# A Categorical Model for the Geometry of Interaction


Esfandiar Haghverdi

*Department of Mathematics and School of Informatics*
*Indiana University Bloomington, Bloomington*
*Indiana 47408, USA*

Philip Scott [1]

*Department of Mathematics and Statistics*
*University of Ottawa, Ottawa, Canada K1S 6N5*


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**Abstract**

We consider the multiplicative and exponential fragment of linear logic (MELL) and give a Geometry of Interaction (GoI) semantics for it based on unique decomposition categories. We prove a Soundness and Finiteness Theorem for this interpretation. We show that Girard's original approach to GoI 1 via operator algebras is exactly captured in this categorical framework.



*Key words:* Geometry of Interaction, Traced monoidal categories, Linear logic, Partially additive categories, Unique decomposition categories.


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## 1 Introduction and Motivation

Girard introduced his Geometry of Interaction (GoI) program in the late 80's. The first proposal appeared in [10], followed by a penetrating series of papers [11,9,12] developing the detailed program.

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*Email addresses:* `ehaghver@indiana.edu` (Esfandiar Haghverdi), `phil@mathstat.uottawa.ca`, `www.site.uottawa.ca/~phil` (Philip Scott).

*URL:* `xavier.informatics.indiana.edu/~ehaghver` (Esfandiar Haghverdi).



[1] Research supported by an NSERC Discovery grant.

The Geometry of Interaction was the first attempt to model, in a mathematically sophisticated way, the dynamics of cut-elimination. Traditional denotational semantics models normalization of proofs (or lambda terms) by static equalities: if $\Pi$ and $\Pi'$ are proofs of a sequent $\Gamma \vdash A$ and if we have a reduction $\Pi \succ \Pi'$ by cut-elimination, then their interpretations $[\![-]\!]$ in any model denote equal morphisms, i.e. $[\![\Pi]\!] = [\![\Pi']\!] : [\![\Gamma]\!] \to [\![A]\!]$. On the other hand *syntax* contains too much irrelevant information and does not yield an independent mathematical modelling of the dynamics of cut-elimination. Thus the goal of GoI is to provide precisely such a mathematical model.

The first implementation of this programme was given by Girard for system $\mathcal{F}$ in [11], based on the $C^*$-algebra of bounded linear operators on the space $\ell^2$ of square summable sequences. For a much more elaborate account of Girard's work, see [11,9,12].

The GoI interpretation was extended to untyped $\lambda$-calculus by Danos in his thesis [7] and further investigated in the thesis of Regnier [27]. Danos and Regnier further developed the GoI interpretation to define a *path-semantics* based on untyped nets, together with a detailed comparison with many $\lambda$-calculus notions of "path" arising from various operational semantics of lambda calculus. The basic reference for their work, with connections to $\ell^2$-models, is in [8]. Further study of the operator algebra interpretation of $\lambda$-calculus and connections with Girard's GoI 2 is in Malacaria and Regnier [24]. In particular, they study the $\ell^2$-convergence of the execution formula.

Abramsky and Jagadeesan gave the first categorical approach to GoI in [4]. Their formulation is based on domain theory and arises from the construction of a categorical model of linear logic. The ideas and techniques used in [4] together with the development of traced monoidal categories, introduced by Joyal, Street and Verity [22], led to more recent abstract formalisations of GoI. Early work connecting traced monoidal categories, compact closedness and GoI was presented in lectures by Hyland and by Abramsky. Our treatment here proceeds via the notion of *GoI Situation* introduced by Abramsky in [2]. GoI Situations give a categorical embodiment of the essential ingredients of GoI, at least for the multiplicative and exponential fragment. Furthermore, in his Siena lecture [2] Abramsky introduced a general GoI construction, equivalent to the Int construction of Joyal, Street and Verity (see [13]), which yields a category whose composition is essentially given by Girard's execution formula. Abramsky's programme was sketched in [2] and completed in [13] and [3]. However, what was still missing was a tighter connection between the abstract GoI frameworks above and the original works of Girard et al. That is, we want our categorical models for GoI to be not only part of well-established categorical logic, but also we want our framework to explicitly connect with the details of the operator algebraic approach, e.g. the execution formula, orthogonality and the notion of type, all found in the original works but which could not be given in the generality of [3].

In this paper, we analyze how the first Girard paper GoI 1 [11] fits into the general theory of GoI Situations. The idea pursued here is to restrict ourselves to a useful class of traced monoidal categories in a GoI situation, namely unique decomposition categories [13,14]. These are monoidal categories whose homsets are enriched with certain infinitary sums, thus allowing us to consider morphisms as matrices, the execution formula as an infinite sum, etc. Such categories are inspired from early categorical analyses of programming languages by Elgot, Arbib and Manes, et al. (e.g. [25] ).

The main contributions of this paper are the following:

(1) We present a categorical model (implementation) for GoI and show that it captures the original Hilbert space model proposed by Girard in [11], including the notions of orthogonality and type.
(2) We show that the execution formula at the heart of modeling computation as cut-elimination is perfectly captured by the categorical notion of trace.
(3) We prove finiteness and soundness results for our model using the categorical properties of trace and GoI Situation.

We believe that our categorical interpretation views the original Girard GoI model in a new light. Not only do the original constructions and proofs appear less ad hoc, but this paper also opens the door towards accommodating other interesting models based on different categories and GoI Situations.

The rest of the paper is organized as follows: In Section 2 we recall the definitions of traced monoidal categories and GoI Situations, following [13,3]. In Section 3 we recall the definition of a unique decomposition category and give some examples. Sections 4 and 5 are the main sections of the paper where we discuss our categorical model for the GoI program and give the main theorems respectively. Section 6 discusses the original model introduced by Girard in [11]. Finally in section 7 we conclude by discussing related and future work.

## 2 Traced Monoidal Categories & GoI Situations

We recall the definitions of traced symmetric monoidal categories and GoI Situations. For more detailed expositions, see [13,3]. The categories introduced below admit a diagrammatic presentation accompanied with a sound and complete diagrammatic reasoning that can be found in the references above and [21]. Though we will not need diagrammatic reasoning in this paper.

Joyal, Street and Verity [22] introduced the notion of abstract trace on a balanced monoidal category (a monoidal category with braidings and twists.) This trace can be interpreted in various contexts where it could be called contraction, feedback, parametrized fixed-point, Markov trace or braid closure. The notion of trace can be used to analyse the cyclic structures encountered in mathematics and physics, most notably in knot theory. Since their introduction, traced symmetric monoidal categories have found applications in many different areas of computer science, for example the model theory of cyclic lambda calculi [16], categorical frameworks for the semantics of asynchronous communication networks [28], full completeness theorems for multiplicative linear logic via GoI models [13,15], analysis of finite state machines [18], relational dataflow [17], and independently arose in Stefanescu's work in network algebra [29].

**Definition 1** A *traced symmetric monoidal category* is a symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ with a family of functions $Tr^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \to \mathbb{C}(X, Y)$ called a *trace*, subject to the following axioms:

- **Natural** in $X$, $Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g \otimes 1_U))$ where $f : X \otimes U \to Y \otimes U$, $g : X' \to X$,
- **Natural** in $Y$, $gTr^U_{X,Y}(f) = Tr^U_{X,Y'}((g \otimes 1_U)f)$ where $f : X \otimes U \to Y \otimes U$, $g : Y \to Y'$,
- **Dinatural** in $U$, $Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$,
- **Vanishing (I,II)**, $Tr^I_{X,Y}(f) = f$ and $Tr^{U \otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \otimes U, Y \otimes U}(g))$ for $f : X \otimes I \to Y \otimes I$ and $g : X \otimes U \otimes V \to Y \otimes U \otimes V$,
- **Superposing**, $Tr^U_{X,Y}(f) \otimes g = Tr^U_{X \otimes W, Y \otimes Z}((1_Y \otimes s_{U,Z})(f \otimes g)(1_X \otimes s_{W,U}))$ for $f : X \otimes U \to Y \otimes U$ and $g : W \to Z$,
- **Yanking**, $Tr^U_{U,U}(s_{U,U}) = 1_U$.

Joyal, Street, and Verity [22] also introduced the *Int* construction on traced symmetric monoidal categories $\mathbb{C}$; $Int(\mathbb{C})$ is a kind of "free compact closure" of the category $\mathbb{C}$. For a traced symmetric monoidal category $\mathbb{C}$, composition in $Int(\mathbb{C})$, which is defined via the trace, is closely related to Girard's Execution Formula. Indeed, one of our goals in this paper is to show that in an appropriate class of models, the categorical execution formula is *exactly* the original Girard formula. In addition, the existence of an object $U$ of $\mathbb{C}$ with $U \otimes U \lhd U$ essentially captures the GoI interpretation for the multiplicatives.

The next problem was how to extend this to the exponential connectives. In the Abramsky program (see [3]) this is achieved by adding certain additional structure to a traced symmetric monoidal category $\mathbb{C}$. This structure involves a monoidal endofunctor $T : \mathbb{C} \to \mathbb{C}$, a reflexive object $U$, and appropriate monoidal retractions, as introduced below. It was shown in [3] that this additional structure is sufficient to guarantee that the endomorphism monoid $\mathbb{C}(U, U)$ forms a *linear combinatory algebra*. Such combinatory algebras model a Hilbert-style presentation of MELL, capturing the essence of the exponentials of Linear Logic.

**Definition 2** A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

(1) $\mathbb{C}$ is a traced symmetric monoidal category
(2) $T : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions (note that the retraction pairs are monoidal natural transformations):
  (a) $TT \lhd T$ $(e, e')$ (Comultiplication)
  (b) $Id \lhd T$ $(d, d')$ (Dereliction)
  (c) $T \otimes T \lhd T$ $(c, c')$ (Contraction)
  (d) $\mathcal{K}_I \lhd T$ $(w, w')$ (Weakening). Here $\mathcal{K}_I$ is the constant $I$ functor.
(3) $U$ is an object of $\mathbb{C}$, called *a reflexive object*, with retractions: (a) $U \otimes U \lhd U$ $(j, k)$, (b) $I \lhd U$, and (c) $TU \lhd U$ $(u, v)$.

Of course, as indicated by the terminology above, the functor $T$ is used to interpret the Exponential rules in linear logic ([3]).

Before introducing examples of GoI situations in the next section, we note that currently there are two "styles" of models for GoI in the literature: *Sum style* and *Product style*. These are determined by the form of the tensor in the underlying TMC. Roughly, in sum style the tensor $\otimes$ is given by a disjoint union on objects; in product style, it is related to a cartesian product. In this paper we exclusively consider sum style models, corresponding to Girard's GoI 1. Sum style GoI admits a semantics based on "particles flowing through a network". For further discussion, see [3,13]. For readers interested in product style (e.g. domain-theoretic models), further information is contained in for example works of Honsell and Lenisa [19].

We now consider an appropriate categorical framework for such sum style GoI.

## 3 Unique Decomposition Categories

We consider monoidal categories whose homsets allow the formation of certain infinite sums. Technically, these are monoidal categories enriched in $\Sigma$-monoids (see below). In the case where the tensor is coproduct and $\Sigma$-monoids satisfy an additional condition, such categories were studied in computer science in the early categorical analyses of flow charts and programming languages by Bainbridge, Elgot, Arbib and Manes, et. al. (e.g. [25] ). The general case, known as unique decomposition categories (UDC's) introduced in [13], are particularly relevant for this paper, since they admit arbitrary tensor product (not necessarily product or coproduct) and traced UDCs have a standard trace given as an infinite sum. For more facts on UDCs see [14].

**Definition 3** A $\Sigma$-*monoid* consists of a pair $(M, \Sigma)$ where $M$ is a nonempty set and $\Sigma$ is a partial operation on the countable families in $M$ (we say that $\{x_i\}_{i\in I}$ is *summable* if $\sum_{i\in I} x_i$ is defined), subject to the following axioms:

(1) *Partition-Associativity Axiom.* If $\{x_i\}_{i\in I}$ is a countable family and if $\{I_j\}_{j\in J}$ is a (countable) partition of $I$ , then $\{x_i\}_{i\in I}$ is summable if and only if $\{x_i\}_{i\in I_j}$ is summable for every $j \in J$ and $\{\sum_{i\in I_j} x_i\}_{j\in J}$ is summable. In that case, $\sum_{i\in I} x_i = \sum_{j\in J}(\sum_{i\in I_j} x_i)$.
(2) *Unary Sum Axiom.* Any family $\{x_i\}_{i\in I}$ in which $I$ is a singleton is summable and $\sum_{i\in I} x_i = x_j$ if $I = \{j\}$.

$\Sigma$-monoids form a symmetric monoidal category (with product as tensor), called $\Sigma$**Mon**. A $\Sigma$**Mon**-category $\mathbb{C}$ is a category enriched in $\Sigma$**Mon**; i.e. the homsets are enriched with an additive structure such that composition distributes over addition from left and right. Note that such categories have non-empty homsets and automatically have zero morphisms, namely $0_{XY} : X \to Y = \sum_{i\in\emptyset} f_i$ for $f_i \in \mathbb{C}(X, Y)$. However having zero morphisms does not imply the existence of a zero object. A key fact about $\Sigma$-monoids is that there are no additive inverses: if $\sum_{i\in I} x_i = 0$ then for all $i \in I$, $x_i = 0$.

Intuitively the terms built from elements in a $\Sigma$-monoid represent computational paths in a model of computation (Turing Machine, Lambda Calculus, fragments of linear logic, etc.) Infinite sums capture iterative behavior which is potentially unbounded, for example a `repeat` or `while` command in an imperative language or an infinite reduction of a lambda term. The fact that there are no additive inverses reflects the property of computational processes in which no computational paths are cancelled; this is in contrast to the paradigm of quantum computing [26] where one can have destructive interference where computational paths cancel each other. In the following we introduce the notion of Unique Decomposition Categories, first introduced in [13]. These are motivated by the Partially Additive Categories (PAC) of Manes and Arbib [25] and models in geometry of interaction. UDCs are more general than PACs in precisely the following senses: they work with any monoidal tensor product whereas PACs are restricted to coproducts and they have less restrictive demands on the additive structures on homsets (see Example 8 below). The price one pays for this generality is that a UDC may not have a trace whereas a PAC always has a trace which is induced by an iteration operator (Elgot dagger). Incidentally all UDCs that we study in this paper are traced with the standard trace (see Proposition 6 below).

**Definition 4** A *unique decomposition category* (UDC) $\mathbb{C}$ is a symmetric monoidal $\Sigma$**Mon**-category which satisfies the following axiom:

(A) For all $j \in I$ there are morphisms called *quasi injections*: $\iota_j : X_j \to \otimes_I X_i$, and *quasi projections*: $\rho_j : \otimes_I X_i \to X_j$, such that

(1) $\rho_k \iota_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise.
(2) $\sum_{i \in I} \iota_i \rho_i = 1_{\otimes_I X_i}$.

**Proposition 5 (Matricial Representation)** *Given $f : \otimes_J X_j \to \otimes_I Y_i$ in a UDC with $|I| = m$ and $|J| = n$, there exists a unique family $\{f_{ij}\}_{i \in I, j \in J} : X_j \to Y_i$ with $f = \sum_{i \in I, j \in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f \iota_j$.*

Thus every $f : \otimes_J X_j \to \otimes_I Y_i$ in a UDC can be represented by its components. We will use the corresponding matrices to represent morphisms. Composition of morphisms in a UDC then corresponds to matrix multiplication.

For example $f$ above (with $|I| = m$ and $|J| = n$) is represented by an $m \times n$ matrix $[f_{ij}]$.

$$f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$$

Given $f : \otimes_K X_k \to \otimes_J Y_j$ and $g : \otimes_J Y_j \to \otimes_I Z_i$, let $h = gf$. Then $h_{ik} = \rho_i h \iota_k = \rho_i (gf) \iota_k = \rho_i g (\sum_{j \in J} \iota_j \rho_j) f \iota_k = \sum_{j \in J} (\rho_i g \iota_j \rho_j) f \iota_k = \sum_{j \in J} (g_{ij} \rho_j f \iota_k) = \sum_{j \in J} g_{ij} f_{jk}$.

**Remark.** Although any $f : \otimes_J X_j \to \otimes_I Y_i$ can be represented by the unique family $\{f_{ij}\}$ of its components, the converse is not necessarily true; that is, given a family $\{f_{ij}\}$ with $I, J$ finite there may not be a morphism $f : \otimes_J X_j \to \otimes_I Y_i$ satisfying $f = \sum_{ij} \iota_i f_{ij} \rho_j$. However, in case such an $f$ exists it will be unique.

**Proposition 6 (Standard Trace Formula, [13])** *Let $\mathbb{C}$ be a unique decomposition category such that for every $X, Y, U$ and $f : X \otimes U \to Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$ exists, where $f_{ij}$ are the components of $f$. Then, $\mathbb{C}$ is traced and $Tr^U_{X,Y}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$.*

The trace formula above is called the standard trace, and a UDC with such a trace is called a traced UDC with standard trace. Note that a UDC can be traced with a trace different from the standard one. In this paper all traced UDCs are the ones with the standard trace. Before proceeding to examples of UDC's, we shall illustrate the trace formula in Proposition 6.

**Example 7 (Calculating Traces)** Let $\mathbb{C}$ be a traced UDC. Then given any $f : X \otimes U \to Y \otimes U$, $Tr^U_{X,Y}(f)$ exists. For example,

- Let $f : X \otimes U \to Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}$. Then

$$Tr^U_{X,Y}(f) = Tr^U_{X,Y}\left(\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}\right) = g + \textstyle\sum_n 00^n h = g + 0h = g + 0 = g.$$

- Let $f : X \otimes U \to Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$. Then

$$Tr^U_{X,Y}(f) = Tr^U_{X,Y}\left(\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}\right) = g + \textstyle\sum_n 0h^n 0 = g + 0 = g.$$

We now give a series of examples of GoI situations. For the exact analysis of Girard's GoI 1, the main examples are the first two: **PInj** and $\mathbf{Hilb_2}$. We end with a list of other models, which are partially additive categories in the sense of Manes and Arbib [25].

**Example 8 (Partial Injections)** The category **PInj** of sets and partial injective functions forms a GoI Situation. We outline the proof; for full details, see [13].

In **PInj** we let $\otimes = \uplus$, i.e. disjoint union, where we define $X \uplus Y = \{1\} \times X \cup \{2\} \times Y$. The tensor unit is the empty set $\emptyset$. Note that disjoint union is not a coproduct; indeed **PInj** does not have coproducts. The UDC structure is given as follows: quasi injections $X_j \xrightarrow{\iota_j} \uplus_{i\in I} X_i$ are defined by $\iota_j(x) = (j, x)$, and quasi projections $\uplus_{i\in I} X_i \xrightarrow{\rho_j} X_j$ are defined by: $\rho_j(j, x) = x$ and $\rho_j(i, x)$ is undefined for $i \neq j$.

We define a countable family of partial injective functions $\{f_i\}_{i\in I}$ to be *summable* iff they have pairwise disjoint domains and codomains. Then $(\sum_{i\in I} f_i)(x)$ $= f_j(x)$ iff $x \in Dom(f_j)$, for some $j \in I$, otherwise undefined.

As for the GoI situation structure, define the endofunctor $T = \mathbb{N}\times -$, with $T = (T, \psi, \psi_I)$, which is a symmetric monoidal functor with natural isomorphism $\psi_{X,Y} : \mathbb{N} \times X \uplus \mathbb{N} \times Y \to \mathbb{N} \times (X \uplus Y)$ given by $(1, (n, x)) \mapsto (n, (1, x))$ and $(2, (n, y)) \mapsto (n, (2, y))$. $\psi$ has an inverse defined by: $(n, (1, x)) \mapsto (1, (n, x))$ and $(n, (2, y)) \mapsto (2, (n, y))$. Also, $\psi_I : \emptyset \to \mathbb{N} \times \emptyset$ given by $1_\emptyset$ is clearly an isomorphism. $T$ is additive, indeed let $\{f_i\}_{i\in I}$ be a summable family in **PInj** $(X, Y)$, then

$$(1_\mathbb{N}\times\textstyle\sum_I f_i)(n, x) = \begin{cases} (n, f_j(x)), & \text{if there exists a } j \in I \text{ such that } x \in Dom(f_j); \\ \text{undefined}, & \text{else.} \end{cases}$$

but this is exactly the definition of $(\sum_I(1_\mathbb{N} \times f_i))(n, x)$ for all $(n, x) \in \mathbb{N} \times$

$X$. Therefore, $\mathbb{N} \times -$ is an additive functor and thus it is also traced, see [13]. In other words, given $f : X \uplus U \to Y \uplus U$ we have $1_{\mathbb{N}} \times Tr^U_{X,Y}(f) = Tr^{\mathbb{N}\times U}_{\mathbb{N}\times X,\mathbb{N}\times Y}(\psi^{-1}(1_{\mathbb{N}} \times f)\psi)$.

We show that $\mathbb{N}$ is a reflexive object.

- $\mathbb{N}\uplus\mathbb{N}\lhd\mathbb{N}(j,k)$ is given as follows: $j : \mathbb{N}\uplus\mathbb{N} \to \mathbb{N}, j(1,n) = 2n, j(2,n) = 2n+1$ and $k : \mathbb{N} \to \mathbb{N}\uplus\mathbb{N}, k(n) = (1, n/2)$ for $n$ even, and $(2,(n-1)/2)$ for $n$ odd. Clearly $kj = 1_{\mathbb{N}\uplus\mathbb{N}}$. Define $j_1 = j\iota_1, j_2 = j\iota_2$, $k_1 = \rho_1 k$, $k_2 = \rho_2 k$ where $\rho_i$ and $\iota_i$ are as defined above.
- $\emptyset \lhd \mathbb{N}$ using the empty partial function as the retract morphisms.
- $\mathbb{N}\times\mathbb{N} \lhd \mathbb{N}(u,v)$ is defined as: $u(m,n) = <m,n> = \frac{(m+n+1)(m+n)}{2} + n$ (Cantor surjective pairing) and $v$ as its inverse, $v(n) = (n_1,n_2)$ with $<n_1,n_2> = n$. Clearly, $vu = 1_{\mathbb{N}\times\mathbb{N}}$.

We next define the necessary monoidal natural transformations.

- $\mathbb{N}\times(\mathbb{N}\times X) \xrightarrow{e_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{e'_X} \mathbb{N}\times(\mathbb{N}\times X)$
  $\mathbb{N}\times(\mathbb{N}\times X) \xrightarrow{e_X} \mathbb{N}\times X$ is defined by, $e_X(n_1,(n_2,x)) = (<n_1,n_2>,x)$. Given $f : X \to Y$, $(1_{\mathbb{N}}\times f)e_X((n_1,(n_2,x)) = (<n_1,n_2>, f(x)) = e_Y(1_{\mathbb{N}} \times (1_{\mathbb{N}}\times f)(n_1,(n_2,x))$ for all $n_1,n_2 \in \mathbb{N}$ and $x \in X$ proving the naturality of $e_X$. $e'_X(n,x) = (n_1,(n_2,x))$ where $<n_1,n_2> = n$. $e'_X e_X(n_1,(n_2,x)) = e'_X(<n_1,n_2>,x) = (n_1,(n_2,x))$ for all $n_1,n_2 \in \mathbb{N}$ and $x \in X$.
- $X \xrightarrow{d_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{d'_X} X$
  $d_X(x) = (n_0,x)$ for a fixed $n_0 \in \mathbb{N}$. Given $f : X \to Y$, $(1_{\mathbb{N}}\times f)d_X(x) = (n_0, f(x)) = d_Y f(x)$ for any $x \in X$, proving the naturality of $d_X$.

$$d'_X(n,x) = \begin{cases} x, & \text{if } n = n_0; \\ \text{undefined}, & \text{else.} \end{cases}$$

  $d'_X d_X(x) = d'_X(n_0,x) = x$ for all $x \in X$.
- $(\mathbb{N}\times X)\uplus(\mathbb{N}\times X) \xrightarrow{c_X} \mathbb{N}\times X$ and $\mathbb{N}\times X \xrightarrow{c'_X} (\mathbb{N}\times X)\uplus(\mathbb{N}\times X)$.

$$c_X = \begin{cases} (1,(n,x)) \mapsto (2n,x) \\ (2,(n,x)) \mapsto (2n+1,x) \end{cases}$$

  Given $f : X \to Y$, $(1_{\mathbb{N}}\times f)c_X(1,(n,x)) = (2n, f(x)) = c_Y(1_{\mathbb{N}}\times f \uplus 1_{\mathbb{N}} \times f)(1,(n,x))$ for all $n \in \mathbb{N}$ and $x \in X$. Similarly $(1_{\mathbb{N}}\times f)c_X(2,(n,x)) = (2n+1, f(x)) = c_Y(1_{\mathbb{N}}\times f \uplus 1_{\mathbb{N}}\times f)(2,(n,x))$ for all $n \in \mathbb{N}$ and $x \in X$, proving the naturality of $c_X$.

$$c'_X(n,x) = \begin{cases} (1,(n/2,x)), & \text{if } n \text{ is even;} \\ (2,((n-1)/2,x), & \text{if } n \text{ is odd.} \end{cases}$$

Finally, $c'_X c_X(1,(n,x)) = c'_X(2n,x) = (1,(n,x))$ and $c'_X c_X(2,(n,x)) = c'_X(2n+1,x) = (2,(n,x))$.

- $\emptyset \xrightarrow{w_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{w'_X} \emptyset$.
  Let $w_X$ and $w'_X$ both be the empty partial function. Clearly for any $f : X \to Y$, $(1_{\mathbb{N}} \times f)w_X = w'_X 1_\emptyset$, proving the naturality of $w_X$. Clearly $w'_X w_X = 1_\emptyset$.

**Example 9 (Hilb$_2$)** This example will provide the connection to operator algebraic models. Given a set $X$ let $\ell_2(X)$ be the set of all complex valued functions $a$ on $X$ for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite. $\ell_2(X)$ is a Hilbert space and its norm is given by $||a|| = (\sum_{x \in X} |a(x)|^2)^{1/2}$ and its inner product is given by $< a, b >= \sum_{x \in X} a(x)\overline{b(x)}$ for $a, b \in \ell_2(X)$.

Barr [6] observed that this construction can be made into a functor, $\ell_2$ [2]. There is a contravariant faithful functor $\ell_2 : \mathbf{PInj}^{op} \to \mathbf{Hilb}$ where $\mathbf{Hilb}$ is the category of Hilbert spaces and linear contractions (norm $\leq 1$). For a set $X$, $\ell_2(X)$ is defined as above and given $f : X \to Y$ in $\mathbf{PInj}$, $\ell_2(f) : \ell_2(Y) \to \ell_2(X)$ is defined by:

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)), & \text{if } x \in Dom(f); \\ 0, & \text{otherwise.} \end{cases}$$

This yields an embedding of partial injective functions into partial isometries on Hilbert spaces ([12,1]), as in the following chart. Observe that the converse is not true, that is to say not every partial isometry on Hilbert spaces is the image of a partial injective map under the functor $\ell_2$.

| $\mathbf{PInj}(X,Y)$ | $\mathbf{Hilb}(\ell_2(Y), \ell_2(X))$ |
|---|---|
| $f$ | $\ell_2(f)$ |
| partial injective function | partial isometry |
| total | isometry |
| total and surjective | unitary |
| $X = Y$ and $f$ is identity on $Dom(f)$ | projection |

It can be shown that $\ell_2(X \times Y) \cong \ell_2(X) \otimes \ell_2(Y)$, where $\otimes$ is the tensor product of Hilbert spaces, see [23]. Also $\ell_2(X \uplus Y) \cong \ell_2(X) \oplus \ell_2(Y)$ where $\oplus$ is the direct sum of Hilbert spaces. To see this, let $a \in \ell_2(X)$ and $b \in \ell_2(Y)$, define $q_{a,b} : X \uplus Y \to \mathbb{C}$ by $q_{a,b}(1,x) = a(x)$ and $q_{a,b}(2,y) = b(y)$; clearly $q_{a,b} \in \ell_2(X \uplus Y)$. The linear transformation $U : \ell_2(X \uplus Y) \to \ell_2(X) \oplus \ell_2(Y)$

[2] Our presentation here is slightly different from Barr's original one in [6].

given by $U(q_{a,b}) = (a, b)$ is an isomorphism.

Let $\mathbf{Hilb_2} = \ell_2[\mathbf{PInj}]$; i.e. its objects are of the form $\ell_2(X)$ for a set $X$ and its morphisms $u : \ell_2(X) \to \ell_2(Y)$ are of the form $\ell_2(f)$ for some partial injective function $f : Y \to X$. $\mathbf{Hilb_2}$ is a (nonfull) subcategory of $\mathbf{Hilb}$ which is naturally equivalent to $\mathbf{PInj}$.

For $\ell_2(X)$ and $\ell_2(Y)$ in $\mathbf{Hilb_2}$, both the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ and the direct sum $\ell_2(X) \oplus \ell_2(Y)$ yield monoidal structures (tensor products) in $\mathbf{Hilb_2}$. This follows from the isomorphisms $\ell_2(X) \oplus \ell_2(Y) \cong \ell_2(X \uplus Y)$ and $\ell_2(X) \otimes \ell_2(Y) \cong \ell_2(X \times Y)$ and the fact that $X \uplus Y$ and $X \times Y$ are tensor products in $\mathbf{PInj}$. Notice however that, although $\ell_2(X) \oplus \ell_2(Y)$ is the direct sum (biproduct) of the Hilbert spaces $\ell_2(X)$ and $\ell_2(Y)$ in $\mathbf{Hilb}$, it fails to be so in $\mathbf{Hilb_2}$, as otherwise this would imply that $X \uplus Y$ is the coproduct in $\mathbf{PInj}$ of $X$ and $Y$, a contradiction.

$\mathbf{Hilb_2}$ is a traced UDC with respect to $\oplus$, where the UDC structure is induced from that of $\mathbf{PInj}$ as follows. We take $\oplus$ as the (monoidal) tensor product with unit $\ell_2(\emptyset)$. We define a sum for operators in $\mathbf{Hilb_2}(\ell_2(X), \ell_2(Y))$. Given a family $\{\ell_2(f_i)\}_I \in \mathbf{Hilb_2}(\ell_2(X), \ell_2(Y))$ with $\{f_i\}_I \in \mathbf{PInj}(Y, X)$, we say that $\{\ell_2(f_i)\}$ is summable iff $\{f_i\}$ is summable in $\mathbf{PInj}$ and in that case $\sum_i \ell_2(f_i) =_{def} \ell_2(\sum_i f_i)$. Clearly, this definition makes $\ell_2$ an additive functor. Quasi injections and projections are the $\ell_2$ images of quasi projections and injections in $\mathbf{PInj}$, respectively. Clearly Axiom (A) holds. As for the trace, given $u : \ell_2(X) \oplus \ell_2(U) \to \ell_2(Y) \oplus \ell_2(U)$, $Tr(u) = \ell_2(Tr^U_{Y,X}(f))$ where $u = \ell_2(f)$ with $f : Y \uplus U \to X \uplus U$.

Let $\ell^2 = \ell_2(\mathbb{N})$. We claim that $(\mathbf{Hilb_2}, \ell^2 \otimes -, \ell^2)$ is a GoI Situation. The proof is taken from [13].

Clearly the functor $\ell^2 \otimes - : \mathbf{Hilb_2} \to \mathbf{Hilb_2}$ is a symmetric monoidal functor. Also, observe that $\ell^2 \otimes \ell_2(X) \cong \ell_2(\mathbb{N} \times X)$ and $1_{\ell^2} \otimes \ell_2(f) \cong \ell_2(1_{\mathbb{N}} \times f)$. Moreover, $\ell^2 \otimes -$ is an additive and hence a traced functor. This follows from the fact that $\mathbb{N} \times -$ is an additive symmetric monoidal endofunctor on $\mathbf{PInj}$.

Also, we have that $\ell^2 \oplus \ell^2 \cong \ell_2(\mathbb{N} \uplus \mathbb{N}) \lhd \ell_2(\mathbb{N})$ $(\ell_2(k), \ell_2(j))$, $\{0\} = \ell_2(\emptyset) \lhd \ell^2$, and finally $\ell^2 \otimes \ell^2 \cong \ell_2(\mathbb{N} \times \mathbb{N}) \lhd \ell_2(\mathbb{N})$ $(\ell_2(v), \ell_2(u))$ for $j, k, u, v$ as the case of $\mathbf{Pinj}$ above. This proves that $\ell^2$ is a reflexive object in $\mathbf{Hilb_2}$.

As for the monoidal natural transformations:

- $\ell^2 \otimes (\ell^2 \otimes \ell_2(X)) \lhd \ell^2 \otimes \ell_2(X)$ $(\ell_2(e'_X), \ell_2(e_X))$ for $e_X, e'_X$ as in the case of $\mathbf{Pinj}$ above.
- $\ell_2(X) \lhd \ell^2 \otimes \ell_2(X)$ $(\ell_2(d'_X), \ell_2(d_X))$ for $d_X, d'_X$ as in the case of $\mathbf{PInj}$ above.
- $(\ell^2 \otimes \ell_2(X)) \oplus (\ell^2 \otimes \ell_2(X)) \lhd \ell^2 \otimes \ell_2(X)$ $(\ell_2(c'_X), \ell_2(c_X))$ for $c_X, c'_X$ as in the case of $\mathbf{PInj}$ above.

- $\{0\} \lhd \ell^2 \otimes \ell_2(X)$ $(\ell_2(w'_X), \ell_2(w_X))$ for $w_X, w'_X$ as in the case of **Pinj** above.

The naturality of the morphisms above follows from the underlying structure of **PInj** and functoriality of $\ell_2$.

**Example 10 (Partially Additive Categories)** Any partially additive category (following Manes and Arbib [25]) forms a traced UDC, with trace given by the formula in Proposition 6 above. The homsets in a partially additive category are $\Sigma$-monoids that in addition have to satisfy a finiteness axiom: an infinite family is summable iff all its finite subfamilies are summable. This yields a more restrictive additive structure compared to $\Sigma$ monoids. In addition, a partially additive category has to have countable coproducts and the sum and coproduct have to satisfy some compatibility axioms. The extra structure present in a PAC, on the other hand, guarantees the existence of a standard trace which is induced by an iteration (Elgot dagger) operation. See [13] for more details on PACs and the relationship between trace and dagger operators. The following partially additive categories actually form GoI situations. We only sketch the constructions, which are either similar to the above examples, or may be found in [3,13].

**$\mathbf{Rel}_+$** (sets and relations). Here $\otimes = \uplus$ (disjoint union, which is a biproduct). In $\mathbf{Rel}_+$, *all* countable families are summable, and $\sum_{i\in I} R_i = \cup_i R_i$.

**Pfn** (sets and partial functions), with $\otimes = \uplus$. Define a countable family of partial functions $\{f_i\}_{i\in I}$ to be summable iff they have pairwise disjoint domains. Then $(\sum_{i\in I} f_i)(x) = f_j(x)$ iff $x \in Dom(f_j)$, for some $j \in I$, otherwise undefined.

The above two examples yield GoI situations $(\mathbf{Rel}_+, T, \mathbb{N})$ and $(\mathbf{Pfn}, T, \mathbb{N})$ with $T = \mathbb{N} \times -$.

**SRel**, the category of stochastic relations. Here the objects are measurable spaces $(X, \mathcal{F}_X)$ and maps $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ are stochastic kernels, i.e. $f : X \times \mathcal{F}_Y \to [0, 1]$ which are bounded measurable in the first variable and subprobability measures in the second. For $f$ as above and $g : (Y, \mathcal{F}_Y) \to (Z, \mathcal{F}_Z)$, composition $g\circ f(x, C) = \int_Y g(y, C)f(x, dy)$, where $f(x, -)$ is the measure for integration. This category has countable coproducts (which form the tensor). A family $\{f_i\}_{i\in I}$ in $\mathbf{SRel}((X, \mathcal{F}_X), (Y, \mathcal{F}_Y))$ is summable iff $\sum_{i\in I} f_i(x, Y) \leq 1$ for all $x \in X$.

$(\mathbf{SRel}, T, \mathbb{N}^{\mathbb{N}})$ forms a GoI Situation, where $T(X, \mathcal{F}_X) = (\mathbb{N} \times X, \mathcal{F}_{\mathbb{N}\times X})$ and $\mathcal{F}_{\mathbb{N}\times X}$ is the $\sigma$-field generated by $\biguplus_{\mathbb{N}} X$.

## 4 Interpretation of Proofs

In this section we define the GoI interpretation for proofs of MELL without the neutral elements. Let $\mathbb{C}$ be a traced UDC, $T$ an additive endofunctor and $U$ an object of $\mathbb{C}$, such that $(\mathbb{C}, T, U)$ is a GoI Situation. We interpret proofs in the homset $\mathbb{C}(U, U)$ of endomorphisms of $U$. Formulas (= types) will be interpreted in the next section (Section 5) as certain subsets of $\mathbb{C}(U, U)$; however, this introduces some novel ideas and is not needed to read the present section.

**Convention**: All identity morphisms are on tensor copies of $U$, however we adopt the convention of writing $1_\Gamma$ instead of $1_{U^n}$ with $|\Gamma| = n$. $U^n$ denotes the $n$-fold tensor product $U \otimes \cdots \otimes U$ ($n$ times). The retraction pairs are fixed once and for all using the names in Definition 2.

Every MELL sequent will be of the form $\vdash [\Delta], \Gamma$ where $\Gamma$ is a sequence of formulas and $\Delta$ is a sequence of cut formulas that have already been made in the proof of $\vdash \Gamma$ (e.g. $A, A^\perp, B, B^\perp$). This is used to keep track of the cuts that are already made in the proof of $\vdash \Gamma$. Suppose that $\Gamma$ consists of $n$ and $\Delta$ consists of $2m$ formulas. Then a proof $\Pi$ of $\vdash [\Delta], \Gamma$ is represented by a morphism $[\![\Pi]\!] \in \mathbb{C}(U^{n+2m}, U^{n+2m})$. Recall that this corresponds to a morphism from $U$ to itself, using the retraction morphisms $U \otimes U \lhd U\ (j, k)$. However, it is much more convenient to work in $\mathbb{C}(U^{n+2m}, U^{n+2m})$ (matrices on $\mathbb{C}(U, U)$). Define the morphism $\sigma : U^{2m} \to U^{2m}$, as $\sigma = s \otimes \cdots \otimes s$ ($m$-copies) where $s$ is the symmetry morphism, the $2 \times 2$ antidiagonal matrix $[a_{ij}]$, where $a_{12} = a_{21} = 1; a_{11} = a_{22} = 0$. Here $\sigma$ represents the cuts in the proof of $\vdash \Gamma$, i.e. it models $\Delta$, it is there to rearrange the cut-formulas so that when we close the loop (see the execution formula in the next section) every cut formula gets connected to its dual formula. If $\Delta$ is empty (that is for a cut-free proof), we define $\sigma : I \to I$ to be the identity morphism $1_I = 0_{II}$ as $I$ is the zero object. Note that $U^0 = I$ where $I$ is the unit of the tensor in the category $\mathbb{C}$.

Let $\Pi$ be a proof of $\vdash [\Delta], \Gamma$. We define the GoI interpretation of $\Pi$, denoted by $[\![\Pi]\!]$, by induction on the length of the proof as follows. There is a corresponding pictorial representation of each morphism in the graphical calculus of boxes and wires where the denotation of a proof $\Pi$ of the sequent $\vdash [\Delta], \Gamma$ is represented as in Figure 1.


Fig. 1. Pictorial representation of a sequent $\vdash [\Delta], \Gamma$

The pictorial representations for the GoI interpretations of axiom, cut, times, of course and contraction rules are collected in Appendix A.

(1) $\Pi$ is an *axiom* $\vdash A, A^{\perp}$, then $m = 0$, $n = 2$ and $[\![\Pi]\!] = s$.

(2) $\Pi$ is obtained using the *cut* rule on $\Pi'$ and $\Pi''$ that is

$$\dfrac{\begin{array}{cc} \Pi' & \Pi'' \\ \vdots & \vdots \\ \vdash [\Delta'], \Gamma', A & \vdash [\Delta''], A^{\perp}, \Gamma'' \end{array}}{\vdash [\Delta', \Delta'', A, A^{\perp}], \Gamma', \Gamma''}\ (cut)$$

Define $[\![\Pi]\!]$ as follows: $[\![\Pi]\!] = \tau^{-1}([\![\Pi']\!] \otimes [\![\Pi'']\!])\tau$, where $\tau : \Gamma' \otimes \Gamma'' \otimes \Delta' \otimes \Delta'' \otimes A \otimes A^{\perp} \to \Gamma' \otimes A \otimes \Delta' \otimes A^{\perp} \otimes \Gamma'' \otimes \Delta''$ is a permutation.

(3) $\Pi$ is obtained using the *exchange* rule on the formulas $A_i$ and $A_{i+1}$ in $\Gamma'$. That is $\Pi$ is of the form

$$\dfrac{\begin{array}{c} \Pi' \\ \vdots \\ \vdash [\Delta], \Gamma' \end{array}}{\vdash [\Delta], \Gamma}\ (exchange)$$

where in $\Gamma'$ we have $A_i, A_{i+1}$. Then, $[\![\Pi]\!]$ is obtained from $[\![\Pi']\!]$ by interchanging the rows $i$ and $i+1$. So suppose that $\Gamma' = \Gamma'_1, A_i, A_{i+1}, \Gamma'_2$, then $\Gamma = \Gamma'_1, A_{i+1}, A_i, \Gamma'_2$ and $[\![\Pi]\!] = \tau^{-1}[\![\Pi']\!]\tau$, where $\tau = 1_{\Gamma'_1} \otimes s \otimes 1_{\Gamma'_2 \otimes \Delta}$.

(4) $\Pi$ is obtained using an application of the *par* rule, that is $\Pi$ is of the form:

$$\dfrac{\begin{array}{c} \Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B \end{array}}{\vdash [\Delta], \Gamma', A \parr B}\ (\parr)$$

Then $[\![\Pi]\!] = g[\![\Pi']\!]f$, where $f = 1_{\Gamma'} \otimes k \otimes 1_{\Delta}$ and $g = 1_{\Gamma'} \otimes j \otimes 1_{\Delta}$, recall that $U \otimes U \lhd U\,(j, k)$.

(5) $\Pi$ is obtained using an application of the *times* rule, that is $\Pi$ has the form:

$$\cfrac{\begin{matrix}\Pi' & & \Pi'' \\ \vdots & & \vdots \\ \vdash [\Delta'], \Gamma', A & & \vdash [\Delta''], \Gamma'', B\end{matrix}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B} \ (times)$$

Then $[\![\Pi]\!] = g\tau^{-1}([\![\Pi']\!] \otimes [\![\Pi'']\!])\tau f$, where $\tau : \Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \Delta' \otimes \Delta'' \to \Gamma' \otimes A \otimes \Delta' \otimes \Gamma'' \otimes B \otimes \Delta''$ is a permutation. $f = 1_{\Gamma' \otimes \Gamma''} \otimes k \otimes 1_{\Delta' \otimes \Delta''}$ and $g = 1_{\Gamma' \otimes \Gamma''} \otimes j \otimes 1_{\Delta' \otimes \Delta''}$ .

(6) $\Pi$ is obtained from $\Pi'$ by an *of course* rule, that is $\Pi$ has the form :

$$\cfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], ?\Gamma', A\end{matrix}}{\vdash [\Delta], ?\Gamma', !A} \ (ofcourse)$$

Then $[\![\Pi]\!] = ((ue_U)^{\otimes n} \otimes u \otimes u^{\otimes 2m})\phi^{-1}T((v^{\otimes n} \otimes 1_A \otimes 1_\Delta)[\![\Pi']\!](u^{\otimes n} \otimes 1_A \otimes 1_\Delta))\phi((e'_U v)^{\otimes n} \otimes v \otimes v^{\otimes 2m})$, where $TT \lhd T\,(e, e')$, $|\Gamma'| = n$, $|\Delta| = 2m$, and $\phi : (T^2U)^{\otimes n} \otimes TU \otimes (TU)^{\otimes 2m} \to T((TU)^{\otimes n} \otimes U \otimes U^{\otimes 2m})$ is the canonical isomorphism.

(7) $\Pi$ is obtained from $\Pi'$ by the *dereliction* rule, that is $\Pi$ is of the form :

$$\cfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A\end{matrix}}{\vdash [\Delta], \Gamma', ?A} \ (dereliction)$$

Then $[\![\Pi]\!] = (1_{\Gamma'} \otimes ud_U \otimes 1_\Delta)[\![\Pi']\!](1_{\Gamma'} \otimes d'_U v \otimes 1_\Delta)$ where $Id \lhd T\,(d, d')$.

(8) $\Pi$ is obtained from $\Pi'$ by the *weakening* rule, that is $\Pi$ is of the form:

$$\cfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{matrix}}{\vdash [\Delta], \Gamma', ?A} \ (weakening)$$

Then $[\![\Pi]\!] = (1_{\Gamma'} \otimes uw_U \otimes 1_\Delta)[\![\Pi']\!](1_{\Gamma'} \otimes w'_U v \otimes 1_\Delta)$, where $\mathcal{K}_I \lhd T\,(w, w')$.

(9) $\Pi$ is obtained from $\Pi'$ by the *contraction* rule, that is $\Pi$ is of the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', ?A, ?A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ (contraction)$$

Then $[\![\Pi]\!] = (1_{\Gamma'} \otimes uc_U(v \otimes v) \otimes 1_\Delta)[\![\Pi']\!](1_{\Gamma'} \otimes (u \otimes u)c'_U v \otimes 1_\Delta)$, where $T \otimes T \lhd T\,(c, c')$.

**Example 11** *Let* $\Pi$ *be the following proof:*

$$\dfrac{\vdash A, A^\perp \quad \vdash A, A^\perp}{\vdash [A^\perp, A], A, A^\perp}\ (cut)$$

*Then the GoI semantics of this proof is given by*

$$[\![\Pi]\!] = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&1&0&0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & Id_2 \\ Id_2 & \mathbf{0} \end{bmatrix}$$

*where* $Id_2$ *is the* $2 \times 2$ *identity matrix and* $\mathbf{0}$ *is the* $2 \times 2$ *zero matrix. Now consider the following proof*

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B, B^\perp \quad \vdash C, C^\perp}{\vdash B, C, B^\perp \otimes C^\perp}}{\vdash B, B^\perp \otimes C^\perp, C}}{\vdash B^\perp \otimes C^\perp, B, C}}{\vdash B^\perp \otimes C^\perp, B \parr C}\ .$$

*Its denotation is given by*

$$\begin{bmatrix} 0 & j_1k_1 + j_2k_2 \\ j_1k_1 + j_2k_2 & 0 \end{bmatrix}.$$

### *4.1 Dynamics*

Dynamics is at the heart of the GoI interpretation as compared to denotational semantics and it is hidden in the cut-elimination process. The mathematical model of cut-elimination is given by the *execution formula* defined as follows:

$$EX([\![\Pi]\!], \sigma) = Tr^{U^{2m}}_{U^n, U^n}((1_{U^n} \otimes \sigma)[\![\Pi]\!]) \tag{1}$$

where $\Pi$ is a proof of the sequent $\vdash [\Delta], \Gamma$ and $\sigma$ models $\Delta$. Pictorially this can be represented as follows:

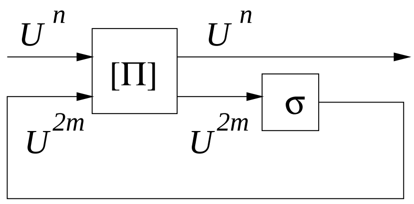


Fig. 2. Pictorial representation of the execution formula.

Note that $EX(\llbracket \Pi \rrbracket, \sigma)$ is a morphism from $U^n \to U^n$, and it makes sense since the trace of any morphism in $\mathbb{C}(U^{n+2m}, U^{n+2m})$ is well-defined. Since we are working with a traced UDC with the standard trace, by Proposition 6 we can rewrite the execution formula (1) in a more familiar form:

$$EX(\llbracket \Pi \rrbracket, \sigma) = \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma\pi_{22})^n(\sigma\pi_{21})$$

where $\llbracket \Pi \rrbracket = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$. Note that in general the execution formula defined in this categorical framework *always* makes sense; that is, we do not need a convergence criterion (e.g. nilpotency or weak nilpotency). This is in contrast to Girard's works where the infinite sum must be made to make sense and this is achieved via proving a nilpotency result.

We later show that formula (1) is the same as Girard's execution formula with $\mathbf{Hilb}_2$ as the underlying category. The intention here is to prove that the result of this formula is what corresponds to the cut-free proof obtained from $\Pi$ using Gentzen's cut-elimination procedure. We will also show that for any proof $\Pi$ of MELL the execution formula is a finite sum, which corresponds to termination of computation as opposed to divergence.

**Example 12** Consider the proof $\Pi$ in Example 11 above. Recall also that $\sigma = s$ in this case ($m = 1$). Then

$$EX(\llbracket \Pi \rrbracket, \sigma) = Tr\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{n \geq 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that in this case we have obtained the GoI interpretation of the cut-free proof obtained by applying Gentzen's Hauptsatz to the proof $\Pi$.

## 5 Soundness of the Interpretation

In this section we shall prove the main result of this paper: the soundness of the GoI interpretation. In other words we have to show that if a proof $\Pi$ is reduced (via cut-elimination) to its cut-free form $\Pi'$, then $EX(\llbracket \Pi \rrbracket, \sigma)$ is a finite sum and $EX(\llbracket \Pi \rrbracket, \sigma) = \llbracket \Pi' \rrbracket$. Intuitively this says that if one thinks of cut-elimination as computation then $\llbracket \Pi \rrbracket$ can be thought of as an algorithm. The computation takes place as follows: if we run $EX(\llbracket \Pi \rrbracket, \sigma)$, it terminates after finitely many steps (cf. finite sum) and yields a datum (cf. cut-free proof). This intuition will be made precise in this section through the definition of type and the main theorems (see Theorems 20,21).

**Lemma 13 (Associativity of cut)** *Let $\Pi$ be a proof of $\vdash [\Delta, \Lambda], \Gamma$ with $|\Delta| = 2m'$, $|\Lambda| = 2m''$ and $\sigma$ and $\tau$ be the morphisms representing the cut-formulas in $\Delta$ and $\Lambda$ respectively. Then*

$$\begin{aligned} EX(\llbracket \Pi \rrbracket, \sigma \otimes \tau) &= EX(EX(\llbracket \Pi \rrbracket, \tau), \sigma) \\ &= EX(EX((1 \otimes s_{U^{2m'}, U^{2m''}}) \llbracket \Pi \rrbracket (1 \otimes s_{U^{2m''}, U^{2m'}}), \sigma), \tau) \end{aligned}$$

**PROOF.**

$$\begin{aligned} &EX(EX(\llbracket \Pi \rrbracket, \tau), \sigma) = \\ &= Tr^{U^{2m'}}((1 \otimes \sigma) Tr^{U^{2m''}}((1 \otimes \tau) \llbracket \Pi \rrbracket)) \text{ definition of } EX \text{ formula} \\ &= Tr(Tr((1 \otimes \sigma \otimes 1)(1 \otimes \tau) \llbracket \Pi \rrbracket)) \text{ naturality of trace} \\ &= Tr^{U^{2(m'+m'')}}((1 \otimes \sigma \otimes \tau) \llbracket \Pi \rrbracket) \text{ vanishing II property of trace} \\ &= EX(\llbracket \Pi \rrbracket, \sigma \otimes \tau). \end{aligned}$$

As for the second equality: (we drop the subscripts for $s$, as there is no danger of confusion!)

$EX(EX((1 \otimes s)[\![\Pi]\!](1 \otimes s), \sigma), \tau) =$

$= Tr^{U^{2m''}}((1 \otimes \tau) Tr^{U^{2m'}}((1 \otimes \sigma)(1 \otimes s)[\![\Pi]\!](1 \otimes s)))$ def. of $EX$ formula

$= Tr(Tr((1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \sigma)(1 \otimes s)[\![\Pi]\!](1 \otimes s)))$ naturality of trace

$= Tr(Tr((1 \otimes \tau \otimes \sigma)(1 \otimes s)[\![\Pi]\!](1 \otimes s)))$ functoriality of tensor

$= Tr(Tr((1 \otimes s)(1 \otimes \sigma \otimes \tau)[\![\Pi]\!](1 \otimes s)))$ naturality of symmetry

$= Tr(Tr((1 \otimes \sigma \otimes \tau)[\![\Pi]\!]))$ dinaturality of trace

$= Tr^{U^{2(m'+m'')}}((1 \otimes \sigma \otimes \tau)[\![\Pi]\!])$ vanishing II property of trace

$= EX([\![\Pi]\!], \sigma \otimes \tau).$

We proceed to defining types. This and similar definitions are directly inspired by the corresponding ones in [11], generalising them to our categorical framework.

**Definition 14** Let $f, g$ be morphisms in $\mathbb{C}(U, U)$. We say that $f$ is *nilpotent* if $f^k = 0$ for some $k \geq 1$. We say that $f$ is *orthogonal* to $g$, denoted $f \perp g$ if $gf$ is nilpotent. Orthogonality is a symmetric relation and it makes sense because $0_{UU}$ exists. Also, $0 \perp f$ for all $f \in \mathbb{C}(U, U)$.

Given a subset $X$ of $\mathbb{C}(U, U)$, we define

$$X^{\perp} = \{f \in \mathbb{C}(U, U) | \forall g (g \in X \Rightarrow f \perp g)\}$$

A *type* is any subset $X$ of $\mathbb{C}(U, U)$ such that $X = X^{\perp\perp}$. Note that types are inhabited, since $0_{UU}$ belongs to every type.

**Definition 15** Consider a GoI situation $(\mathbb{C}, T, U)$ as in the beginning of Section 4 with $j_1, j_2, k_1, k_2$ components of $j$ and $k$ respectively. Let $A$ be an MELL formula. We define *the GoI interpretation of* $A$, denoted $\theta A$, inductively as follows:

(1) If $A \equiv \alpha$ that is $A$ is an atom, then $\theta A = X$ an arbitrary type.
(2) If $A \equiv \alpha^{\perp}$, $\theta A = X^{\perp}$, where $\theta\alpha = X$ is given by assumption.
(3) If $A \equiv B \otimes C$, $\theta A = Y^{\perp\perp}$, where $Y = \{j_1 a k_1 + j_2 b k_2 \mid a \in \theta B, b \in \theta C\}$.
(4) If $A \equiv B \invamp C$, $\theta A = Y^{\perp}$, where $Y = \{j_1 a k_1 + j_2 b k_2 \mid a \in (\theta B)^{\perp}, b \in (\theta C)^{\perp}\}$.
(5) If $A \equiv\, !B$, $\theta A = Y^{\perp\perp}$, where $Y = \{uT(a)v \mid a \in \theta B\}$.
(6) If $A \equiv ?B$, $\theta A = Y^{\perp}$, where $Y = \{uT(a)v \mid a \in (\theta B)^{\perp}\}$.

It is an easy consequence of the definition that $(\theta A)^{\perp} = \theta A^{\perp}$ for any formula $A$.

**Definition 16** Let $\Gamma = A_1, \cdots, A_n$. A *datum* of type $\theta\Gamma$ is a morphism $M : U^n \to U^n$ such that for any $\beta_1 \in \theta(A_1^\perp), \cdots, \beta_n \in \theta(A_n^\perp)$, $(\beta_1 \otimes \cdots \otimes \beta_n)M$ is nilpotent. An *algorithm* of type $\theta\Gamma$ is a morphism $M : U^{n+2m} \to U^{n+2m}$ for some non-negative integer $m$ such that for $\sigma : U^{2m} \to U^{2m}$ defined in the usual way, $EX(M, \sigma) = Tr((1 \otimes \sigma)M)$ is a finite sum and a datum of type $\theta\Gamma$.

**Lemma 17** *Let $M : U^n \to U^n$ and $a : U \to U$. Define $CUT(a, M) = (a \otimes 1_{U^{n-1}})M : U^n \to U^n$. Note that the matrix representation of $CUT(a, M)$ is the matrix obtained from $M$ by multiplying its first row by $a$. Then $M = [m_{ij}]$ is a datum of type $\theta(A, \Gamma)$ iff for any $a \in \theta A^\perp$, $am_{11}$ is nilpotent and the morphism $ex(CUT(a, M)) = Tr^A(s_{\Gamma,A}^{-1} CUT(a, M) s_{\Gamma,A})$ is in $\theta(\Gamma)$. Here $s_{\Gamma,A}$ is the symmetry morphism from $\Gamma \otimes A$ to $A \otimes \Gamma$.*

We shall need the following lemmas for the proof of Lemma 17.

**Lemma 18** *Let $f, g : U^k \to U^k$ for some positive integer $k$. Then $(f + g)$ is nilpotent iff $f$ is nilpotent and $\sum_{n \geq 0} gf^n$ is nilpotent.*

**PROOF.** The proof of this lemma is implicit in [11] (page 242, Lemma 5). Let $n$ be a positive integer and $q \subseteq \{1, 2, \cdots, n\}$ and define $\mu_{n,q} = \rho_1 \rho_2 \cdots \rho_n$ where $\rho_i = g$ if $i \in q$ and $\rho_i = f$ otherwise. Hence for a fixed $n$, $(f + g)^n = \sum_q \mu_{n,q}$, a sum with $2^n$ summands. Clearly $(f + g)$ is nilpotent iff there is an $m$ such that $\mu_{m,q} = 0$ for all $q$. Now suppose $f$ and $\sum_{n \geq 0} gf^n$ are nilpotent. Thus there exists a $k$ such that $f^k = 0$ and there is a $p$ such that $(g + gf + \cdots + gf^{k-1})^p = 0$. Then we claim that $(f + g)^{kp}$ is zero, note that a term of $(g + gf + \cdots + gf^{k-1})^p$ is of the form $\mu_{i_1,\{1\}} \cdots \mu_{i_p,\{1\}}$ with $i_j \in \{1, \cdots, k\}$, for $j = 1, \cdots, p$. Also note that this latter term itself is of the form $\mu_{a,b}$ for some $a$ and $b$. The $2^{kp}$ terms in $(f + g)^{kp}$ can be generated using $f^k$ and the $\mu_{a,b}$ by pre and post compositions with $f$ and $g$ and are all null. Hence $\mu_{kp,q} = 0$ for all $q$ and so $(f + g)$ is nilpotent.

Conversely suppose $(f+g)$ is nilpotent, then there exists $m$ such that $\mu_{m,q} = 0$ for all $q$, in particular $\mu_{m,\emptyset} = f^m = 0$ and so $f$ is nilpotent. Also all the terms in $(g + gf + \cdots + gf^{m-1})^m$ are null and thus the desired result follows.

**Lemma 19** *Let $M : U \otimes U \to U \otimes U$ be given by the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $M$ is nilpotent iff $a$ is nilpotent and $Tr(sMs) = d + \sum_{n \geq 0} ca^n b$ is nilpotent.*

**PROOF.** Let $M_1 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. Clearly $M = M_1 + M_2$. Now $M$ is nilpotent iff $M_1$ and $\sum_{n \geq 0} M_2 M_1^n$ are nilpotent by Lemma 18. Now $M_1$

is nilpotent iff $a$ and $d$ are nilpotent and $\sum_{n\geq 0} M_2 M_1^n = \begin{bmatrix} 0 & \sum_n bd^n \\ \sum_n ca^n & 0 \end{bmatrix}$, is nilpotent iff $\sum_{n,m} ca^n bd^m$ is nilpotent. Finally note that $Tr(sMs) = d + \sum_{n\geq 0} ca^n b$ is nilpotent iff $d$ and $\sum_{n,m} ca^n bd^m$ are nilpotent. Hence the desired result.

**PROOF. (Lemma 17)** Consider the simple case where $\Gamma$ is a single formula $B$, the more general case follows using a similar argument in higher dimension. So $M$ is a $2\times 2$ matrix, say $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

Suppose $M$ is a datum of type $\theta(A, B)$. Then for any $a \in \theta(A^\perp)$ and $b \in \theta(B^\perp)$, the matrix $P = \begin{bmatrix} a\alpha & a\beta \\ b\gamma & b\delta \end{bmatrix}$ is nilpotent and thus so is $a\alpha$ by Lemma 19.

Let $\delta' = ex(CUT(a, M))$, then $b\delta' = b\,Tr^A(s_{A,B} CUT(a, M) s_{B,A}) = b\,\delta + \sum_{n\geq 0} b\,\gamma(a\alpha)^n a\beta$. Note that $b\delta' = Tr(sPs)$ and hence is nilpotent by Lemma 19. Hence $\delta' \perp b$, so $\delta' \in \theta(B)$.

Conversely, suppose $a \in \theta(A^\perp)$ and $b \in \theta(B^\perp)$, note that by assumption $a\alpha$ is nilpotent and $\delta' = ex(CUT(a, M)) \in \theta(B)$ and so $b\delta'$ is nilpotent. But $b\delta' = Tr(sPs)$ and so by Lemma 19, $P$ is nilpotent. So $(a \otimes b)M$ is nilpotent and $M$ is a datum of type $\theta(A, B)$.

**Theorem 20** *Let $\Pi$ be a proof of $\vdash [\Delta], \Gamma$. Then $[\![\Pi]\!]$ is an algorithm of type $\theta\Gamma$.*

**PROOF.** The main ideas behind the proof parallel those in [11]. However the techniques used are different and we have included all the cases in detail.

- $\Pi$ is an axiom, where $\Gamma = A, A^\perp$, then we need to prove that $EX([\![\Pi]\!], 0) = [\![\Pi]\!]$ is a datum of type $\theta\Gamma$. That is, for all $a \in \theta A^\perp$ and $b \in \theta A$, $M = (a \otimes b)[\![\Pi]\!] = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ must be nilpotent. Observe that $M^n = \begin{bmatrix} (ab)^{n/2} & 0 \\ 0 & (ba)^{n/2} \end{bmatrix}$ for $n$ even and $M^n = \begin{bmatrix} 0 & (ab)^{(n-1)/2}a \\ (ba)^{(n-1)/2}b & 0 \end{bmatrix}$ for $n$ odd. But $a \perp b$ and hence $ab$ and $ba$ are nilpotent. Therefore $M$ is nilpotent.
- Suppose $\Pi$ is obtained by applying the cut rule to the proofs $\Pi'$ and $\Pi''$ of $\vdash [\Delta'], \Gamma', A$ and $\vdash [\Delta''], A^\perp, \Gamma''$ respectively. We assume first that $\Delta'$ and $\Delta''$

are empty and $\Gamma' = B'$ and $\Gamma'' = B''$ are single formulas. Then we must show that $[\![\Pi]\!]$ is an algorithm of type $\theta(B', B'')$; that is to say $EX([\![\Pi]\!], \sigma)$ (with $\sigma = s$) is a finite sum and for all $b' \in \theta(B'^{\perp})$ we must have that the 11th entry of $CUT(b', EX([\![\Pi]\!], \sigma))$ is nilpotent and $ex(CUT(b', EX([\![\Pi]\!], \sigma))) \in \theta(B'')$.

By inductive hypothesis we have that $[\![\Pi']\!], [\![\Pi'']\!]$ are algorithms of types $\theta(B', A)$ and $\theta(A^{\perp}, B'')$ respectively. Hence given $b' \in \theta(B'^{\perp})$ we have that $b'\pi'_{11}$ is nilpotent and $a = ex(CUT(b', [\![\Pi']\!])) \in \theta(A)$. Note that $a = \pi'_{22} + \sum_n \pi'_{21}(b'\pi'_{11})^n b'\pi'_{12}$ and $a\pi''_{11}$ is nilpotent and $ex(CUT(a, [\![\Pi'']\!])) \in \theta(B'')$. This means that $\pi'_{22}\pi''_{11}$ is nilpotent which implies that $\begin{bmatrix} 0 & \pi''_{11} \\ \pi'_{22} & 0 \end{bmatrix}$ is nilpotent and hence $EX([\![\Pi]\!], \sigma)$ is a finite sum. Recall that

$$[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & 0 & \pi'_{12} & 0 \\ 0 & \pi''_{22} & 0 & \pi''_{21} \\ \pi'_{21} & 0 & \pi'_{22} & 0 \\ 0 & \pi''_{12} & 0 & \pi''_{11} \end{bmatrix}.$$

The 11th entry of $CUT(b', EX([\![\Pi]\!], \sigma))$ is given by $b'\pi'_{11} + \sum_n b'\pi'_{12}(\pi''_{11}\pi'_{22})^n \pi''_{11}\pi'_{21}$ which is nilpotent as follows: we shall write it as $Tr(sMs)$ for $M = \begin{bmatrix} \pi''_{11}\pi'_{22} & \pi''_{11}\pi'_{21} \\ b'\pi'_{12} & b'\pi'_{11} \end{bmatrix}$ which can be written as $M_1 + M_2$ with $M_1 = \begin{bmatrix} \pi''_{11}\pi'_{22} & 0 \\ 0 & b'\pi'_{11} \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & \pi''_{11}\pi'_{21} \\ b'\pi'_{12} & 0 \end{bmatrix}$. Now note that $M_1$ is nilpotent by inductive hypothesis and

$$\sum_n M_2 M_1^n = \begin{bmatrix} 0 & \sum_n \pi''_{11}\pi'_{21}(b'\pi'_{11})^n \\ \sum_n b'\pi'_{12}(\pi''_{11}\pi'_{22})^n & 0 \end{bmatrix}$$

which is nilpotent because $a\pi''_{11}$ is nilpotent. Thus $M$ is nilpotent by Lemma 18 and so $Tr(sMs)$ is nilpotent by Lemma 19.

Finally we shall show that $ex(CUT(b', EX([\![\Pi]\!], \sigma))) \in \theta(B'')$. Many steps in the following equations have been compressed, they all follow from trace properties and naturality of symmetry morphisms.

$ex(CUT(a, [\![\Pi'']\!])) =$

$= Tr^U(s(a \otimes 1)[\![\Pi'']\!]s)$

$= Tr^U(s(Tr^U(s(b' \otimes 1)[\![\Pi']\!]s) \otimes 1)[\![\Pi'']\!]s)$

$= Tr^U((1 \otimes s)(b' \otimes 1)(Tr^U((1 \otimes s)([\![\Pi']\!] \otimes 1)(1 \otimes s)$

$([\![\Pi'']\!] \otimes 1)(s \otimes 1)(1 \otimes s))))$

$= Tr^U((1 \otimes s)(b' \otimes 1)Tr^{U^2}((1 \otimes s \otimes 1)(1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$

$([\![\Pi']\!] \otimes [\![\Pi'']\!])(s \otimes s)(1 \otimes s \otimes 1)(s \otimes 1 \otimes 1)))$

$= ex(CUT(b', EX([\![\Pi]\!], \sigma)))$

However, by inductive hypothesis $ex(CUT(a, [\![\Pi'']\!])) \in \theta(B'')$. The case of nonsingleton $\Gamma'$ and $\Gamma''$ is similar except that it is done in higher dimension with block matrices.

In this and all the following cases we assume that $\Delta$ is empty. The nonempty case can be reduced to the empty case using the associativity of cut. More explicitly, we would like to prove the result for $EX([\![\Pi]\!], \sigma)$, where $\sigma$ represents the cut formulas in $\Delta$. We remove all the cuts in $\Pi$ except the one occurring as the last rule by first pre- and post-composing $[\![\Pi]\!]$ with appropriate permutations (see the rightmost formula in Lemma 13) and then applying the execution formula. Then we apply this theorem and get back to $EX([\![\Pi]\!], \sigma)$ using the associativity of cut.

- Suppose that $\Pi$ is obtained from a proof $\Pi'$ of $\Gamma'$ by an application of an exchange rule. Let $\Gamma' = A_1, \cdots, A_i, A_{i+1}, \cdots, A_n$ and $\Gamma = A_1, \cdots, A_{i+1}, A_i, \cdots, A_n$. By inductive hypothesis $[\![\Pi']\!]$ is a datum of type $\theta(\Gamma')$, so for $a_i \in \theta(A_i^\perp)$, we have that $(a_1 \otimes \cdots \otimes a_n)[\![\Pi']\!]$ is nilpotent. Recall that $[\![\Pi]\!] = \tau^{-1}[\![\Pi']\!]\tau$ for $\tau = (1 \otimes s \otimes 1)$ with identity morphisms of appropriate types. Then for $a_i \in \theta(A_i^\perp)$, $(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes a_n)[\![\Pi]\!] = \tau^{-1}(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes a_n)[\![\Pi']\!]\tau$ which is clearly nilpotent.
- Suppose that $\Pi$ is obtained from the proofs $\Pi'$ and $\Pi''$ of $\vdash \Gamma', A$ and $\vdash \Gamma'', B$ respectively by an application of a times rule. We let $\Gamma' = C'$ and $\Gamma'' = C''$ be single formulas, the general case is similar. We need to show that $[\![\Pi]\!]$ is a datum of type $\theta(C', C'', A \otimes B)$ that is, given $\gamma' \in \theta(C'^\perp)$ and $\gamma'' \in \theta(C''^\perp)$ we need to show that $\gamma'\pi_{11}$ and $\gamma''\pi_{22}$ are nilpotent and that $ex(CUT(\gamma', \gamma'', [\![\Pi]\!])) \in \theta(A \otimes B)$.

  By the inductive hypothesis we have $\gamma'\pi'_{11}$ and $\gamma''\pi''_{11}$ are nilpotent. Recall that

$$[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & 0 & \pi'_{12}k_1 \\ 0 & \pi''_{11} & \pi''_{12}k_2 \\ j_1\pi'_{21} & j_2\pi''_{21} & j_1\pi'_{22}k_1 + j_2\pi''_{22}k_2 \end{bmatrix}$$

  Hence, $\gamma'\pi_{11}$ and $\gamma''\pi_{22}$ are nilpotent. Also by the inductive hypothesis we have that $a = ex(CUT(\gamma', [\![\Pi']\!])) = \pi'_{22} + \sum_n \pi'_{21}(\gamma'\pi'_{11})^n\gamma'\pi'_{12} \in \theta(A)$ and

$b = ex(CUT(\gamma'', [\![\Pi'']\!])) = \pi''_{22} + \sum_n \pi''_{21}(\gamma'\pi''_{11})^n\gamma''\pi''_{12} \in \theta(B)$. It is immediate that $j_1ak_1 + j_2bk_2 \in \theta(A \otimes B)$, but a simple calculation shows that $ex(CUT(\gamma', \gamma'', [\![\Pi]\!])) = j_1ak_1 + j_2bk_2$.

- Suppose $\Pi$ is obtained from a proof $\Pi'$ of $\vdash \Gamma', A, B$ by an application of a par rule. We assume that $\Gamma' = C$ a single formula, the general case being similar. By the inductive hypothesis we have that $[\![\Pi']\!] \in \theta(C, A, B)$. Note that

$$[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & \pi'_{12}k_1 + \pi'_{13}k_2 \\ j_1\pi'_{21} + j_2\pi'_{31} & j_1\pi'_{22}k_1 + j_2\pi'_{32}k_1 + j_1\pi'_{23}k_2 + j_2\pi'_{33}k_2 \end{bmatrix}$$

By the inductive hypothesis for any $\gamma \in \theta(C^\perp)$, $\gamma\pi'_{11}$ is nilpotent and hence $\gamma\pi_{11}$ is nilpotent. $ex(CUT(\gamma, [\![\Pi]\!])) = \alpha + \beta$ where $\alpha = j_1\pi'_{22}k_1 + j_2\pi'_{32}k_1 + j_1\pi'_{23}k_2 + j_2\pi'_{33}k_2$ and $\beta = \sum_n (j_1\pi'_{21} + j_2\pi'_{31})(\gamma\pi'_{11})^n(\gamma\pi'_{12}k_1 + \gamma\pi'_{13}k_2)$.

Now let $a \in \theta(A^\perp)$ and $b \in \theta(B^\perp)$. Then by assumption $(\gamma \otimes a \otimes b)[\![\Pi']\!]$ is nilpotent. We need to show that $ex(CUT(\gamma, [\![\Pi]\!])) \perp (j_1ak_1 + j_2bk_2)$ and conclude $ex(CUT(\gamma, [\![\Pi]\!])) \in \theta(A \invamp B)$. Let $\epsilon = (j_1ak_1 + j_2bk_2)$. Using the assumption above we see that $\epsilon\alpha$ and $\sum_n \epsilon\beta(\epsilon\alpha)^n$ are nilpotent and hence by Lemma 18 we get the desired result.

- Suppose $\Pi$ is obtained from a proof $\Pi'$ of $\vdash ?\Gamma', A$ by an application of an *of course* rule. For simplicity we assume that $\Gamma' = B$, a single formula, the general case being similar. Recall that

$$[\![\Pi]\!] = \begin{bmatrix} ue_UT(v\pi'_{11}u)e'_Uv & ue_UT(v\pi'_{12})v \\ uT(\pi'_{21}u)e'_Uv & uT(\pi'_{22})v \end{bmatrix}.$$

Now let $b \in \theta(B^\perp)$, then $uT(b)v \in \theta(!B^\perp) = \theta((?B)^\perp)$. By inductive hypothesis $uT(b)v\pi'_{11}$ is nilpotent and hence so is $uT(b)vue_UT(v\pi'_{11}u)e'_Uv$, because

$$(uT(b)vue_UT(v\pi'_{11}u)e'_Uv)^n = ue_UT((T(b)v\pi'_{11}u)^n)e'_Uv$$

by functoriality of $T$, naturality of $e$ and $vu = 1$. It remains to show that $ex(CUT(uT(b)v, [\![\Pi]\!])) \in \theta(!A)$. By inductive hypothesis $\alpha = ex(CUT(uT(b)v, [\![\Pi']\!])) = \pi'_{22} + \sum_n \pi'_{21}(uT(b)v\pi'_{11})^n uT(b)v\pi'_{12} \in \theta(A)$. One computes that $ex(CUT(uT(b)v, [\![\Pi]\!])) = uT(\alpha)v$ because of $(e, e'), (u, v)$ being retraction pairs, naturality of $e$ and functoriality of $T$. Hence $ex(CUT(uT(b)v, [\![\Pi]\!])) \in \theta(!A)$.

- Suppose $\Pi$ is obtained from a proof $\Pi'$ of $\vdash \Gamma'$ by an application of a weakening rule. As usual we assume that $\Gamma' = B$. Let $b \in \theta(B^\perp)$, and recall that

$$[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

By inductive hypothesis $b\pi'_{11}$ is nilpotent and $ex(CUT(b, [\![\Pi]\!])) = 0 + \sum_n 0(b\pi'_{11})^n 0 = 0$, which clearly belongs in $\theta(?A)$.

- Suppose $\Pi$ is obtained from a proof $\Pi'$ of $\vdash \Gamma', A$ using an application of a dereliction rule. As usual we assume that $\Gamma' = B$. Recall that

$$[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & \pi'_{12} d'_U v \\ u d_U \pi'_{21} & u d_U \pi'_{22} d'_U v \end{bmatrix}$$

and by inductive hypothesis we have that for any $b \in \theta(B^\perp)$, $b\pi'_{11}$ is nilpotent and hence so is $b\pi_{11}$. Also $\alpha = ex(CUT(b, [\![\Pi']\!])) = \pi'_{22} + \sum_n \pi'_{21}(b\pi'_{11})^n b\pi'_{12} \in \theta(A)$. Now let $a \in \theta(A^\perp)$, then $a\alpha$ is nilpotent. Let $\epsilon = ex(CUT(b, [\![\Pi]\!])) = u d_U \alpha d'_U v$, and note that $u(Ta)v\epsilon = uTa(d_U \alpha d'_U)v = u d_U a\alpha d'_U v$ by naturality of $d$ and hence $u(Ta)v\epsilon$ is nilpotent as $(d, d'), (u, v)$ are retraction pairs and $a\alpha$ is nilpotent. Therefore, $\epsilon \in \theta(?A)$.
- Suppose $\Pi$ is obtained from a proof $\Pi'$ of $\vdash \Gamma', ?A, ?A$ using an application of the contraction rule. As usual we assume that $\Gamma' = B$. We use $c_1, c_2, c'_1$ and $c'_2$ to denote the components of $c_U$ and $c'_U$. Recall that $[\![\Pi]\!] = \begin{bmatrix} \pi'_{11} & \pi'_{12} u c'_1 v + \pi'_{13} u c'_2 v \\ u c_1 v \pi'_{21} + u c_2 v \pi'_{31} & \omega \end{bmatrix}$ where $\omega = u c_1 v \pi'_{22} u c'_1 v + u c_2 v \pi'_{32} u c'_1 v + u c_1 v \pi'_{23} u c'_2 v + u c_2 v \pi'_{33} u c'_2 v$. Let $b \in \theta B^\perp$, then $b\pi'_{11}$ is nilpotent and thus so is $b\pi_{11}$. Let $\alpha = ex(CUT(b, [\![\Pi']\!])) = \begin{bmatrix} \pi'_{22} & \pi'_{23} \\ \pi'_{32} & \pi'_{33} \end{bmatrix} + \sum_n \begin{bmatrix} \pi'_{21} \\ \pi'_{31} \end{bmatrix} (b\pi'_{11})^n \begin{bmatrix} b\pi'_{12} & b\pi'_{13} \end{bmatrix} \in \theta(?A, ?A)$.

  We need to show that $\epsilon = ex(CUT(b, [\![\Pi]\!])) \in \theta(?A)$. Let $a \in \theta(A^\perp)$, we have to show that $\epsilon \perp u(Ta)v$. $uT(a)v\epsilon = uc_1(Ta)v\alpha_{11}uc'_1 v + uc_1(Ta)v\alpha_{12}uc'_2 v + uc_2(Ta)v\alpha_{21}uc'_1 v + uc_2(Ta)v\alpha_{22}uc'_2 v$ which is nilpotent because by inductive hypothesis we have that $(u(Ta)v \otimes u(Ta)v)\alpha$ is nilpotent, $(c, c')$ is a retraction pair, that is $c'_j c_i = 0$ for $i \neq j$ and 1 for $i = j$ and $(u, v)$ is a retraction pair.

**Theorem 21** *Let $\Pi$ be a proof of a sequent $\vdash [\Delta], \Gamma$ in MELL. Then*

(i) *$EX([\![\Pi]\!], \sigma)$ is a finite sum.*

(ii) *If $\Pi$ reduces to $\Pi'$ by any sequence of cut-elimination steps and $\Gamma$ does not contain any formulas of the form $?A$, then $EX([\![\Pi]\!], \sigma) = EX([\![\Pi']\!], \tau)$. So $EX([\![\Pi]\!], \sigma)$ is an invariant of reduction. In particular, if $\Pi'$ is any cut-free proof obtained from $\Pi$ by cut-elimination, then $EX([\![\Pi]\!], \sigma) = [\![\Pi']\!]$.*

**PROOF.** Part (i) is an easy corollary of Theorem 20. We proceed to the proof of part (ii). As explained in Girard's proof of the same theorem, see pages 235-239 and 248-249 in [11], it suffices to check the following cases:

(1) Suppose $\Pi'$ is a cut-free proof of $\vdash \Gamma, A$ and $\Pi$ is obtained by applying the cut rule to $\Pi'$ and the axiom $\vdash A^{\perp}, A$. Then

$$EX([\![\Pi]\!],\sigma) = Tr\left((1\otimes\sigma)\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}\pi'_{11}&\pi'_{12}&0&0\\\pi'_{21}&\pi'_{22}&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\right)$$

$$= Tr\left(\begin{bmatrix}\pi'_{11}&0&\pi'_{12}&0\\0&0&0&1\\0&1&0&0\\\pi'_{21}&0&\pi'_{22}&0\end{bmatrix}\right) = \begin{bmatrix}\pi'_{11}&\pi'_{12}\\\pi'_{21}&\pi'_{22}\end{bmatrix} = [\![\Pi']\!]$$

(2) Similarly to [11], in the following five cases we shall consider proofs $\Pi$ of the form

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash \Gamma', A\end{matrix} \qquad \begin{matrix}\Pi'' \\ \vdots \\ \vdash A^{\perp}, \Gamma''\end{matrix}}{\vdash [A, A^{\perp}], \Gamma', \Gamma''}\ (cut)$$

Also we assume that the last rules in $\Pi'$ and $\Pi''$ are logical rules applied to $A$ or $A^{\perp}$. Hence in the syntax the cut rule for $A$ will be replaced by other cuts. We use $\sigma$ to represent the cuts of $\Pi$ and $\tau$ for those of $\Xi$, which is obtained from $\Pi$ by one step reduction (cut-elimination). We shall ignore the exchange rule hereafter.

First we consider the case where $A \equiv B \otimes C$ and hence $A^{\perp} \equiv B^{\perp} \parr C^{\perp}$. Hence $\Pi'$ is obtained from $\Pi'_1$ of $\vdash \Gamma'_1, B$ and $\Pi'_2$ of $\vdash \Gamma'_2, C$ using the times rule. Also $\Pi''$ is obtained from $\Pi''_1$ of $\vdash B^{\perp}, C^{\perp}, \Gamma''$ using the par rule. We shall, without loss of generality, assume that $\Gamma'_1, \Gamma'_2$ and $\Gamma''$ consist of single formulas. The more general case follows in similar ways only using block matrices.

$$[\![\Pi'_1]\!] = \begin{bmatrix}a&b\\c&d\end{bmatrix}, \quad [\![\Pi'_2]\!] = \begin{bmatrix}e&f\\g&h\end{bmatrix} \quad \text{and} \quad [\![\Pi''_1]\!] = \begin{bmatrix}l&m&o\\p&q&r\\x&y&z\end{bmatrix}$$

Then

$$[\![\Pi'']\!] = \begin{bmatrix} j_1lk_1 + j_2pk_1 + j_1mk_2 + j_2qk_2 & j_1o + j_2r \\ xk_1 + yk_2 & z \end{bmatrix} \quad \text{and}$$

$$[\![\Pi']\!] = \begin{bmatrix} a & 0 & bk_1 \\ 0 & e & fk_2 \\ j_1c & j_2g & j_1dk_1 + j_2hk_2 \end{bmatrix}.$$

$$\text{And } [\![\Pi]\!] = \begin{bmatrix} a & 0 & 0 & bk_1 & 0 \\ 0 & e & 0 & fk_2 & 0 \\ 0 & 0 & z & 0 & xk_1 + yk_2 \\ j_1c & j_2g & 0 & j_1dk_1 + j_2hk_2 & 0 \\ 0 & 0 & j_1o + j_2r & 0 & j_1lk_1 + j_2pk_1 + j_1mk_2 + j_2qk_2 \end{bmatrix}.$$

Finally $EX([\![\Pi]\!], \sigma) = Tr((1 \otimes s)[\![\Pi]\!])$.

Now apply the cut rule to $\Pi_1'$ and $\Pi_1''$ to get $\Pi_0$ of $\vdash [B, B^\perp], C^\perp, \Gamma_1', \Gamma''$ and apply cut again to $\Pi_2'$ and $\Pi_0$ to get $\Xi$ of $\vdash [B, B^\perp, C, C^\perp], \Gamma_1', \Gamma_2', \Gamma''$. We have

$$[\![\Xi]\!] = \begin{bmatrix} a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & f & 0 \\ 0 & 0 & z & 0 & x & 0 & y \\ c & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & o & 0 & l & 0 & m \\ 0 & g & 0 & 0 & 0 & h & 0 \\ 0 & 0 & r & 0 & p & 0 & q \end{bmatrix}.$$

Define the matrices $K = (1 \otimes s \otimes 1)(k \otimes k)$, $J = (j \otimes j)(1 \otimes s \otimes 1)$, and $L = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & K \end{bmatrix}$, $L' = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix}$. An easy matrix multiplication shows that $L'(1 \otimes s \otimes s)[\![\Xi]\!]L = (1 \otimes s)[\![\Pi]\!]$.

$$\begin{aligned} EX([\![\Xi]\!], \tau) &= Tr^{U \otimes U \otimes U \otimes U}((1 \otimes s \otimes s)[\![\Xi]\!]) \\ &= Tr^{U \otimes U}(L'(1 \otimes s \otimes s)[\![\Xi]\!]L) \\ &\quad \text{by dinaturality of trace and } LL' = 1_{U^7} \\ &= Tr^{U \otimes U}((1 \otimes s)[\![\Pi]\!]) \\ &= EX([\![\Pi]\!], \sigma) \end{aligned}$$

As a matter of fact we could have proven both cases above using algebraic methods (properties of trace and structural morphisms, etc). We shall do so in the following cases to avoid long calculations. In the following: $\beta = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$, $\alpha = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)(s \otimes 1 \otimes 1)$ and $\rho = (1 \otimes s)(s \otimes 1)$. Note that the permutations above are instantiated at different types in the formulas where they appear. Also, we let $\beta' = s_{U^2,U^2}$ and $\delta = s_{U,U^4}$.

(3) Suppose that $A \equiv\ !B$ and so $A^\perp \equiv\ ?B^\perp$, and $\Pi$ is given by the following proof.

$$\dfrac{\dfrac{\begin{array}{c}\Pi'_1\\ \vdots\\ \vdash ?\Gamma'_1, B\end{array}}{\vdash ?\Gamma'_1, !B}\,(!) \qquad \dfrac{\begin{array}{c}\Pi''_1\\ \vdots\\ \vdash ?B^\perp, ?B^\perp, \Gamma''\end{array}}{\vdash ?B^\perp, \Gamma''}\,(contraction)}{\vdash [!B, ?B^\perp], ?\Gamma'_1, \Gamma''}\,(cut)$$

The proof $\Xi$ is obtained by first cutting $\Pi'$ against $\Pi''_1$ to get a proof $\Pi_0$ of $\vdash [!B, ?B^\perp], ?B^\perp, ?\Gamma'_1, \Gamma''$, next cutting $\Pi'$ against $\Pi_0$ to get $\Xi_0$ ending with $\vdash [!B, ?B^\perp, !B, ?B^\perp], ?\Gamma'_1, ?\Gamma'_1, \Gamma''$ and finally doing a sequence of contractions on formulas in $?\Gamma'_1$.

Recall that by assumption, $\Gamma'_1$ has to be empty. In fact the following equations are not valid otherwise. Without loss of generality we let $\Gamma''$ be a single formula.

$EX([\![\Pi]\!], \sigma) =$

$= Tr^{U \otimes U}\left[(1 \otimes s)\rho^{-1}(uT([\![\Pi'_1]\!])v \otimes (uc_U(v \otimes v) \otimes 1_U)[\![\Pi''_1]\!]((u \otimes u)c'_U v \otimes 1_U))\rho\right]$

$= Tr^{U \otimes U \otimes U}\left[(1 \otimes \beta)\alpha^{-1}(T([\![\Pi'_1]\!])c_U \otimes (v \otimes v \otimes 1)[\![\Pi''_1]\!]((u \otimes u)c'_U \otimes 1))\alpha\right]$

dinaturality of trace, $vu = 1$

$= Tr^{U \otimes U \otimes U}[(1 \otimes \beta)\alpha^{-1}(c_U(T([\![\Pi'_1]\!]) \otimes T([\![\Pi'_1]\!])) \otimes (v \otimes v \otimes 1)[\![\Pi''_1]\!]$

$((u \otimes u)c'_U \otimes 1))\alpha]$ naturality of $c$

$= Tr^{U^4}\left[(1 \otimes \beta')\delta^{-1}((u \otimes u)(T([\![\Pi'_1]\!]) \otimes T([\![\Pi'_1]\!]))(v \otimes v) \otimes [\![\Pi''_1]\!])\delta\right]$

dinaturality of trace, $c'c = 1$

$= Tr^{U^4}\left[(1 \otimes \beta')\delta^{-1}([\![\Pi']\!] \otimes [\![\Pi']\!] \otimes [\![\Pi''_1]\!])\delta\right]$

$= EX([\![\Xi]\!], s \otimes s)$

(4) Suppose $\Pi$ is given as:

$$\dfrac{\dfrac{\begin{array}{c}\Pi'_1\\ \vdots\\ \vdash B\end{array}}{\vdash !B}\,(!) \qquad \dfrac{\begin{array}{c}\Pi''_1\\ \vdots\\ \vdash B^\perp, \Gamma''\end{array}}{\vdash ?B^\perp, \Gamma''}\,(dereliction)}{\vdash [!B, ?B^\perp], \Gamma''}\,(cut)$$

and $\Xi$ is obtained as

$$\dfrac{\begin{array}{c}\Pi_1' \\ \vdots \\ \vdash B\end{array} \quad \begin{array}{c}\Pi_1'' \\ \vdots \\ \vdash B^\perp, \Gamma''\end{array}}{\vdash [B, B^\perp], \Gamma''}\ (cut)$$

We have

$EX([\![\Pi]\!], \sigma) =$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(uT([\![\Pi_1']\!])v \otimes (ud_U \otimes 1)[\![\Pi_1'']\!](d_U' v\otimes 1))\rho\right]$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(T([\![\Pi_1']\!])d_U \otimes [\![\Pi_1'']\!](d_U' \otimes 1))\rho\right]$

dinaturality of trace and $vu = 1$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(d_U[\![\Pi_1']\!] \otimes [\![\Pi_1'']\!](d_U' \otimes 1))\rho\right]$ naturality of $d$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}([\![\Pi_1']\!] \otimes [\![\Pi_1'']\!])\rho\right]$ dinaturality of trace and $d'd = 1$

$= EX([\![\Xi]\!], s)$

(5) Suppose $\Pi$ is given as:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1' \\ \vdots \\ \vdash B\end{array}}{\vdash !B}\ (!) \qquad \dfrac{\begin{array}{c}\Pi_1'' \\ \vdots \\ \vdash \Gamma''\end{array}}{\vdash ?B^\perp, \Gamma''}\ (weakening)}{\vdash [!B, ?B^\perp], \Gamma''}\ (cut)$$

and $\Xi = \Pi_1''$.

We have

$EX([\![\Pi]\!], \sigma) =$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(uT([\![\Pi_1']\!])v \otimes (uw_U \otimes 1)(1_I \otimes [\![\Pi_1'']\!])(w_U' v\otimes 1))\rho\right]$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(T([\![\Pi_1']\!])w_U \otimes (1_I \otimes [\![\Pi_1'']\!])(w_U' \otimes 1))\rho\right]$

dinaturality of trace and $vu = 1$

$= Tr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(w_U \otimes (1_I \otimes [\![\Pi_1'']\!])(w_U' \otimes 1))\rho\right]$ naturality of $w$

$= Tr^{I}\left[s_{I,U}(1_I \otimes [\![\Pi_1'']\!])s_{U,I}\right]$ dinaturality of trace and $w'w = 1$

$= [\![\Pi_1'']\!]$ vanishing I.

(6) The last case is where $\Pi$ is given by

$$\dfrac{\dfrac{\begin{matrix}\Pi_1' \\ \vdots \\ \vdash B\end{matrix}}{\vdash !B}(!) \qquad \dfrac{\begin{matrix}\Pi_1'' \\ \vdots \\ \vdash ?B^\perp, C\end{matrix}}{\vdash ?B^\perp, !C}(!)}{\vdash [!B, ?B^\perp], !C}(cut)$$

and $\Xi$ is

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1' \\ \vdots \\ \vdash B\end{matrix}}{\vdash !B}(!) \qquad \begin{matrix}\Pi_1'' \\ \vdots \\ \vdash ?B^\perp, C\end{matrix}}{\vdash [!B, ?B^\perp], C}}{\vdash [!B, ?B^\perp], !C}(!)$$

We have

$EX([\![\Pi]\!], \sigma) =$

$= Tr^{U\otimes U}[(1\otimes s)\rho^{-1}(uT([\![\Pi_1']\!])v \otimes (ue_U \otimes u)\phi^{-1}T((v\otimes 1)[\![\Pi_1'']\!]$

$(u\otimes 1))\phi(e_U' v \otimes v))\rho]$

$= Tr^{U\otimes U}[(1\otimes s)\rho^{-1}(T[\![\Pi_1']\!]e_U \otimes (1\otimes u)\phi^{-1}T((v\otimes 1)[\![\Pi_1'']\!]$

$(u\otimes 1))\phi(e_U' \otimes v))\rho]$ dinaturality of trace and $vu = 1$

$= Tr^{U\otimes U}[(1\otimes s)\rho^{-1}(e_U T^2([\![\Pi_1']\!]) \otimes (1\otimes u)\phi^{-1}T((v\otimes 1)[\![\Pi_1'']\!]$

$(u\otimes 1))\phi(e_U' \otimes v))\rho]$ naturality of $e$

$= Tr^{U\otimes U}[(1\otimes s)\rho^{-1}(T^2([\![\Pi_1']\!]) \otimes (1\otimes u)\phi^{-1}T((v\otimes 1)[\![\Pi_1'']\!]$

$(u\otimes 1))\phi(1 \otimes v))\rho]$ dinaturality of trace and $e'e = 1$

$= uTr^{U\otimes U}\left[(1\otimes s)\rho^{-1}(T^2([\![\Pi_1']\!]) \otimes \phi^{-1}T((v\otimes 1)[\![\Pi_1'']\!](u\otimes 1))\phi)\rho\right] v$

naturality of trace and symmetry

$= EX(\Xi, s)$

## 6 Girard's Operator Algebraic model

In this section we observe that Girard's original $C^*$-algebra model (implementation) in GoI 1 is captured in our categorical framework using the category

$\mathbf{Hilb}_2$. First, recall Examples 8 and 9 which show that $(\mathbf{PInj}, \mathbb{N} \times -, \mathbb{N})$ and $(\mathbf{Hilb}_2, \ell^2 \otimes -, \ell^2)$ are GoI situations.

**Proposition 22** $(\mathbf{Hilb}_2, \ell^2 \otimes -, \ell^2)$ *is a GoI Situation which agrees with Girard's $C^*$-algebraic model, where $\ell^2 = \ell_2(\mathbb{N})$. Its structure is induced via $\ell_2$ from* $\mathbf{PInj}$.

**PROOF.** The structural retractions have already been shown in the two examples above. The correspondence of our set-up with that of Girard and in fact with dynamic algebras [11,9,7,27] is given by the dictionary below. Here $j_1, j_2, k_1, k_2$ are the components of $j : U \otimes U \to U$ and $k : U \to U \otimes U$, respectively, $c_1, c_2$ are the components of $c : TU \otimes TU \to TU$ and $c'_1, c'_2$ are the components of $c' : TU \to TU \otimes TU$.

| *Girard* | *This Paper* |
|---|---|
| $1 \otimes a$ | $uT(a)v$ |
| $p, p^*$ | $j_1, k_1$ |
| $q, q^*$ | $j_2, k_2$ |
| $(1 \otimes r), (1 \otimes r^*)$ | $uc_1v, uc'_1v$ |
| $(1 \otimes s), (1 \otimes s^*)$ | $uc_2v, uc'_2v$ |
| $t, t^*$ | $ue_U(Tv)v, u(Tu)e'_Uv$ |
| $d, d^*$ | $ud_U, d'_Uv$ |

What remains to be shown, to convince the reader that we have really obtained Girard's model, is to show our execution formula is the same as his. We do this in the next Proposition.

We now show that we obtain the same execution formula as Girard. Note that in Girard's original execution formula $[\![\Pi]\!]$ and $\sigma$ are both $2m+n$ by $2m+n$ matrices. To connect up with our previous notation, let $\tilde{\sigma} = s \otimes \cdots \otimes s$ ($m$-times.)

**Proposition 23** *Let $\Pi$ be a proof of $\vdash [\Delta], \Gamma$. Then in Girard's model* $\mathbf{Hilb}_2$ *above,*

$$((1-\sigma^2)\sum_{n=0}^{\infty}[\![\Pi]\!](\sigma[\![\Pi]\!])^n(1-\sigma^2))_{n\times n} = Tr((1\otimes\tilde{\sigma})[\![\Pi]\!])$$

*where $(A)_{n\times n}$ is the submatrix of $A$ consisting of the first $n$ rows and the first $n$ columns.*

**PROOF.** First note that by Lemma 4, page 242 of [11], the left hand side, when it exists, is a morphism in $\mathbf{Hilb_2}(\ell^{2\otimes n}, \ell^{2\otimes n})$. Also, we considered our matrices to be indexed as $n+2m$ and not $2m+n$ as Girard does, so we have just flipped the indexing in his execution formula to have indexing as $n+2m$. We next show that Girard's execution formula is the same as our execution formula. Recall $\tilde{\sigma} = s \otimes \cdots \otimes s$ ($m$-copies) and $\sigma = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\sigma} \end{bmatrix}$.

Now a simple calculation shows that the two sides are the same as follows:

First note that $(1-\sigma^2) = \begin{bmatrix} \mathbf{Id}_{n\times n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Hence the multiplication by $(1-\sigma^2)$ from left and right selects the 11th block matrix, which is an $n \times n$ matrix and sets all other entries to zero. Then one retrieves this $n \times n$ matrix out of the execution formula by removing the $2m$ zero rows and $2m$ zero columns.

$$\sigma[\![\Pi]\!] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\sigma} \end{bmatrix} \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \tilde{\sigma}\Pi_3 & \tilde{\sigma}\Pi_4 \end{bmatrix}.$$

Hence for $n \geq 1$,

$$(\sigma[\![\Pi]\!])^n = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\tilde{\sigma}\Pi_4)^{n-1}\tilde{\sigma}\Pi_3 & (\tilde{\sigma}\Pi_4)^n \end{bmatrix}.$$

Finally $(1-\sigma^2)\sum_{n=0}^{\infty}[\![\Pi]\!](\sigma[\![\Pi]\!])^n(1-\sigma^2) = \begin{bmatrix} \Pi_1 + \sum_{n\geq 1}\Pi_2(\tilde{\sigma}\Pi_4)^{n-1}\tilde{\sigma}\Pi_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

So the 11th block entry of this matrix is nothing but $Tr\left(\begin{bmatrix} \Pi_1 & \Pi_2 \\ \tilde{\sigma}\Pi_3 & \tilde{\sigma}\Pi_4 \end{bmatrix}\right) = Tr((1 \otimes \tilde{\sigma})[\![\Pi]\!])$.

Therefore in the categorical setting we need not remove $2m$ all-zero columns and rows; they are automatically removed!

## 7 Conclusions and Further Work

In this paper we have given a categorical model and axiomatization for the GoI semantics of MELL and have proven the necessary theorems. We also showed how Girard's original operator algebra model fits into this framework. We did not discuss the work by Abramsky and Jagadeesan [4] for the simple reason that it does not fit the unique decomposition category framework; that is, the category of domains does not form a UDC. This already suggests the necessity for a suitable generalization of the ideas presented in this paper. More precisely, we observe that the necessary ingredients for a categorical interpretation (model) are provided in the definition of a GoI Situation. However one still needs to give general meaning to the notions of *orthogonality* and *type* as well as provide a notion of "nilpotency", "finite sum" or "convergence". Observe that these notions found natural meanings in UDCs but a general traced category does not always have the corresponding notions.

We should note that there are many concrete GoI situations based on partially additive and unique decomposition categories; thus there are many models of this paper ([14]). However, to obtain *exactly* Girard's GoI 1, we also used Barr's $\ell_2$ representation of **PInj** in **Hilb**. We do not yet know of any operator-algebra representations for other models. That is an interesting open problem. However, independently of operator algebras, one may also investigate the GoI interpretation in any *specific* model (for example, the partially additive category **SRel**) of stochastic kernels, perhaps with respect to alternative notions of orthogonality (as in [20]) and convergence.

In [9], Girard addresses the issue of non-terminating algorithms and proves a convergence theorem for the execution formula (note that in this case nilpotency is out of the question). This work is further studied in [24]. It would be interesting to see how this can be captured in our categorical framework where all existing infinite sums make sense. The challenge would be to have a means of distinguishing good and bad infinite sums, that is the ones corresponding to non-termination and to divergence respectively.

Moreover in [12], Girard extended GoI to the full case, including the additives and constants. He also proved a nilpotency theorem for this semantics and its soundness (for a slightly modified sequent calculus) in the case of exponential-free conclusions. This too constitutes one of the main parts of our future work.

Last but certainly not least, we believe that GoI could be further used in its capacity as a new kind of semantics to analyze PCF and other fragments of functional and imperative languages and be compared to usual denotational and operational semantics through full abstraction theorems. The work on full completeness theorems for MLL via GoI in [13,15] is just a first step. Further related results, including those of Abramsky and Lenisa (e.g. [5]), should be examined.

## 8 Acknowledgements

We would like to thank the anonymous referees for their very illuminating and insightful remarks. We would also like to thank many colleagues for enlightening observations during our presentation of these results. We especially wish to thank Samson Abramsky, Vincent Danos, and Martin Hyland for helpful remarks.

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## A Pictorial Representation for GoI Interpretation

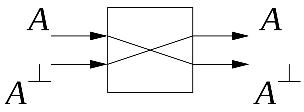


Figure A.1: Axiom

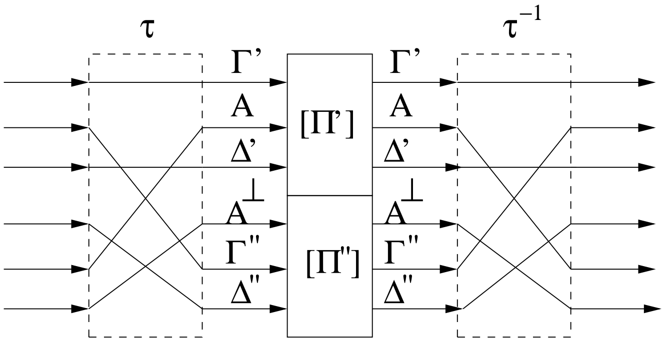


Figure A.2: Cut

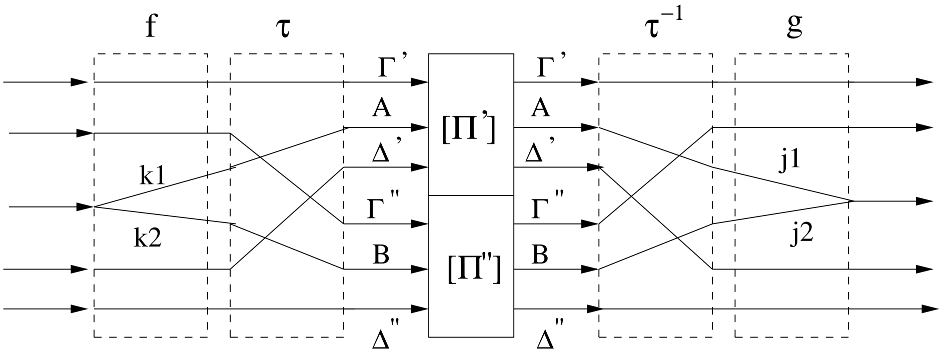


Figure A.3: Times

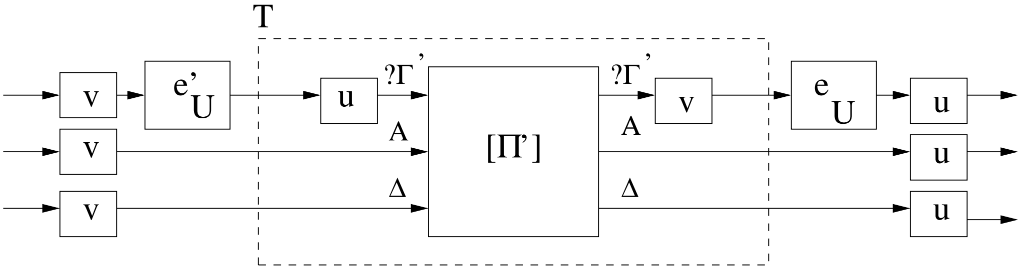


Figure A.4: of Course

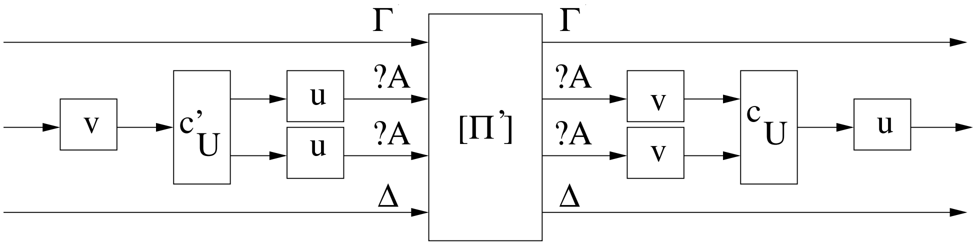

Figure A.5: Contraction