. Then there is a provable MLL formula φ built from the A_i, with a proof Π such that $[\![\Pi]\!] = M$.*

Motivated by this result, we can also prove a completeness theorem for MLL in any traced Unique Decomposition Category with a reflexive object, under (uni-object) GoI semantics [17]. This will appear in the full journal article.

7 Conclusion and Future Work

In this work we introduce a new semantics called multiobject Geometry of Interaction (MGoI). This semantics, while inspired by GoI, differs from it in significant points. Namely, we deal with many objects in the underlying category, we make use of a denotational semantics to define the interpretation of logical formulas and we develop the execution formula based on a new theory of partial traces and trace classes. Moreover, there is an orthogonality relation linked to the notion of trace class, which allows us to develop Girard's theory of types, data and algorithms in our setting. This permits a structured approach to Girard's concept of solving feedback equations [14], and an axiomatization of the critical features needed for showing that the execution formula is an invariant of cut-elimination. Computationally, GoI provides a kind of algorithm for normalization based on the execution formula. In future work, we hope to explore the algorithmic and convergence properties of the execution formula in various models, independently of the syntax.

An advantage of the approach taken here is that we are able to carry out our MGoI interpretation in categories of finite dimensional vector spaces and the other examples mentioned above. This is not possible for the earlier theory of uni-object GoI (for example, $\mathbf{Vec_{fd}}$ does not have non-trivial reflexive objects). Our examples illustrate that both "sum-style" and "product-style" GoI (as discussed in [3]) are compatible with our multiobject approach.

An obvious direction for future research is to extend our MGoI interpretation to the exponentials and additives of linear logic: this is under active development. As well, the thorny problem of how to handle the units (as mentioned in the Introduction) is being explored. New directions in GoI semantics now arise with the introduction of partial traces and abstract orthogonalities. For example, we are pursuing the correspondence of trace class/nuclear morphisms as achieved in [2] for their examples. We are also currently exploring MGoI interpretations in Banach spaces and related categories, to find appropriate trace class structures.

It is natural to seek examples of traces that are induced by more general notions of orthogonalities, especially those arising in functional analysis. We hope this may lead to new classes of MGoI models, perhaps connected to current work in operator algebras and general solutions to feedback equations, as in [14].

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