Tutorial on Geometry of Interaction

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Reporting on recent work with E. Haghverdi

Traditional (model theory & categorical logic)

$$[\![-]\!] \; : \quad \text{Logic} \longrightarrow \text{Model}$$

$$\text{formulas} \mapsto \text{objects}$$

$$\text{Proofs} \mapsto \text{arrows (functions)}$$

$$A \overset{\vdots\, \pi}{\vdash} B \quad \mapsto \quad [\![A]\!] \xrightarrow{[\![\pi]\!]} [\![B]\!]$$

More generally:

$$A_1, \cdots, A_m \overset{\vdots\, \rho}{\vdash} B_1, \cdots, B_n \quad \mapsto \quad \bigotimes_i [\![A_i]\!] \xrightarrow{[\![\rho]\!]} \bigodot_j [\![B_j]\!]$$

Cut-Elimination: denotations are equal !

(rewriting) (no dynamics)

$$\pi_1 \succ \pi_2 \quad \mapsto [\![\pi_1]\!] = [\![\pi_2]\!] : [\![A]\!] \to [\![B]\!]$$

Girard's GoI Program (GoI 1-GoI 3 (1989-1995), GoI 4 (2004)) aims to mathematically model the dynamics of cut-elimination via operator algebras. One Goal: dynamical invariants.

Recall Gentzen's Cut-Rule

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

Gentzen proved the following theorem (which applies to many systems of logic):

Cut-elimination (Gentzen's Haupsatz, 1934): If Π is a proof of $\Gamma \vdash \Delta$, then there is a proof Π' of $\Gamma \vdash \Delta$ which does not use the cut-rule.

For usual sequent calculus, Gentzen gave an ND algorithm $\Pi \rightsquigarrow \Pi'$ (the cut-elimination procedure)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B, B\end{array}}{\Gamma \vdash B} \quad \begin{array}{c}\vdots\\ B \vdash \Delta\end{array}}{\Gamma \vdash \Delta}\,\text{Cut}$$

reduces to (w.r.t. appropriate measure)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B, B\end{array} \quad \begin{array}{c}\vdots\\ B \vdash \Delta\end{array}}{\Gamma \vdash B, \Delta}\,\text{Cut} \quad \begin{array}{c}\vdots\\ B \vdash \Delta\end{array}}{\begin{array}{c}\Gamma \vdash \Delta, \Delta\\ \hline \vdots\\ \Gamma \vdash \Delta\end{array}}\,\text{Cut}$$

In Girard's papers:

Proofs $\mapsto$ matrix operators on a C^*-algebra $\mathcal{B}(\mathcal{H})$

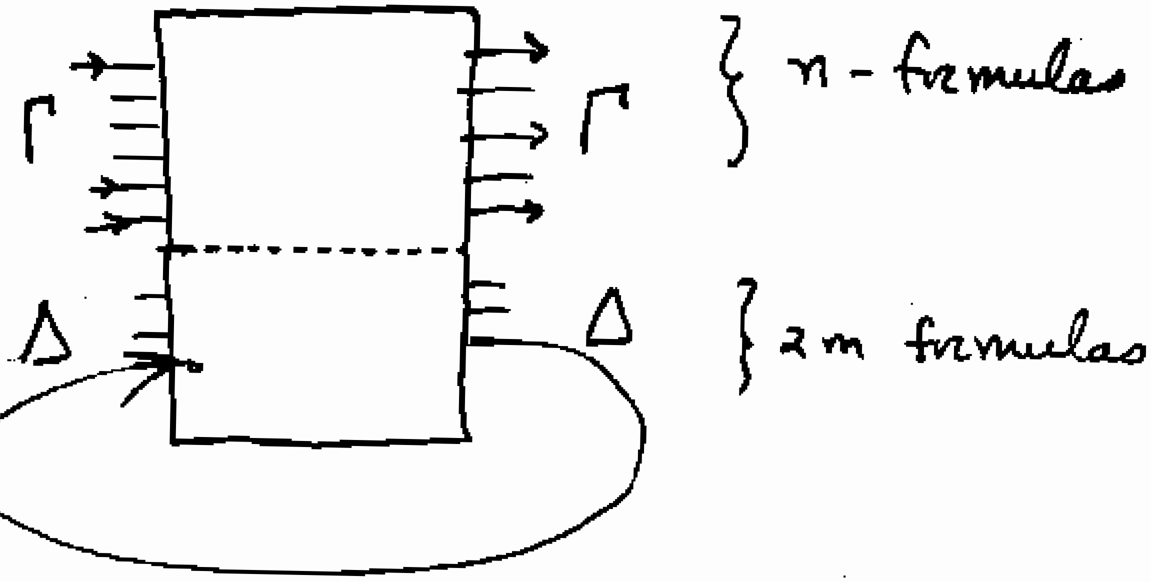
Idea of Girard's work (Details Later!) for MELL:

Proofs $\pi : \vdash [\Delta], \Gamma$ where Δ is a list of Cut formulas (e.g. $\langle A, A^\perp, B, B^\perp, C, C^\perp \cdots \rangle$) $|\Delta| = 2m$
$|\Gamma| = n$

Dynamic View: A Proof $=$ I/O box (with feedback) in a graphical network.

(*)

$[\![\pi]\!] =$



Models Concrete $\otimes$-categories $\mathcal{C}$ with distinguished "reflexive" object $U \in \mathcal{C}$, with additional structure.

Proofs are modeled as follows:

$$\pi : \vdash [\Delta], \Gamma \qquad \mapsto \qquad ([\![\pi]\!], \sigma)$$

where $[\![\pi]\!] : U^{n+2m} \longrightarrow U^{n+2m}$, and $U^{2m} \xrightarrow{\sigma} U^{2m}$ represents the cuts Δ, where $|\Delta| = 2m$ and $|\Gamma| = n$. (Here $U^k = U \otimes \cdots \otimes U \quad k$-times).

If $\Delta = \emptyset$, π is cut-free and $\sigma = 0$ will be a zero map ($\mathcal{C}$ is a semi-additive $\otimes$-category).

Write $[\![\pi]\!]$ as a block matrix:

$$[\![\pi]\!] = \left(\begin{array}{c|c} \pi_{11} & \pi_{12} \\ \hline \pi_{21} & \pi_{22} \end{array}\right)$$

A version of feedback/trace

(The Execution Formula)

$$Ex([\![\pi]\!], \sigma) = \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma\pi_{22})^n(\sigma\pi_{21})$$

Here we can think of $Ex([\![\pi]\!], \sigma) : U^n \to U^n$.

In Girard's Hilbert-space models, the Execution Formula has a special form (described later).

Theorem(Girard): (MELL and System $\mathcal{F}$)

- $Ex([\![\pi]\!],\sigma)$ is a finite sum.
- $Ex([\![\pi]\!],\sigma)$ is an invariant of cut-elimination. (Under certain restrictions on types for MELL)
- If π' is a cut-free normal form of π, then

$$Ex([\![\pi]\!],\sigma) = Ex([\![\pi']\!],0) = [\![\pi']\!]$$

- Above suggests new idea: GoI computing (à la Curry-Howard). (Recently studied in Complexity Theory, by Harry Mairson).

Girard also introduced fundamental operator algebra encodings which we need to categorize for:

- Types and Orthogonalities (cf. Hyland-Schalk)
- Algorithms
- Data

Later Works on GoI:

Girard-Style:

- Danos (1990)
- Danos-Regnier (1992–96)
- Malacaria-Regnier (1991)

GoI-style normalization & Complexity:

- Abadi,Gonthier,Levy (1992): Optimal Reduction (Lamping)
- Girard-Scedrov-Scott (1992): Bounded LL
- H. Mairson (2002–): GoI & Complexity Classes

Categorical Frameworks:

- Abramsky-Jagadeesan (1994): New Foundations for GoI
- Abramsky (1997): Siena Lectures
- Esfan Haghverdi (2000): Phd Thesis
- AHS (2002): GoI & LCA's
- Lenisa-Honsell: λ-Calc. & "wave-style" GoI
- H-S (2004): 2 papers on UDC-based GoI .

Basic Algebraic Framework

- *GoI Situations* (Abramsky '97, AHS'02)
  - Traced Monoidal Category $\mathcal{C}$
  - Endofunctor $T : \mathcal{C} \to \mathcal{C}$ with monoidal retracts: $TT \lhd T$, $Id \lhd T$, $T \otimes T \lhd T$, $\mathcal{K}_I \lhd T$
  - Reflexive object $U \in \mathcal{C}$ with retractions $U \otimes U \lhd U$, $I \lhd U$, $TU \lhd U$

GoI Situations isolate basic algebraic structure of GoI. We obtain Linear Combinatory Algebras on $\mathcal{C}(U, U)$ "modelling" full LL.

Variants of GoI

- $\otimes = +$ (Sum or "particle"-style)
- $\otimes = \times$ (Product or "wave"-style)

Theorem: ℓ_2[**PInj**] is *exactly* Girard's GoI 1.

Unique Decomposition Categories (UDC's)

- Symmetric $\otimes$-Category $\mathcal{C}$

- Axioms saying: homsets have infinitary partially-additive-monoid operation $\sum_{i\in I} f_i$ for countable families (compatible with composition). In particular zero morphisms $0_{XY} \in \mathcal{C}(X,Y)$.

- Finite tensors are *quasi biproducts*: there are *quasi-injections* $\iota_j : X_j \to \otimes_I X_i$ and *quasi-projections* $\rho_j : \otimes_I X_i \to X_j$ satisfying:

1. $\rho_k \iota_j = \begin{cases} 1_{X_j} & \text{if } j = k \\ 0_{X_j X_k} & \text{else} \end{cases}$

2. $\sum_{i\in I} \iota_i \rho_i = 1_{\otimes X_i}$

Arrows as Matrices: Given $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$ in a UDC with $|I| = m$ and $|J| = n$, there exists a unique family $\{f_{ij}\}_{i\in I, j\in J} : X_j \longrightarrow Y_i$ with $f = \sum_{i\in I, j\in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f \iota_j$. We write f as a matrix $f = [f_{ij}]$.

Composition in UDC's = matrix multiplication.

Traced Monoidal Categories (Joyal-Street-Verity)

Symmetric monoidal categories $(\mathcal{C}, I, \otimes, \sigma)$, equipped with a family of functions called a *trace* $Tr^U_{X,Y} : \mathcal{C}(X \otimes U, Y \otimes U) \longrightarrow \mathcal{C}(X, Y)$ subject to some axioms. In our models, think of $Tr^U_{X,Y}(f)$ given by "feedback".

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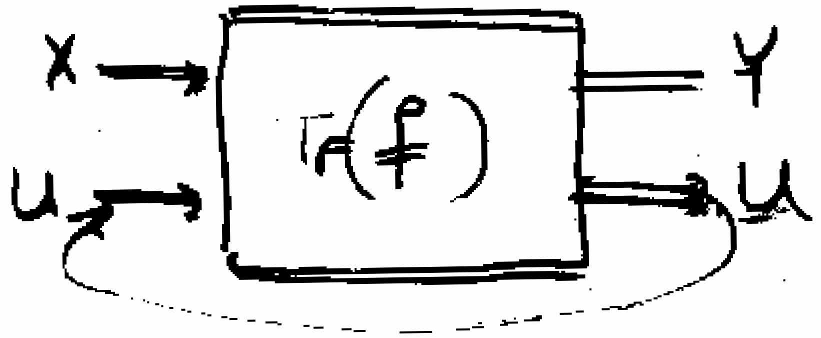


Natural in X, $Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g \otimes 1_U))$

where $f : X \otimes U \longrightarrow Y \otimes U$, $g : X' \longrightarrow X$,

Natural in Y, $gTr^U_{X,Y}(f) = Tr^U_{X,Y'}((g \otimes 1_U)f)$

where $f : X \otimes U \longrightarrow Y \otimes U$, $g : Y \longrightarrow Y'$,

Dinatural in U,

$Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,

Vanishing (I,II),

$Tr^I_{X,Y}(f) = f$ and

$Tr^{U \otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \otimes U, Y \otimes U}(g))$ (cf. Bekič)

Superposing,

$$Tr^U_{X,Y}(f) \otimes g =$$
$$Tr^U_{X\otimes W, Y\otimes Z}((1_Y \otimes \sigma_{U,Z})(f \otimes g)(1_X \otimes \sigma_{W,U}))$$

for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$,

Yanking $Tr^U_{U,U}(\sigma_{U,U}) = 1_U$.

There a general geometric calculus for reasoning about such TMC's...

Some Examples of TMC's

- **Rel**$_\times$, **Vec**$_{\mathrm{fd}}$, more generally any *compact* category (where $\otimes \cong \wp$) has a *canonical trace*

- ω-**CPO**$_\perp$ where trace given by Y combinator

- **Unique Decomposition Categories**

  (Iterative Traces:) **Rel**$_+$, **SRel** , **Pfn** , **PInj**, and in general all partially additive categories (Manes-Arbib), etc.

- (Selinger) : Quantum TMC's .

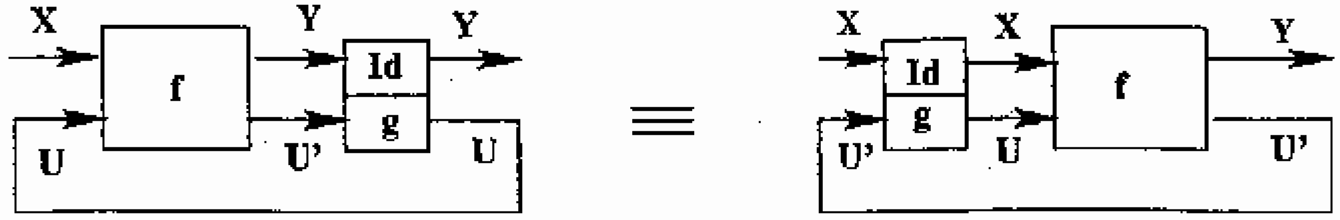


Figure 4: Dinaturality in U

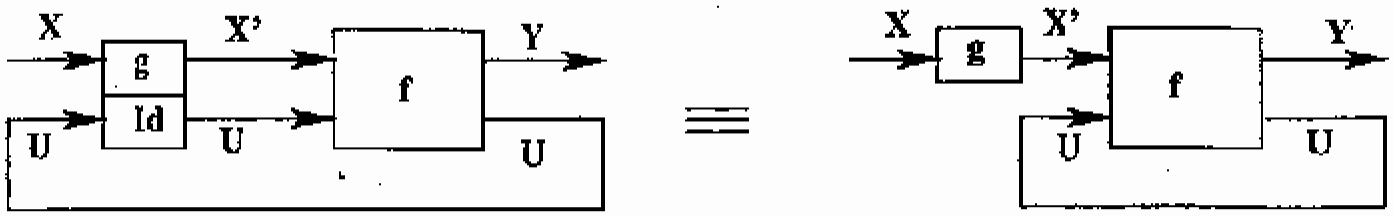


Figure 5: Naturality in X

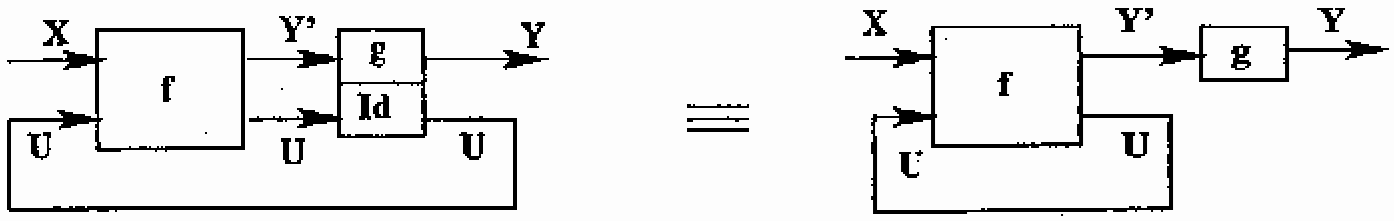


Figure 6: Naturality in Y

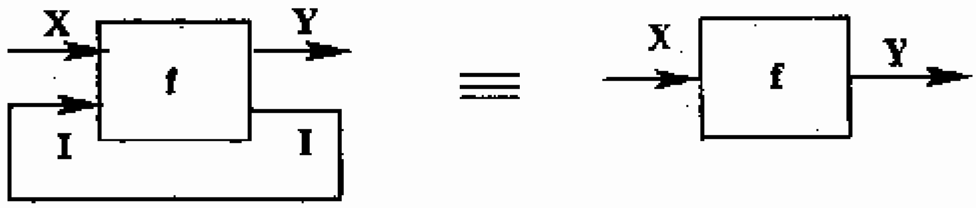


Figure 7: Vanishing I

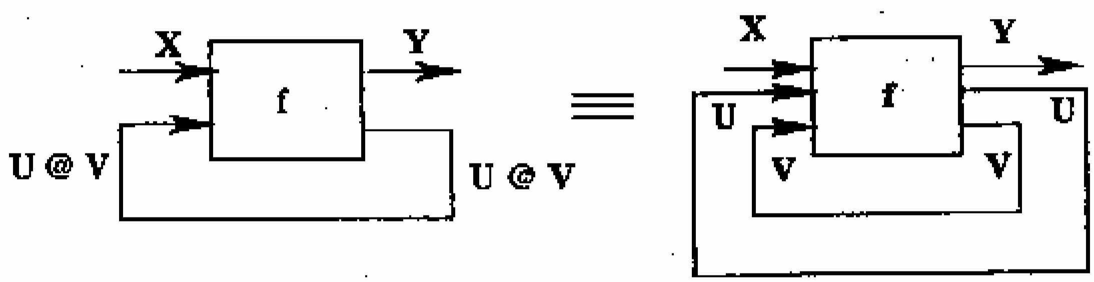


Figure 8: Vanishing II, $U@V$ denotes the simultaneous feedback on the lines U and V

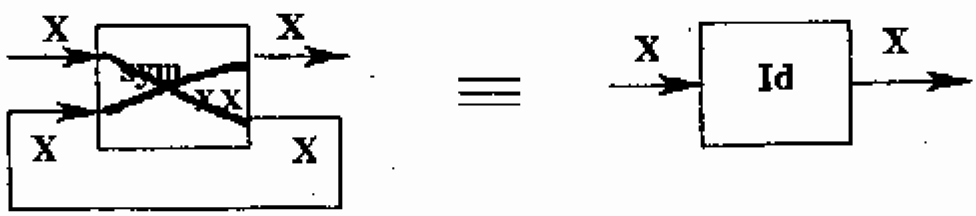


Figure 9: Yanking

Execution / Trace Formula in UDC's

Traced UDC's : Let $\mathcal{C}$ be a U.D.C. If for every

$$f : X \otimes U \longrightarrow Y \otimes U$$

the sum

$$f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21} \quad \text{exists}$$

then
- $\mathcal{C}$ is traced
- $Tr^{U}_{X,Y}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21}$

E.g. $X \otimes U \xrightarrow{f} Y \otimes U \quad = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$

$$Tr^{U}_{X,Y}(f) = Tr \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$$

$$= g + \sum_n 0 h^n 0 = g + 0 = g$$

E.g.'s : Traced UDC's

- PAC's :

  Rel_+ : $\sum_i R_i = \bigcup_i R_i$

  Pf^n : $\{f_i\}$ summable

  $\iff$ prwise disj. domains

  $$\left(\sum_i f_i\right)(x) = f_j(x) \qquad x \in Dom(f_j)$$
  $$= \perp \qquad \text{else}$$

  sRel

- PInj : as for Pf^n but prwise disjoint domains & codomains.

Building a UDC from Hilbert Spaces

Let **Hilb** = the category of Hilbert spaces and linear contractions (norm ≤ 1). M. Barr defined a contravariant faithful functor $\ell_2 : \mathbf{Pinj}^{op} \rightarrow \mathbf{Hilb}$ as follows:

On Objects: $X \mapsto \ell_2(X)$ = set of all complex valued functions a on X for which the (unordered) sum $\sum_{x\in X} |a(x)|^2$ is finite.

$\ell_2(X)$ is a Hilbert space with

- $||a|| = (\sum_{x\in X} |a(x)|^2)^{1/2}$
- Inner product $< a, b >= \sum_{x\in X} a(x)\overline{b(x)}$ for $a, b \in \ell_2(X)$.

On Maps Given $f : X \longrightarrow Y$ in **PInj**,

$\ell_2(f) : \ell_2(Y) \longrightarrow \ell_2(X)$ is defined by

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)), & \text{if } x \in Dom(f); \\ 0, & \text{otherwise.} \end{cases}$$

So we get a correspondence

$\boxed{\text{partial inj. functions}} \leftrightarrow \boxed{\text{partial isometries in } \mathbf{Hilb}}$.

Various isomorphisms:

$$\ell_2(X \uplus Y) \cong \ell_2(X) \oplus \ell_2(Y)$$

$$\ell_2(X \times Y) \cong \ell_2(X) \otimes \ell_2(Y)$$

Define **Hilb**$_2$ $= \ell_2$[**PInj**]. More precisely:

define the subcategory **Hilb**$_2$ of **Hilb**:

<u>Objects</u> $\ell_2(X)$ for a set X

<u>Morphisms</u> $u : \ell_2(X) \longrightarrow \ell_2(Y)$ of the form $\ell_2(f)$ for some $f : Y \longrightarrow X \in$ **PInj**.

There are two $\otimes$-structures on **Hilb**$_2$ induced from tensors on **PInj**

$$\ell_2(X) \otimes \ell_2(Y) \cong \ell_2(X \times Y)$$

$$\ell_2(X) \oplus \ell_2(Y) \cong \ell_2(X \uplus Y) \qquad (\longleftarrow \text{UDC tensor})$$

Note: $\ell_2(X) \oplus \ell_2(Y)$ is direct sum (biproduct) in **Hilb** but only a tensor product in **Hilb**$_2$, (otherwise $X \uplus Y$ would be coproduct in **PInj**, a contradiction.)

Hilb$_2$ is a traced UDC (with UDC structure induced from **PInj**)

$\oplus$ is the tensor product, with unit $\ell_2(\emptyset)$.

Consider a family $\{\ell_2(f_i)\}_I \in$**Hilb**$_2(\ell_2(X), \ell_2(Y))$ with $\{f_i\}_I \in$**PInj**(Y, X)

Define: $\{\ell_2(f_i)\}$ is *summable* if $\{f_i\}$ is summable in **PInj** and in that case $\sum_i \ell_2(f_i) =_{def} \ell_2(\sum_i f_i)$.

Clearly, ℓ_2 is an additive functor.

Quasi injections and projections are the ℓ_2 images of quasi projections and injections in **PInj**.

Hilb$_2$ is traced. Given

$$u : \ell_2(X) \oplus \ell_2(U) \longrightarrow \ell_2(Y) \oplus \ell_2(U)$$

$$Tr(u) =_{def} \ell_2(Tr^U_{Y,X}(f))$$

where $u = \ell_2(f)$ with $f : Y \uplus U \longrightarrow X \uplus U \in$ **PInj**.

PInj and **Hilb$_2$** form GoI situations.

GoI Situations:

1. $TT \lhd T$ (e, e') (Comult.)

$$TTU \underset{e_U}{\overset{e'_U}{\leftrightarrows}} TU$$

2. $Id \lhd T$ (d, d') (Dereliction)

$$U \underset{d_U}{\overset{d'_U}{\leftrightarrows}} TU$$

3. $T \otimes T \lhd T$ (c, c') (Contraction)

$$TU \otimes TU \underset{c_U}{\overset{c'_U}{\leftrightarrows}} TU$$

4. $\mathcal{K}_I \lhd T$ (w, w') (Weakening).

$$I \underset{w_U}{\overset{w'_U}{\leftrightarrows}} TU$$

5. $U \in \mathcal{C}$, *a reflexive object,*

   (a) $U \otimes U \lhd U$ (j, k)

   $$U \otimes U \underset{j}{\overset{k}{\leftrightarrows}} U$$

   (b) $I \lhd U$

   (c) $TU \lhd U$ (u, v).

   $$TU \underset{u}{\overset{v}{\leftrightarrows}} U$$

PInj is GoI situation

$U = \mathbb{N}$, $T(\_) = \mathbb{N} \times (\_)$

- T is additive, mon. functor
- $\mathbb{N} \cup \mathbb{N} \underset{j}{\overset{k}{\rightleftarrows}} \mathbb{N}$ $\quad j(1,n) = 2n$, $\quad j(2,n) = 2n+1$

$$k(n) = \begin{cases} (1, n/2) & n \text{ even} \\ (2, \frac{n-1}{2}) & n \text{ odd} \end{cases}$$

- $T(\mathbb{N}) \lhd \mathbb{N}$ i.e. $\mathbb{N} \times \mathbb{N} \lhd \mathbb{N}$ (Cantor)

etc. (retracts are monoidal...)

$\ell_2[\mathrm{PInj}]$ is also a GoI situation

SRel $\left(\begin{array}{ll} \text{Lawvere} & \text{'62} \\ \text{Giry} & \text{'81} \end{array} \right)$

Stochastic Rel$^{\text{n}}$s

<u>Objects</u> : (X, Σ_X) sets w/ σ-alg

<u>arrows</u> : Stochastic Kernels

$f : X \times \Sigma_Y \longrightarrow [0,1]$

(i) $\forall B \in \Sigma_Y$, $f(-, B)$ measurable

(ii) $\forall x \in X$, $f(x, -) : \Sigma_Y \to [0,1]$

is subprobability meas. $f(x, Y) \leq 1$

composition : $(g \circ f)(x, C) = \int_Y g(y, C)\, d\{f(x, \cdot)\}$

Forms PAC, $U = \mathbb{N}^{\mathbb{N}}$ (Baire Space)

$T(X, \Sigma_X) = (\mathbb{N} \times X, \Sigma_{\mathbb{N} \times X})$.

Matrix Notations

$$A \otimes B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

e.g. if $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

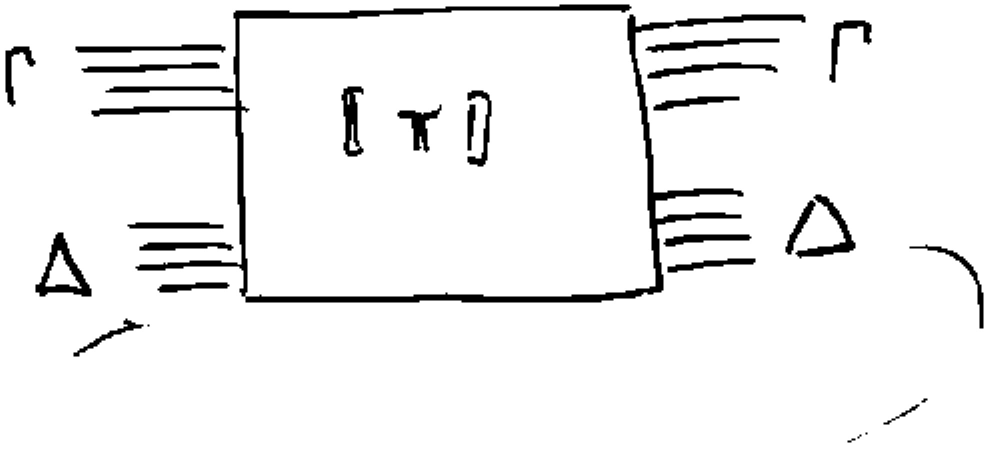
$\sigma = S \otimes \cdots \otimes S$ (m times)

is

$$= \begin{bmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \ddots & \\ 0 & & & & S \end{bmatrix}_{2m \times 2m}$$

$$\left[\!\left[\begin{array}{c} \vdots\ \pi \\ \vdash \underbrace{[\Delta]}_{2m}, \underbrace{\Gamma}_{n} \end{array} \right]\!\right] : U^{n+2m} \longrightarrow U^{n+2m} \quad \&$$

where $\sigma : U^{2m} \to U^{2m} = s \otimes \cdots \otimes s$ (m-times)

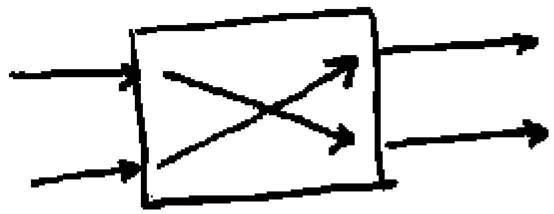


<u>Axiom</u> : $\vdash A, A^{\perp}$ $\quad m=0,\ n=2$

$$[\![Ax]\!] : U \otimes U \xrightarrow{\text{twist}} U \otimes U$$

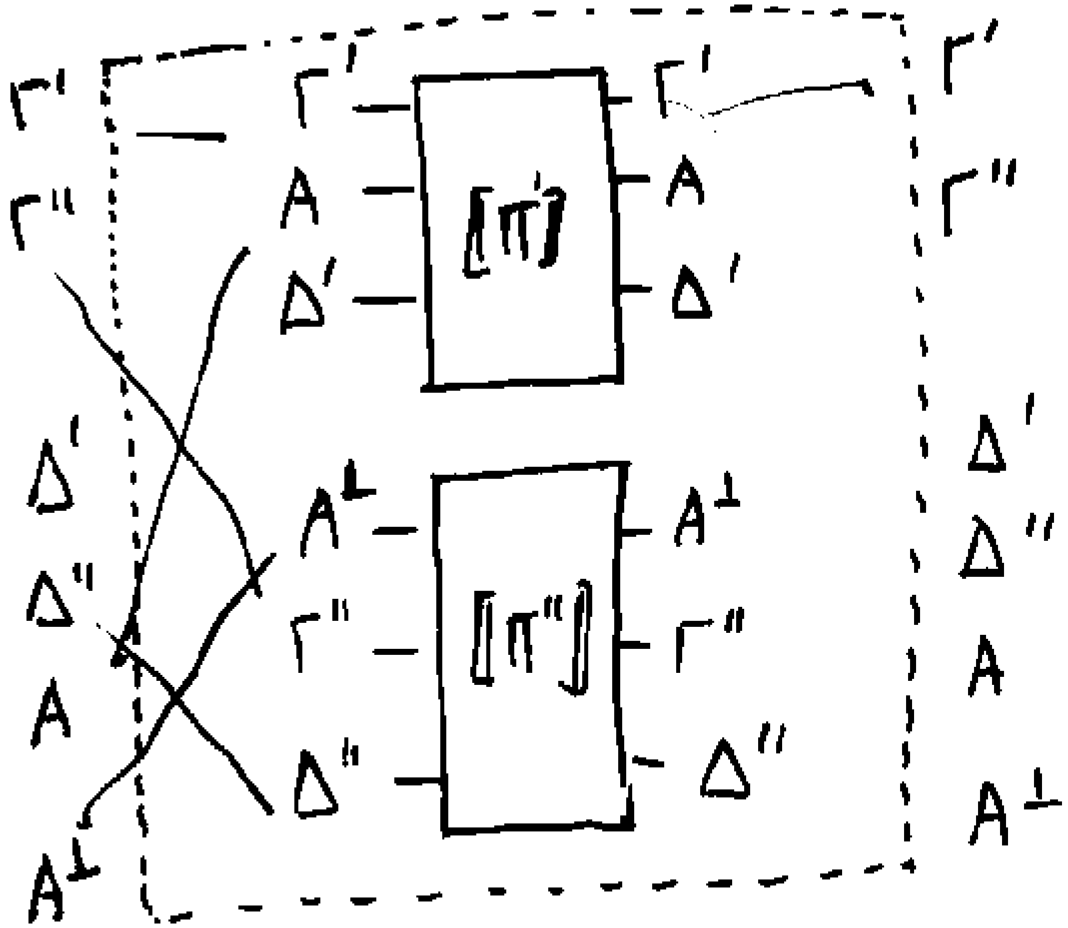
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s$$

=



Tensor/Cut interpretat$^{\text{n}}$:

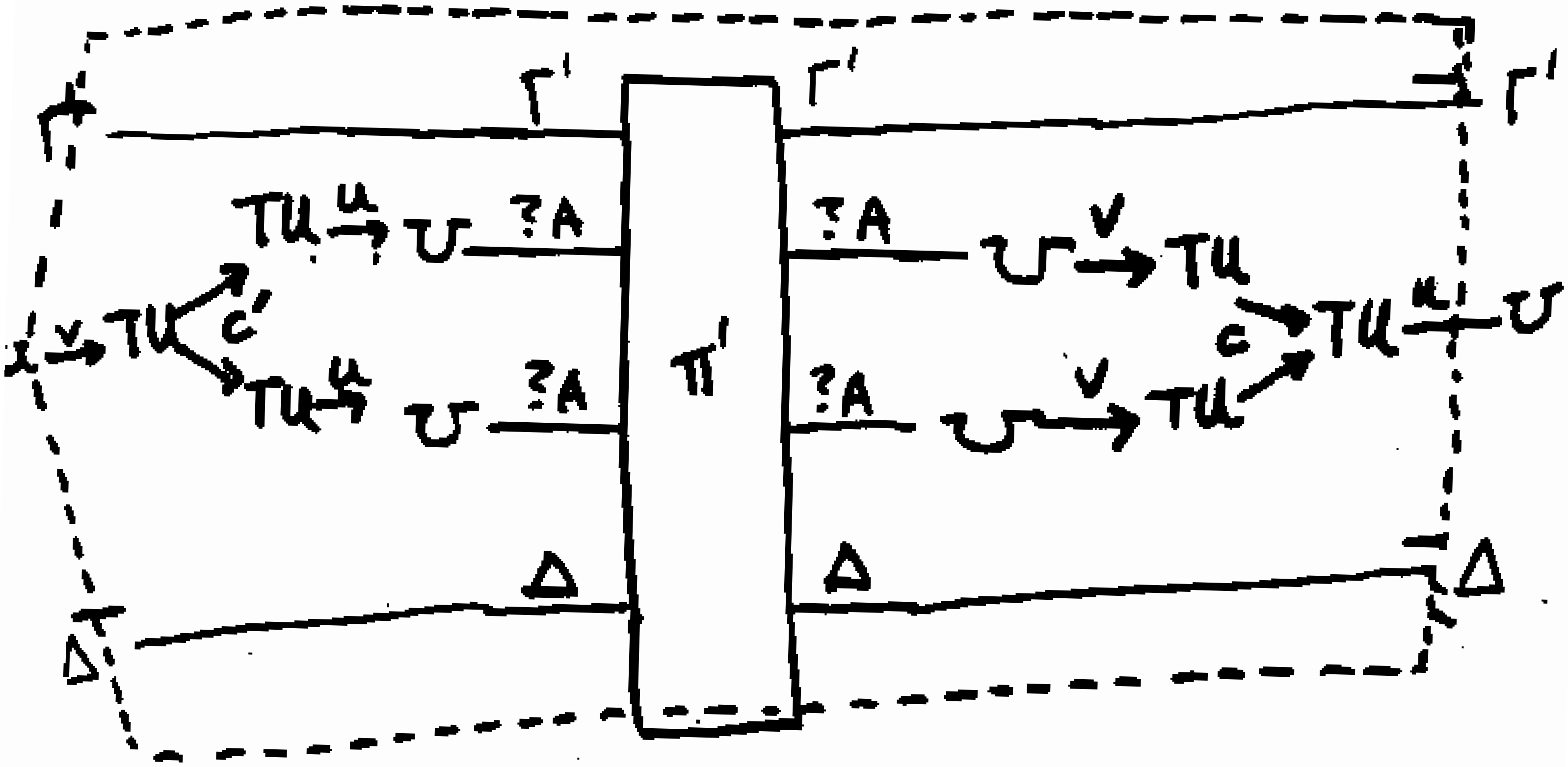
$$
\dfrac{\begin{matrix} \Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A \end{matrix} \qquad \begin{matrix} \Pi'' \\ \vdots \\ \vdash [\Delta''], A^\perp, \Gamma'' \end{matrix}}{\vdash [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma''}\ \text{cut}
$$



$$\tau^{-1}\big([\![\Pi']\!] \otimes [\![\Pi'']\!] \big)\, \tau \;=\; [\![\text{cut}]\!]$$

Contraction

$$\dfrac{\begin{matrix}\vdots\ \pi' \\ \vdash [\Delta], \Gamma', ?A, ?A\end{matrix}}{\pi:\ \vdash [\Delta], \Gamma', ?A}$$



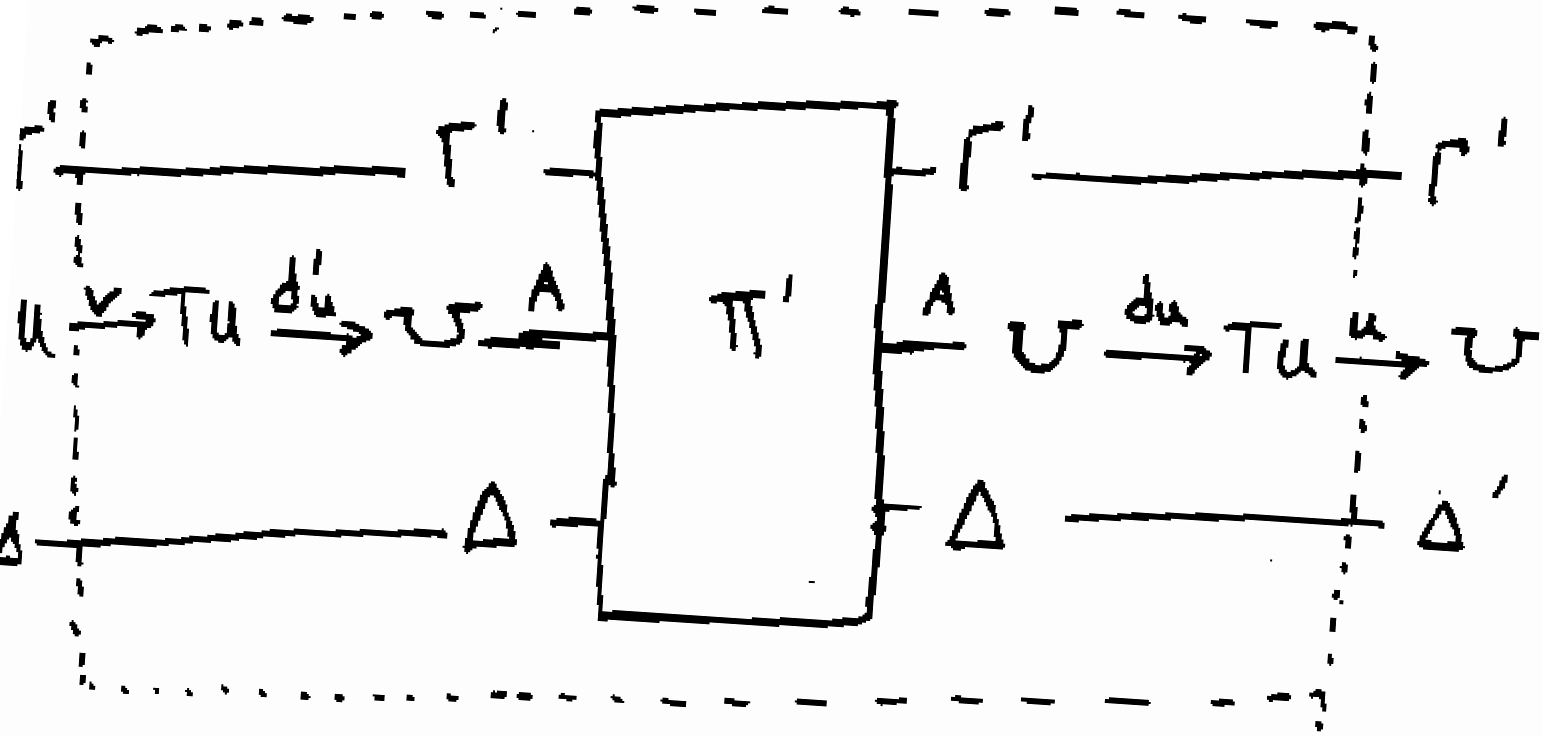
$$[\![\pi]\!] = \left(1_{\Gamma'} \otimes \left(u \cdot (c_U \cdot v \otimes v)\right) \otimes 1_\Delta\right) \circ$$

$$[\![\pi']\!] \cdot \left(1_{\Gamma'} \otimes (u \otimes u)\, c'_U\, v \otimes 1_\Delta\right)$$

<u>idea</u>: $?A : \mho$ "really" is $?A : TU$

Dereliction

$$\dfrac{\begin{array}{c}\vdots\ \pi' \\ \vdash [\Delta], \Gamma', A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{dereliction}$$



$$[\![\pi]\!] = \left(1_{\Gamma'} \otimes u\, d_u \otimes 1_{\Delta'}\right) \cdot [\![\pi']\!] \cdot \left(1_{\Gamma'} \otimes d'_u v \otimes 1_{\Delta}\right)$$

Example of GoI Semantics

Let Π be the following proof:

$$\frac{\vdash A^{\perp}, A \qquad \vdash A^{\perp}, A}{\vdash [A, A^{\perp}], A^{\perp}, A}\,(cut)$$

Then the GoI semantics of this proof is given by

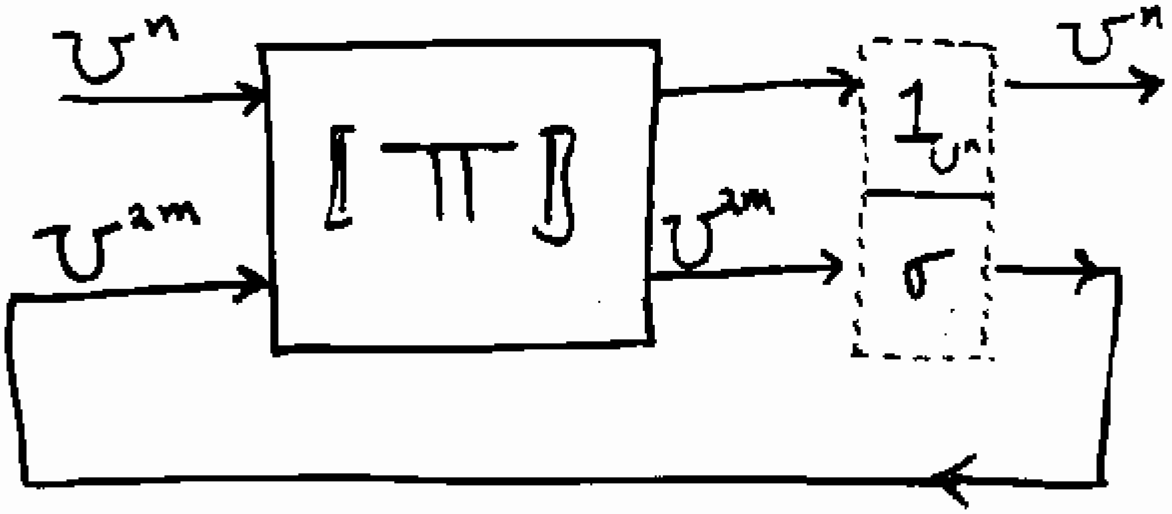
$$[\Pi] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{Id} \\ \hline \mathbf{Id} & \mathbf{0} \end{array}\right]_{4\times 4}$$

Here $m = 1$ & $\sigma = S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Dynamics

Let π be a proof of $\vdash [\Delta], \Gamma$. Consider



Execution Formula

$$\mathrm{Ex}([\![\pi]\!], \sigma) =_{\mathrm{def}} \mathrm{Tr}\left((1_{U^n} \otimes \sigma)[\![\pi]\!]\right)$$

$$= \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma\pi_{22})^n(\sigma\pi_{21})$$

in any traced UDC

Example

Recall

$$\dfrac{\dfrac{\vdots\ \pi}{\vdash A^\perp, A \qquad \vdash A^\perp, A}}{\vdash [A, A^\perp], A^\perp, A}\ \text{cut}$$

$$[\![\pi]\!] = \left[\begin{array}{c|c} 0_2 & Id_2 \\ \hline Id_2 & 0_2 \end{array}\right]_{4\times 4} \quad ; \quad \sigma = s$$

$$Ex([\![\pi]\!], \sigma) \underset{\text{def}}{=} Tr((1 \otimes \sigma)[\![\pi]\!])$$

$$= Tr\left(\left[\begin{array}{c|c} Id & 0_2 \\ \hline 0_2 & s \end{array}\right] \begin{bmatrix} 0_2 & Id_2 \\ Id_2 & 0_2 \end{bmatrix} \right)$$

$$= \cdots = \pi_{11} + \sum_{n \geq 0} \pi_{12} (\sigma \pi_{22})^n (\sigma \pi_{21})$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{n \geq 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [\![\vdash A^\perp, A]\!] = [\![\text{n.f. of } \pi]\!]$$

The <u>Main Idea</u>: Cut-Elim. is computation, so $[\![\Pi]\!]$ should be given by an algorithm:

- Run $Ex([\![\Pi]\!], \sigma)$. It should terminate in finitely many steps.

- It terminates in a <u>datum</u> i.e. a cut-free proof.

<u>Lemma</u> (Associativity of Cut)
(Gives <u>Soundness</u>, i.e. Church-Rosser)

$$\left.\begin{array}{l} Ex([\![\Pi]\!], \sigma \otimes \tau) = \\ \qquad Ex(Ex([\![\Pi]\!], \tau), \sigma) \end{array}\right\} \begin{array}{l}\text{where}\\ \vdash [\Gamma, \Delta],\\ \Lambda\end{array}$$

<u>Pf</u>: Properties of trace.

Types

- Let $f, g \in \mathcal{C}(V, V)$. We say <u>f is orthogonal to g</u> $(f \perp g)$ if gf is nilpotent.

(Recall $0_{VV} \in \mathcal{C}(V,V)$ $\therefore$ $\perp$ makes sense & is symmetric)

Let $X \subseteq \mathcal{C}(V,V)$.

$$X^{\perp} = \{ f \in \mathcal{C}(V,V) \mid f \perp X \}$$

(where $f \perp X \underset{\text{def}}{\equiv} f \perp g,\ \forall g \in X$)

<u>Type</u> $\underset{\text{def}}{\equiv}$ X s.t. $X = X^{\perp\perp}$

MELL Formulas as Types

$$\text{formula } A \overset{\Theta}{\longmapsto} \Theta A \quad \text{a type}$$

- $\alpha \longmapsto X$
- $\alpha^{\perp} \longmapsto X^{\perp}$
- $B \otimes C \longmapsto Y^{\perp\perp}$

  where $Y = \left\{ j(a \otimes b)k \;\middle|\; \begin{array}{l} a \in \Theta B \\ b \in \Theta C \end{array} \right\}$

  with $U \otimes U \underset{j}{\overset{k}{\rightleftarrows}} U$

- $B \parr C \longmapsto Z^{\perp}$

  where $Z = \left\{ j(a \otimes b)k \;\middle|\; \begin{array}{l} a \in (\Theta B)^{\perp} \\ b \in (\Theta C)^{\perp} \end{array} \right\}$

$$A = {!B} \longmapsto Y^{\perp\perp}$$
$$A = {?B} \longmapsto Z^{\perp}$$

where $Y = \{ u\,T(a)\,v \mid a \in \Theta B \}$

$$\frac{U \xrightarrow{a} U}{U \xrightarrow{v} TU \xrightarrow{Ta} TU \xrightarrow{u} U}$$

and $Z = \{ u\,T(a)\,v \mid a \in (\Theta B)^{\perp} \}$

Data & Algorithms

Let $\Gamma = A_1, \ldots, A_n$. $\theta\Gamma = \theta A_1, \ldots, \theta A_n$.

A <u>datum of type $\theta\Gamma$</u>

$= M : \mathcal{U}^n \longrightarrow \mathcal{U}^n$ s.t.

for any $\beta_i \in \theta(A_i^\perp)$,

$(\beta_1 \otimes \cdots \otimes \beta_n) \perp M$

An <u>algorithm of type $\theta\Gamma$</u>

$= M : \mathcal{U}^{n+2m} \longrightarrow \mathcal{U}^{n+2m}$

(for some m s.t. $\sigma : \mathcal{U}^{2m} \longrightarrow \mathcal{U}^{2m}$)

s.t. $Ex(M, \sigma)$ is finite sum

& datum of type $\theta\Gamma$. Here

$$Ex(M, \sigma) = Tr^{\mathcal{U}^{2m}}_{\mathcal{U}^n, \mathcal{U}^n}((1 \otimes \sigma)M)$$

Examples:

$\Gamma = \alpha \quad , \quad \Theta\Gamma = X$

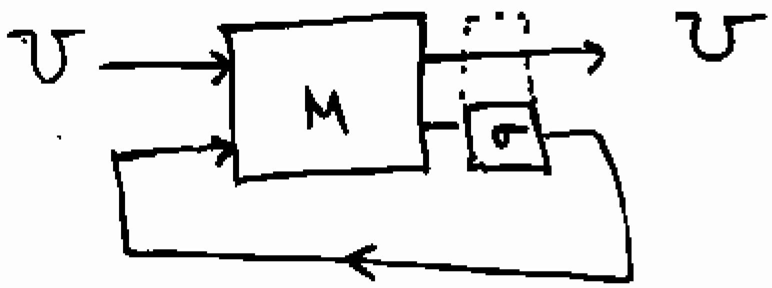
$M : \mathcal{U} \to \mathcal{U}$ is datum of type X

iff $\forall \beta \in X^{\perp}, \quad \beta \perp M$

i.e. $M \in X^{\perp\perp} = X$

$M : \mathcal{U}^{1+2m} \longrightarrow \mathcal{U}^{1+2m}$ is algorithm of type X (for some $\sigma : \mathcal{U}^{2m} \circlearrowleft$)

iff



$= \mathrm{Tr}((1 \otimes \sigma)M)$

is finite & $\in X^{\perp\perp} = X$

Using technical lemmas on nilpotence one obtains:

<u>Characterizat$\underline{n}$ Lemma</u>. Consider

$M: U^n \longrightarrow U^n$, $a: U \to U$.

$M = (m_{ij})_{n\times n}$ is a <u>datum of</u>

type $\Theta(A,\Gamma)$ $\Leftrightarrow$ for any

$a \in \Theta A^{\perp}$, $a m_{11}$ is nilpotent

& $\mathrm{Tr}\left(S^{-1}_{\Gamma,A}(a \otimes id_{n-1}) M S_{\Gamma,A}\right) \in \Theta(\Gamma)$

where $S_{\Gamma,A}: \Gamma \otimes A \longrightarrow A \otimes \Gamma$

<u>Theorem</u> (Girard). Let Π be a proof of $\vdash [\Delta], \Gamma$ in MELL. Then

① $[\![\Pi]\!]$ is an algorithm of type $\Theta\Gamma$; in particular $Ex([\![\Pi]\!], \sigma)$ is a <u>finite</u> sum.

② If $\Pi \succcurlyeq \Pi'$ by Cut-Elim and ? does not occur in Γ then $Ex([\![\Pi]\!], \sigma) = Ex([\![\Pi']\!], \tau)$.

$\therefore$ $Ex([\![\Pi]\!], -)$ is <u>invariant</u> to Cut-Elim.

③ If Π' is n.f. of Π then $Ex(\Pi, \sigma) = Ex(\Pi', 0) = [\![\Pi']\!]$.

④ In $Hilb_2$, we get Girard's original execution formula:

$$Ex([\![\Pi]\!], \sigma) =$$

$$\left((1-\tilde{\sigma}^2) \sum_{n=0}^{\infty} [\![\Pi]\!] \left(\tilde{\sigma}([\![\Pi]\!]) \right)^n (1-\tilde{\sigma}^2) \right)_{n \times n}$$

where $\tilde{\sigma} = 0_n \otimes \sigma = \left(\begin{array}{c|c} 0_n & \\ \hline & \sigma \end{array} \right)_{n+2m}$

and $(A)_{n \times n}$ = the $n \times n$ submatrix of A.

<u>Example of the proof in (1):</u>

By induction on proofs.

<u>Axiom</u> : $\vdash A, A^\perp \quad = \pi$

Show $Ex(⟦\pi⟧, 0) = ⟦\pi⟧$
is a datum of type $\Theta\Gamma$.

$\therefore \forall\ a \in \Theta A^\perp,\ b \in \Theta A,$

$M = ⟦\pi⟧(a \otimes b) = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$ must
be Nil potent.

E.g. If n even

$$M^n = \begin{bmatrix} (ba)^{n/2} & 0 \\ 0 & (ab)^{n/2} \end{bmatrix}$$

But $a \perp b$. $\therefore$ ba, ab are
nilpotent. $\therefore$ so is M.

Denotational Models from GoI

Let $(\mathcal{C}, T, U)$ be a U.D.C.-GoI situation. Define a category $\mathcal{O}(\mathcal{C})$ as follows:

Objects = Types (i.e. Subsets $A \subseteq \mathcal{C}(U, U)$ s.t. $A^{\perp\perp} = A$).

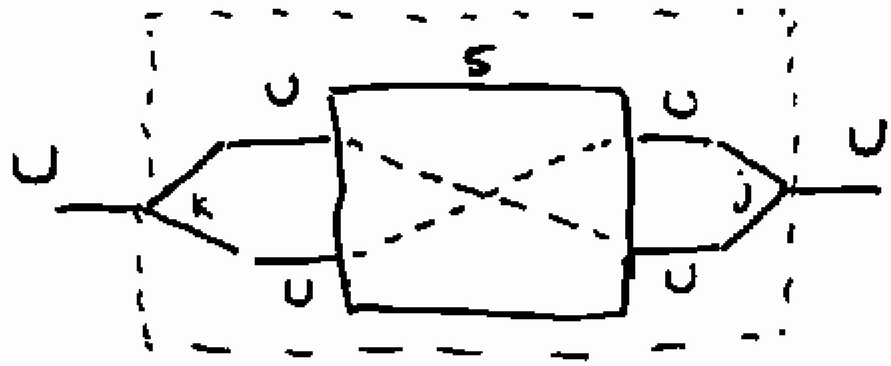
Arrows = $A \xrightarrow{f} B$ is a morphism $f \in \mathcal{C}(U, U)$ s.t. $\forall a \in A,\ f \bullet a \in B$ where

$$f \bullet a = Tr^{U}_{U\,U}\left(s(a \otimes 1)(k f_j) s \right)$$

<u>Motivation</u> : a morphism =
cut-free proof of $\vdash A^\perp, B$
= datum of type $\Theta(A^\perp, B)$.

<u>identity</u> : $j\, s_{uu}\, k$

$$U \xrightarrow{k} U \otimes U \xrightarrow{s} U \otimes U \xrightarrow{j} U$$



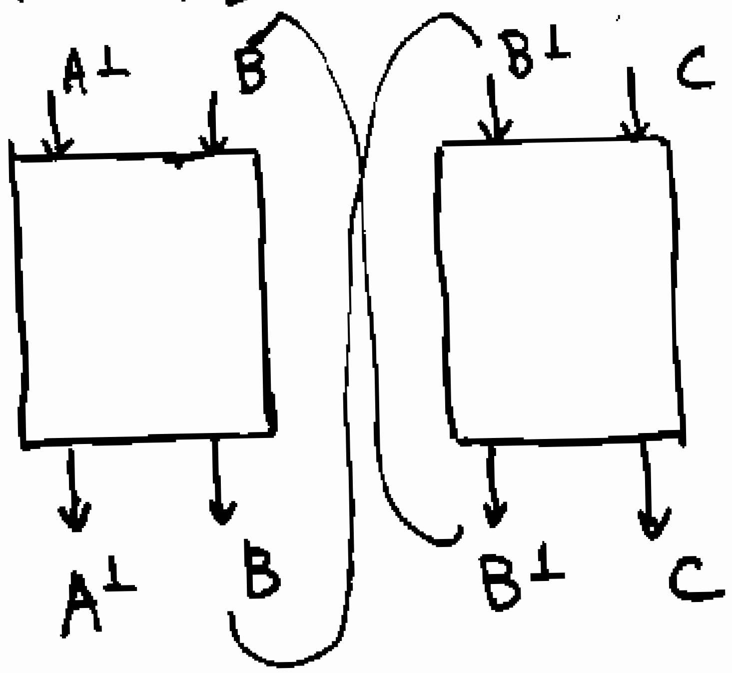
(<u>intuition</u> : cut-free proof of
$\vdash A^\perp, A$)

<u>Composition</u> : consider

$$\frac{\vdash A^\perp, B \qquad \vdash B^\perp, C}{\vdash [B, B^\perp], A^\perp, C}$$ & Run <u>EX</u>.

This gives a cut-free proof of $\vdash A^{\perp}, C$ (i.e. datum of type $\Theta(A^{\perp}, C)$)

In terms of $\mathrm{Int}(\mathcal{C}) \cong \mathcal{G}(\mathcal{C})$



$$g \circ f = j\,\mathrm{Tr}\left[\quad A \quad\right]^{*}_{4\times 4}$$

where A is complicated 4×4 matrix

<u>Thm</u>: Let $(\mathcal{C}, T, U)$ be a GoI situation & suppose $U \otimes U \simeq U$ (j, k). Then $\theta(\mathcal{C})$ is $*$-aut. category without units

Tensor, Par, etc. given by operations on types.

Trouble with units:

$A \lhd A \otimes I$, but not iso.

Denotational Models of MELL

- MLL : $*$-aut. category
  $(\mathcal{C}, \otimes, I, s, (-)^{\perp})$
- Exponentials : $! : \mathcal{C} \to \mathcal{C}$
- monoidal n.t.'s

$der_A : !A \to A$

$\delta_A : !A \to !!A$

$Weak_A : !A \to I$

$Con_A : !A \to !A \otimes !A$

- $(!, der, \delta)$ comonoid
- $(!A, weak_A, con_A)$ cocomm. comon.
- $weak_A$, Con_A : co-alg maps, δ_A comon. map

Thm : $(\mathcal{C}, T, U)$ UDC-GoI situat^{n}. Define

$! : \mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{O}(\mathcal{C})$ by

$!A = \{ u\, T(a)\, v \mid a \in A \}^{\perp\perp}$

$$\frac{U \xrightarrow{a} U}{U \xrightarrow{v} TU \xrightarrow{Ta} TU \xrightarrow{u} U}$$

Suppose
- $U \otimes U \simeq U$, $TU \simeq U$.
- (T, d', e') comonad
- (TA, w'_A, c'_A) comm comon.
- e'_A is map of comm. comonoids
- w'_A, c'_A : maps of coalgs

Then $(\mathcal{O}(\mathcal{C}), !)$ = denotat^{n}l model MELL

GoI / Int Construction $\mathcal{G}(-)$

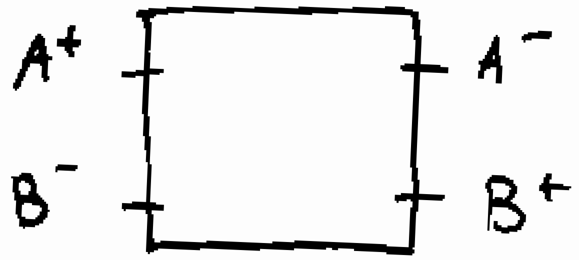
$\mathbb{C}$ = T.M.C.

$\mathcal{G}(\mathbb{C})$: <u>objects</u> = (A^+, A^-)

player opponent

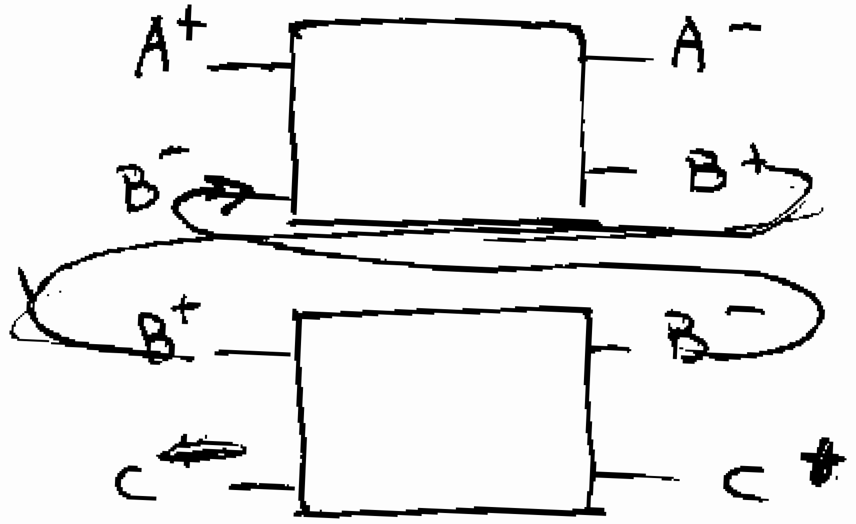
<u>arrows</u>: $(A^+, A^-) \xrightarrow{f} (B^+, B^-)$

=



= $A^+ \otimes B^- \longrightarrow A^- \otimes B^+$

<u>Comp^n</u> : Symmetric feedback



<u>Thm</u> (JSV/Ab.) If $\mathbb{C}$ is traced symm. monoidal cat, $\mathcal{G}(\mathbb{C})$ is compact closed

(2-categorically: Compact-closure of $\mathbb{C}$)

Let Gl = double gluing

<u>Prop^n</u>: There is a faithful $(-)^{\perp}$-preserving embedding
$F : \mathcal{O}(\mathbb{C}) \to Gl(\mathcal{G}\mathbb{C})$

Future Directions

- GoI 2: Non-converging algms (untyped λ-calc / PCF)
  - uses more topological info on operator algs
- GoI 3: Uses additives & additive proof nets —
- GoI 4 (last month): von Neumann algebras: $Ex(f, \tau)$ for f arb (not necessarily coming from proof)
- Quantum GoI ?