Coordinatizing some Concrete MV Algebras and a Decomposition Theorem

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Abstract

MV algebras are the Lindenbaum-Tarski algebras of Łukasiewicz many-valued logics. From work of D. Mundici, countable such algebras correspond to certain AF C*-algebras. M. Lawson and P. Scott gave a coordinatization theorem for them, representing any countable MV-algebra as the lattice of principal ideals of an AF Boolean inverse monoid. In this note, we give two concrete examples of such a coordinatization, one for $\mathbb{Q} \cap [0, 1]$ and another for the so-called Chang algebra.

1 Introduction

Many-valued logics were introduced by Łukasiewicz in the 1920s and studied by the Polish school of logicians (e.g. Łukasiewicz, Tarski, et. al) during that period. The algebras corresponding to these logics, MV algebras, were introduced by the logician C.C. Chang in the 1950s [Cha58]. Chang's completeness theorems showed that the (equational) variety of MV algebras is generated by the interval $[0, 1]$. For a modern treatment of MV algebras, see the book [CDM00].

Major advances in the subject of MV algebras began in the 1980's (and continue today) with the work of D. Mundici and his school. Starting with the seminal paper [Mun86], this work sets up surprising categorical correspondences between the category of countable MV algebras and the category of AF C*-algebras whose K_0 groups are lattice-ordered. This work has shown MV algebras to have deep connections with several areas of contemporary mathematics, quantum physics, and theoretical computer science [Mun13, Mun93].

Recently, Lawson and Scott [LS17] introduced a coordinatization program for countable MV algebras, based on inverse semigroup theory. Such results are inspired (in spirit) by von Neumann's Continuous Geometry [vN98]; the latter represents a modular lattice (of subspaces of a given space) as a lattice of principal ideals of some (von Neumann) regular ring. In the case of MV algebras, Lawson and Scott show that every countable MV algebra arises as the lattice of principal ideals of a special class of Boolean inverse monoids. These monoids, called AF inverse

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monoids, are the inverse monoid analogs of AF C*-algebras. The paper [LS17] gives two examples of coordinatizations: (a) the MV algebra of dyadic rationals in $[0,1]$ (which correspond, via Mundici's work, to the AF C*-algebra for the CAR algebra of a Fermi gas) and (b) finite MV algebras. In this paper, based on the first author's MSc thesis, we will look at the coordinatization problem for two important cases: (i) the rationals in $[0,1]$ (corresponding, by Mundici [Mun93], to Glimm's UHF Algebra), and (ii) Chang's algebra (corresponding, by Mundici, loc. cit, to the AF Behncke-Leptin algebra $A_{0,1}$.)

2 Preliminaries

We recall some definitions and results about MV algebras and boolean inverse semigroups needed for the main work of this paper. We will also need to mention effect algebras, as our approach to coordinatization involves viewing MV algebras as MV-effect algebras. Briefly, effect algebras (as defined below) were discovered by mathematical physicists in the 1990's as part of quantum measurement theory. They were independently and intensively studied for their connections with dimension groups and operator algebras [DP00]. It was later realized that certain lattice-ordered effect algebras were closely related to MV algebras, as discussed below. We remark that the original coordinatization paper of Lawson-Scott [LS17] also transits through the larger category of effect algebras.

Details of omitted proofs and/or references for results in this section can be found in the first author's thesis, [Lu16]. For a detailed treatment of MV algebras, we recommend [CDM00], for inverse semigroups, we recommend [Law98], and for effect algebras see [DP00, Jac14].

Definition 2.1 (MV algebra)**.** An *MV algebra* is a quadruple $(A, \oplus, \neg, 0)$ wherein A is a set, $\oplus\colon A \times A \to A$ is a (total) binary operation, $\neg\colon A \to A$ is a (total) unary operation, and $0 \in A$ is a distinguished element, such that the following hold for all $a, b, c \in A$:

MV 1. *Associativity*: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

MV 2. *Commutativity*: $a \oplus b = b \oplus a$.

MV 3. *Zero law*: $a \oplus 0 = a$.

MV 4. *Involution*: $\neg\neg a = a$.

MV 5. *Absorption law*: $a \oplus \neg 0 = \neg 0$.

MV 6. *Lukasiewicz axiom*: $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

Example 2.2. The unit interval $[0,1]$ with $\oplus$ as the cutoff addition: for $a, b \in [0,1]$, define $a \oplus b = \min(1, a+b)$, and $\neg a = 1 - a$, is an MV algebra. Another example is to take an ℓ-group G, i.e. a lattice-ordered abelian group with Archimedean order unit u. The poset "interval" $[0,u]_G = \{x \in G \mid 0 \leq x \leq u\}$ is an MV-algebra, in which $a \oplus b = u \wedge (a+b)$ and $\neg a = u - a$.

Definition 2.3 (Effect algebra)**.** An *effect algebra* is a partial algebra, defined as a quadruple $(E, \widetilde{\oplus}, (-)^{\perp}, 0)$ wherein $(E, \widetilde{\oplus}, 0)$ is a partial commutative monoid and $(-)^{\perp}\colon E \to E$ is a (total) unary operation called *orthocomplement* (we write $0^{\perp} = 1$, call 0 the *zero* of E, and call 1 the *unit* of E), such that the following hold for all $a \in E$:

EA 1. *Orthocomplement law*: $a^\perp$ is the unique element satisfying $a \mathbin{\widetilde{\oplus}} a^\perp = 1$.

EA 2. *Zero-one law*: if $a \mathbin{\widetilde{\oplus}} 1 \downarrow$, then $a = 0$.

Note that in an effect algebra, $\widetilde{\oplus}$ is only a *partial* binary operation. We write "$a \mathbin{\widetilde{\oplus}} b \downarrow$" to mean "$a \mathbin{\widetilde{\oplus}} b$ is defined" and "$a \mathbin{\widetilde{\oplus}} b \uparrow$" to mean "$a \mathbin{\widetilde{\oplus}} b$ is undefined".

Example 2.4. The closed interval $[0, 1] \subseteq \mathbb{R}$, with $\widetilde{\oplus}$ being the usual addition of real numbers and $0, 1$ in their usual roles, is an effect algebra. We have $a \mathbin{\widetilde{\oplus}} b \downarrow$ if and only if $a + b \leq 1$ and the orthocomplement is given by $a^\perp = 1 - a$. Observe that $a \mathbin{\widetilde{\oplus}} b$ is simply undefined if $a + b > 1$ (cf. Example 2.2 above). There are also analogous examples to interval MV algebras in Example 2.2, using partially ordered abelian groups (G, G^+, u) where $u \in G^+$. Namely, define $[0, u]_{G^+} = \{a \in G^+ \mid 0 \leq a \leq u\}$ with partial operation $a \mathbin{\widetilde{\oplus}} b \downarrow$ if and only if $a + b \leq u$. Similarly, $\neg a = u - a$. Such effect algebras $[0, u]_{G^+}$ cover a wide range of examples from quantum effects (see [DP00]).

There is a natural partial order on both MV and effect algebras given by $a \leq c$ if and only if there exists b such that $a \oplus b = c$, or respectively, $a \mathbin{\widetilde{\oplus}} b = c$.

Definition 2.5 (MV-effect algebra)**.** An *MV-effect algebra* is an effect algebra which is lattice ordered and additionally satisfies the *Riesz decomposition property* (that is, the property that for all $a, b_1, b_2 \in E$, if $a \leq b_1 \mathbin{\widetilde{\oplus}} b_2$, then there exist $a_1, a_2 \in E$ such that $a = a_1 \mathbin{\widetilde{\oplus}} a_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$).

Theorem 2.6. *There is a one-to-one correspondence between MV algebras and MV-effect algebras. In particular:*

1. *Given an MV-effect algebra* $(E, \widetilde{\oplus}, (-)^\perp, 0)$*, we can form an MV algebra* $\mathcal{A}(E) = (E, \oplus, \neg, 0)$ *with* $\neg a = a^\perp$ *and*
$$a \oplus b = a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b).$$
2. *Given an MV algebra* $(A, \oplus, \neg, 0)$*, we can form an MV-effect algebra* $\mathcal{E}(A) = (A, \widetilde{\oplus}, (-)^\perp, 0)$ *with* $a^\perp = \neg a$*, and*
$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \oplus b & \text{if } a \leq \neg b, \\ \uparrow & \text{otherwise.} \end{cases}$$

Proof. See [Lu16, Chapter 1.4]. □

Definition 2.7 (Homomorphism (of effect algebras))**.** Let E, F be effect algebras. A function $f : E \to F$ is called an *effect algebra homomorphism* if it satisfies the following.

EH 1. $f(1_E) = 1_F$.

EH 2. If $a, b \in E$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$ and $f(a \mathbin{\widetilde{\oplus}} b) = f(a) \mathbin{\widetilde{\oplus}} f(b)$.

We denote by **EA** the *category of effect algebras*, with objects effect algebras and arrows effect algebra homomorphisms.

Definition 2.8 (Homomorphism (of MV algebras))**.** Let A, B be MV algebras. A function $f : A \to B$ is called an *MV algebra homomorphism* if it satisfies the following for all $a, b \in A$.

MH 1. $f(0_A) = 0_B$.

MH 2. $f(a \oplus b) = f(a) \oplus f(b)$.

MH 3. $f(\neg a) = \neg f(a)$.

We denote by **MV** the *category of MV algebras*, with objects MV algebras and arrows MV algebra homomorphisms.

We wish to view MV algebras as MV-effect algebras for the calculations to follow, but one must be careful about the morphisms. From [Lu16, Propositions 1.1.16 & 1.2.11], there are infinitely many effect algebra maps from $[0,1]^2$ to $[0,1]$, but only two MV algebra maps (the two projections). The issue is that the preservation condition for effect algebra maps only applies to summands that are defined. Thus, the full subcategory of **EA** consisting of all MV-effect algebras is not equivalent or isomorphic to **MV**. What we need is the following restriction.

Definition 2.9 (MV-effect homomorphism)**.** Let E, F be MV-effect algebras. Then, an effect algebra homomorphism $f : E \to F$ is called an *MV-effect homomorphism* if it additionally preserves the lattice operations $\wedge$ and $\vee$.

We denote the *category of MV-effect algebras*, with objects MV-effect algebras and arrows MV-effect homomorphisms, by **MVEA**.

Theorem 2.10. *Extending the maps $\mathcal{E}$ and $\mathcal{A}$ in the statement of Theorem 2.6 to morphisms by sending each MV algebra homomorphism (respectively, each MV-effect homomorphism) to the same underlying function yields an isomorphism of categories* $\mathbf{MV} \cong \mathbf{MVEA}$.

Proof. See [Lu16, Chapter 1.4]. □

Definition 2.11 (Inverse semigroup)**.** An *inverse semigroup* is a pair $(S, *)$ consisting of a set S and a binary operation $* : S \times S \to S$ satisfying (writing simply xy for $x * y$):

IS 1. *Associativity*: for all $x, y, z \in S$, $(xy)z = x(yz)$.

IS 2. *Pseudoinverse*: for all $x \in S$, there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

For an inverse semigroup S, we write $E(S) = \{x \in S \mid x \text{ is idempotent}\}$, and for $x, y \in S$, we define $x \le y$ to mean there exists $e \in E(S)$ such that $x = ye$. We note that for any element $x \in S$, xx^{-1} and $x^{-1}x$ are both idempotents. Moreover, in an inverse semigroup, idempotents commute.

Example 2.12. Let X be a set. Then the *partial bijections* on X (that is, partially defined functions $X \to X$ which are injective), $\mathcal{I}(X)$, is an inverse semigroup called the *symmetric inverse monoid* on X. Given $f, g \in \mathcal{I}(X)$, we have $gf = g \circ f$, where $\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$, and when $x \in \operatorname{dom}(g \circ f)$, then $(g \circ f)(x) = g(f(x))$.

When X is a finite set of n elements, we write $\mathcal{I}(X)$ as $\mathcal{I}_n$. We call the elements of X *letters*. The idempotents of $\mathcal{I}(X)$ are the partial identity maps; indeed, if $i \in \mathcal{I}(X)$ is idempotent, then $i^3 = i$, from which uniqueness of pseudoinverses forces $i^{-1} = i$, so $i = i^2 = i \circ i^{-1} = \mathrm{id}_{\operatorname{dom} i}$.

Definition 2.13 (Compatible, orthogonal elements)**.** Let S be an inverse semigroup and $a, b \in S$. We say that a and b are *compatible*, and write $a \sim b$, to mean that ab^{-1} and $a^{-1}b$ are both idempotents.

If S has a zero (that is, an element 0 such that $0x = x0 = 0$ for all $x \in S$), we say that a and b are *orthogonal*, and write $a \perp b$, to mean that $ab^{-1} = 0 = ab^{-1}$.

For example, in $\mathcal{I}(X)$, 0 denotes the empty partial function. Note that for $f, g \in \mathcal{I}(X)$, $f \sim g$ iff $f \cup g$ is again in $\mathcal{I}(X)$. Also $f \perp g$ iff f and g have disjoint domains and ranges.

Definition 2.14 ((Distributive, boolean) inverse monoid)**.** An *inverse monoid* is an inverse semigroup S together with an element $1 \in S$ such that $1x = x = x1$ for all $x \in S$. Note that it is *not* necessary for xx^{-1} to equal 1.

An inverse monoid with zero is *distributive* if the following hold.

DIM 1. $E(S)$ is a distributive lattice.

DIM 2. For all $a, b \in S$, if $a \sim b$, then $a \vee b$ exists.

DIM 3. Multiplication distributes over binary joins; for all $a, b, c \in S$ such that $b \vee c$ exists, we have $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$.

A distributive inverse monoid S where $E(S)$ is a boolean algebra is a *boolean inverse monoid*. If binary meets also always exist, then S is a *boolean inverse $\wedge$-monoid*.

Definition 2.15 (Factorizable inverse monoid)**.** Let S be an inverse monoid. We say that S is *factorizable* if every element is beneath an element in the group of units (i.e. for all $x \in S$, there is $y \in S$ satisfying $yy^{-1} = 1 = y^{-1}y$, and $x \leq y$).

Definition 2.16 (Foulis monoid)**.** A *Foulis monoid* is a factorizable boolean inverse monoid.

Example 2.17. Symmetric inverse monoids are boolean inverse $\wedge$-monoids. The finite ones are also factorizable. So, finite symmetric inverse monoids are Foulis monoids.

Definition 2.18 (Green's relations)**.** We now define *Green's relations* $\mathcal{D}$ and $\mathcal{J}$ for an inverse semigroup S as follows. We say that, for $a, b \in S$, that $a\mathcal{J}b$ if the two-sided ideals SaS and SbS are equal.

We define $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$ and call them the domain and range of a, respectively.

We say that $a\mathcal{D}b$ if there exists $c \in S$ such that $a = \mathbf{d}(c)$ and $b = \mathbf{r}(c)$, and write $a \xrightarrow{c} b$. Note that if $a\mathcal{D}b$, then $a = c^{-1}c$ and $b = cc^{-1}$ so that both a and b are idempotent.

We wish to extend the relation $\mathcal{D}$ to elements which are not necessarily idempotent. Define $a\mathcal{D}'b$ if $\mathbf{d}(a)\mathcal{D}\mathbf{d}(b)$. In the case of idempotent elements, $\mathcal{D}$ and $\mathcal{D}'$ coincide.

Proposition 2.19. *If S is a Foulis monoid, then the relations $\mathcal{D}$ and $\mathcal{J}$ coincide. We can regard S as an effect algebra by defining*

$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \vee b, & \text{if } a \perp b, \\ \uparrow, & \text{otherwise.} \end{cases}$$

The quotient $S/\mathcal{D}$ equipped with the induced operation on equivalence classes is an effect algebra satisfying the refinement property. So, if additionally, $S/\mathcal{D}$ is a lattice, then $S/\mathcal{D}$ is an MV effect algebra.

Proof. See [LS17, Sec. 2.3]. □

Definition 2.20 (Coordinatizable)**.** An MV algebra A is called *coordinatizable* if there is a Foulis monoid S such that $S/\mathcal{D} = S/\mathcal{J} \cong A$. We also say that S *coordinatizes* A.

Theorem 2.21 (Coordinatization of MV algebras)**.** *Every countable MV algebra A can be coordinatized. Moreover, S can be taken to be an AF inverse monoid (see [LS17, Section 3] for what this means).*

Proof. The proof of this theorem is the main subject of [LS17]. □

Remark 2.22. According to [Weh15, Theorem 5.2.10], the coordinatization theorem also extends to uncountable MV algebras, with what appears to be a generalization of AF inverse monoids applying at cardinality $\aleph_1$, but not beyond. However, this monograph is not primarily about coordinatization or MV algebras, and uses a large amount of different definitions and technical machinery, so it is difficult to directly compare this work with the work cited here.

3 The coordinatization of $\mathbb{Q} \cap [0, 1]$

In [LS17], some concrete examples of coordinatization are given. The finite subalgebras of $[0, 1]$, the *Łukasiewicz chains*, $Ł_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$, are coordinatized by the symmetric inverse monoids $\mathcal{I}_n$.

Recall that $\mathbb{Q}_{\mathrm{Dyad}}$, the *dyadic rationals*, are rational numbers of the form $\frac{a}{2^b}$. In [LS17, Sec. 5], $\mathbb{Q}_{\mathrm{Dyad}} \cap [0, 1]$ is shown to be coordinatized by a construction called the *dyadic inverse monoid*, which turns out to be isomorphic to the directed colimit of

$$I_1 \xrightarrow{\tau_0} I_2 \xrightarrow{\tau_1} I_4 \xrightarrow{\tau_2} I_8 \xrightarrow{\tau_3} I_{16} \xrightarrow{\tau_4} \dots,$$

where the τ_i are inclusion maps (we will make precise what this means shortly). The idea is that for $f \in \mathcal{I}_{2^l}$, the $\mathcal{D}$-class of idempotents of f is associated to the number $\frac{|\mathrm{dom}(f)|}{2^l}$. We will now generalize this to the coordinatization of all rationals in $[0, 1]$.

Definition 3.1 (Omnidivisional sequence, D-canonical form)**.** A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is *omnidivisional* if it satisfies the following properties.

- For all i, $n_i \mid n_{i+1}$.
- For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.

By the second condition, every rational number can be written with one of the n_i as the denominator. If $\frac{a}{b} \in \mathbb{Q}$ is in lowest terms, then there is c such that $bc = n_i$ and so $\frac{a}{b} = \frac{ac}{n_i}$.

For a fixed omnidivisional sequence D and $q \in \mathbb{Q}$, the smallest i such that $q = \frac{d}{n_i}$ for some $d \in \mathbb{N}$ is called the *D-canonical form* of q.

Example 3.2. The sequence $\{n!\}_{n=1}^{\infty}$ is easily seen to be omnidivisional.

Example 3.3. Let p_i be the i^{th} prime number. Then $\{\Pi_{i=1}^{n} p_i^{n-i+1}\}_{n=1}^{\infty}$ is an omnidivisional sequence. The first few members of the sequence are $2, 2^2 3, 2^3 3^2 5, 2^4 3^3 5^2 7, \ldots$.

Clearly each member of the sequence divides the next, and for $m \in \mathbb{N}$, if one looks at the prime factorization of m, then for sufficiently large k, the k^{th} member of the sequence will contain all prime factors of m, taking multiplicity into account.

We now describe what we mean by an inclusion map $\mathcal{I}_n \xrightarrow{\tau} \mathcal{I}_m$ for $n, m \in \mathbb{N}$ where $n \mid m$. Write $na = m$. We denote the underlying set of $\mathcal{I}_n$ by $X_n = \{x_1, \ldots, x_n\}$ and similarly for $\mathcal{I}_m$. We wish to identify each of the n elements of the X_n with a subset of a elements of X_m in a systematic way as follows.

$$\begin{aligned}
&X_n \ni x_1 \mapsto y_1 := \{x_a, x_{a-1}, \ldots, x_1\} \subseteq X_m \\
&X_n \ni x_2 \mapsto y_2 := \{x_{2a}, x_{2a-1}, \ldots, x_{a+1}\} \subseteq X_m \\
&\qquad\quad \cdots \\
&X_n \ni x_n \mapsto y_n := \{x_{na}, x_{na-1}, \ldots, x_{(n-1)a+1}\} \subseteq X_m.
\end{aligned}$$

If f is a partial bijection on X_n and it maps x_i to x_j, then $\tau(f)$ should be a bijection from y_i to y_j in the obvious way. If $f(x_i) = x_j$, then we denote $f^*(i) = j$. For all $i \in \{1, \ldots, m\}$, we write $i = qa - r$ for some $1 \leq q \leq n$ and $0 \leq r < a$. Then, define

$$(\tau(f))(x_i) = (\tau(f))(x_{aq-r}) = x_{a(f^*(q))-r}.$$

This is clearly a partial bijection on X_m, and $|dom(\tau(f))| = a|dom(f)|$. If we identify f with its representing *rook matrix* (that is, the $n \times n$ matrix $(a_{i,j})$ with entries in $\{0, 1\}$ where $a_{i,j} = 1 \Leftrightarrow f(x_i) = x_j$), then the rook matrix for $\tau(f)$ is simply the $na \times na$ expansion of the matrix for f obtained by replacing all 0s with the $a \times a$ zero matrix and replacing all 1s by the $a \times a$ identity matrix.

Theorem 3.4 (Coordinatization of the rationals)**.** *Let $D = \{n_i\}_{n=1}^{\infty}$ be an omnidivisional sequence. Then, the directed colimit of the sequence*

$$Q\colon \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \ldots,$$

where the τ_i are inclusion maps in the sense described above, coordinatizes $\mathbb{Q} \cap [0, 1]$.

Proof. We denote the directed colimit of Q by Q_∞. We define a map $w\colon Q_\infty/\mathcal{D} \to \mathbb{Q} \cap [0, 1]$ as follows. For $s \in \mathcal{I}_{n_i}$, define

$$w([s]/\mathcal{D}) = \frac{|\operatorname{dom}(s)|}{n_i}.$$

We claim w is well defined on $\mathcal{D}$ classes, and is an isomorphism of MV effect algebras.

First, we will prove that, for $a \in \mathcal{I}_{n_i}$ and $b \in \mathcal{I}_{n_j}$, $[a] = [b]$ in Q_∞ implies $w([a]/\mathcal{D}) = w([b]/\mathcal{D})$. That $[a] = [b]$ implies the existence of $e_{n_k} \in \mathcal{I}_{n_k}$ such that $a \cdot e_{n_k} = b \cdot e_{n_k}$. This means

$$\tau^{n_i}_{n_i \vee n_k}(a)\tau^{n_k}_{n_i \vee n_k}(e_{n_k}) = \tau^{n_j}_{n_j \vee n_k}(b)\tau^{n_k}_{n_j \vee n_k}(e_{n_k}). \tag{3.1}$$

In the case that $n_i = n_j$, (3.1) yields $|\operatorname{dom}(\tau^{n_i}_{n_i \vee n_k}(a))| = |\operatorname{dom}(\tau^{n_i}_{n_i \vee n_k}(b))|$. Let $m \in \mathbb{N}$ such that $n_i m' = n_i \vee n_k$ (this exists because we chose these numbers to be from an omnidivisional sequence). Then, we have $m'|\operatorname{dom}(a)| = m'|\operatorname{dom}(b)|$, so $w([a]/\mathcal{D}) = \frac{|\operatorname{dom}(a)|}{n_i} = \frac{|\operatorname{dom}(b)|}{n_i} = w([b]/\mathcal{D})$.

Now suppose $n_i \neq n_j$. Since the left side of the equation (3.1) takes place in $S_{n_i \vee n_k}$ and the right side takes place in $S_{n_j \vee n_k}$ and these must be the same, we must have $n_i \vee n_k = n_j \vee n_k$, from which it follows that $n_k \geq n_i$ and $n_k \geq n_j$. Then, we have $|\operatorname{dom}(\tau_{n_k}^{n_i}(a))| = |\operatorname{dom}(\tau_{n_k}^{n_j}(b))|$. Pick $m_1, m_2 \in \mathbb{N}$ such that $m_1 n_i = n_k$ and $m_2 n_j = n_k$. Then, $m_1|\operatorname{dom}(a)| = m_2|\operatorname{dom}(b)|$. But $m_1 = \frac{n_k}{n_i}$ and $m_2 = \frac{n_k}{n_j}$, so it follows that $w([a]/\mathcal{D}) = \frac{|\operatorname{dom}(a)|}{n_i} = \frac{|\operatorname{dom}(b)|}{n_j} = w([b]/\mathcal{D})$.

Now we are ready to prove well definedness. If $[s]/\mathcal{D} = [t]/\mathcal{D}$, then there is some a such that $[s^{-1}s] = [a^{-1}a]$ and $[aa^{-1}] = [t^{-1}t]$. But then, by the above observation,

$$w(s) = w(s^{-1}s) = w(a^{-1}a) = w(aa^{-1}) = w(t^{-1}t) = w(t)$$

For injectivity, suppose $w([s]/\mathcal{D}) = w([t]/\mathcal{D})$. Then, if $s \in \mathcal{I}_{n_i}$ and $t \in \mathcal{I}_{n_j}$, and without loss of generality we let $n_i \leq n_j$, we have $\frac{|\operatorname{dom}(s)|}{n_i} = \frac{|\operatorname{dom}(t)|}{n_j}$. Write $n_i m = n_j$. We have

$$|\operatorname{dom}(\tau_{n_i \vee n_j}^{n_i}(s))| = |\operatorname{dom}(\tau_{n_j}^{n_i}(s))| = m|\operatorname{dom}(s)| = |\operatorname{dom}(t)|.$$

But then in $\mathcal{I}_{n_j}$, we know elements are $\mathcal{D}$-related if and only if they have the same cardinality, so $\tau_{n_j}^{n_i}(s)/\mathcal{D} = t/\mathcal{D}$. It follows that $[\tau_{n_j}^{n_i}(s)]/\mathcal{D} = [t]/\mathcal{D}$. Clearly,

$$e_{n_j} \cdot s = \tau_{n_j}^{n_i}(e_j)\tau_{n_j}^{n_i}(s) = e_{n_j}\tau_{n_j}^{n_i}(s),$$

so $[s] = [\tau_{n_j}^{n_i}(s)]$. So then $[s]/\mathcal{D} = [t]/\mathcal{D}$.

Surjectivity follows from omnidivisionality of the chosen sequence; if $q \in \mathbb{Q} \cap [0,1]$, we write $q = \frac{a}{n_q}$ in D-canonical form. Let $f \in \mathcal{I}_{n_q}$ be the partial identity on $\{x_1, \ldots, x_a\}$. Then, $w(f) = \frac{a}{n_q} = q$.

Finally, we argue that w is an MV-effect homomorphism and hence an MV homomorphism. Since MV algebras and homomorphisms are defined from an algebraic theory and we know w is bijective, it suffices to show that its inverse is an MV-effect homomorphism. Let $q \in \mathbb{Q} \cap [0,1]$ and write q in D-canonical form as $q = \frac{d}{n_i}$. Then, it is easy to see that

$$\begin{aligned} w^{-1} \colon \mathbb{Q} \cap [0,1] &\to Q_\infty/\mathcal{D}, \\ \frac{d}{n_i} &\mapsto [\mathrm{id}^{n_i}_{\{x_1,\ldots,x_d\}}]/\mathcal{D}, \end{aligned}$$

where by $\mathrm{id}^{n_i}_{\{x_1,\ldots,x_d\}}$ we mean the partial identity map on $\{x_1, \ldots, x_d\}$ considered as an element of $\mathcal{I}_{n_i}$, is really the two-sided inverse to w.

We have that $w(1) = w(\frac{n_1}{n_1}) = [\mathrm{id}_{I_{n_1}}]/\mathcal{D}$, which we argue is the top element of Q_∞. Clearly the τ maps take total identity functions to total identity functions, so for all i we have $[\mathrm{id}_{I_{n_1}}] = [\mathrm{id}_{I_{n_i}}]$; as it is enough to consider $\mathcal{D}$-classes of idempotents, if $s \in E(I_{n_i})$, then denoting the partial identity with domain $X \setminus \operatorname{dom}(s)$ by $s^\perp$, we have that $s \mathbin{\widetilde{\oplus}} s^\perp = s \vee s^\perp = \mathrm{id}_{X_{n_i}}$, so $s \leq \mathrm{id}_{X_{n_i}}$ whence $[s]/\mathcal{D} \leq [\mathrm{id}_{X_{n_1}}]/\mathcal{D}$.

Now let $a, b \in \mathbb{Q} \cap [0,1]$. Suppose $a \mathbin{\widetilde{\oplus}} b \downarrow$. Then $a + b \leq 1$. Write in D-canonical form $a = \frac{d_1}{n_i}$ and $b = \frac{d_2}{n_j}$. Without loss of generality, suppose $n_i \leq n_j$. Then $n_i \mid n_j$, so there is $m \in \mathbb{N}$ with

$n_i m = n_j$, and we have $a = \frac{d_1 m}{n_j}$, and $d_1 m + d_2 \leq n_j$. We have

$$\begin{aligned} w^{-1}(a) \mathbin{\widetilde{\oplus}} w^{-1}(b) &= [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_1 m}}]/\mathcal{D} \mathbin{\widetilde{\oplus}} [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_2}}]/\mathcal{D} \\ &= [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_1 m}}]/\mathcal{D} \mathbin{\widetilde{\oplus}} [\mathrm{id}^{n_j}_{x_{d_1 m+1},\dots,x_{d_1 m + d_2}}]/\mathcal{D} \\ &= [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_1 m + d_2}}]/\mathcal{D} \\ &= w^{-1}\left(\frac{d_1 m + d_2}{n_j}\right) \\ &= w^{-1}(a \mathbin{\widetilde{\oplus}} b). \end{aligned}$$

Now let a and b be written in D-canonical form as above. Then, we have that

$$w^{-1}(a) \vee w^{-1}(b) = [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_1 m}}]/\mathcal{D} \vee [\mathrm{id}^{n_j}_{x_1,\dots,x_{d_2}}]/\mathcal{D}.$$

But the join of the above is the $\mathcal{D}$-class of the partial identity defined on the union of the domains, hence the above is equal to $w^{-1}(\max(a,b))$, which is $w^{-1}(a \vee b)$.

Repeating the above argument with meet in place of join, intersection in place of union, and min in place of max, yields that w^{-1} also preserves meets.

Thus, w^{-1} is an effect algebra isomorphism which preserves the lattice structure, hence an MV-effect isomorphism, and hence an MV isomorphism, and so Q_∞ coordinatizes $\mathbb{Q} \cap [0,1]$. □

4 Coordinatization decomposition theorem

We now turn to generalizing the approach we took in coordinatizing $\mathbb{Q} \cap [0,1]$. We knew the Łukasiewicz chains Ł_n were coordinatized by $\mathcal{I}_n$, and if we fix an omnidivisional sequence $\{n_1, n_2, \dots\}$, we know $\mathbb{Q} \cap [0,1] = \bigcup_{i=1}^\infty \text{Ł}_{n_i}$ and its coordinatization turned out to be the direct limit of the sequence of I_{n_i}.

We have the following decomposition theorem, which gives us a general way to coordinatize those MV algebras which can be written as unions of subalgebras, provided the latter can each be coordinatized by a single inverse semigroup. The proof is similar to that of Theorem 3.4.

Theorem 4.1 (Decomposition Theorem I)**.** *Let A be an MV algebra. Suppose that A has subalgebras forming a chain of inclusions*

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$$

such that $A = \bigcup_{i=1}^\infty A_i$ and that each A_i is coordinatized by an inverse semigroup S_i. Denote the inclusions by $\ell_i \colon A_i \to A_{i+1}$. Choose an explicit (MV algebra) isomorphism $f_i \colon A_i \to S_i/\mathcal{D}$ for each i. Suppose there are injective maps $\tau_i \colon S_i \to S_{i+1}$ such that the maps

$$\begin{aligned} t_i \colon S_i/\mathcal{D} &\to S_{i+1}/\mathcal{D}, \\ s/\mathcal{D} &\mapsto \tau_i(s)/\mathcal{D} \end{aligned}$$

are well defined on $\mathcal{D}$-classes, and that we have $t_i = f_{i+1} \circ j_i \circ f_i^{-1}$; i.e. the following diagram commutes for all i.

$$\begin{tikzcd} A_i \arrow[r, "\ell_i"] & A_{i+1} \arrow[d, "f_{i+1}"] \\ S_i/\mathcal{D} \arrow[u, "f_i^{-1}"] \arrow[r, "t_i"'] & S_{i+1}/\mathcal{D} \end{tikzcd}$$

Then, A is coordinatized by the directed colimit of

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots .$$

Proof. We define $F\colon S_\infty/\mathcal{D} \to A$ as follows. Let $[s]/\mathcal{D} \in S_\infty/\mathcal{D}$. So $[s] \in S_\infty$, and hence $s \in S_i$ for some i. Define

$$F([s]/\mathcal{D}) = f_i^{-1}(s/\mathcal{D}).$$

We first claim that, for $s \in S_i$ and $u \in S_j$, that $[s] = [u]$ in S_∞ implies $F([s]/\mathcal{D}) = F([u]/\mathcal{D})$; that is, that $f_i^{-1}(s/\mathcal{D}) = f_j^{-1}(u/\mathcal{D})$. Without loss of generality, we let $i \leq j$. Now, $[s] = [u]$ means there exists $e_k \in S_k$ for some k such that $s \cdot e_k = u \cdot e_k$, where e_k is the identity of S_k. This means

$$\tau^i_{i\vee k}(s)\tau^k_{i\vee k}(e_k) = \tau^j_{j\vee k}(u)\tau^k_{j\vee k}(e_k). \tag{4.1}$$

If $i = j$, the above equation becomes $\tau^i_{i\vee k}(s) = \tau^i_{i\vee k}(u)$. But $\tau^i_{i\vee k}$ is a composite of injective maps and hence itself injective, so $s = u$, thus $f_i^{-1}(s/\mathcal{D}) = f_i^{-1}(u/\mathcal{D})$.

On the other hand, if $i \neq j$, the equation (4.1) occurs in $S_{i\vee k}$ on the left and $S_{j\vee k}$ on the right, so we must have $i \vee k = j \vee k = k$, and $k \geq i$, $k \geq j$. We have

$$\tau^j_k \tau^i_j(s) = \tau^i_k(s) = \tau^j_k(u).$$

By injectivity of the τ maps, we have $\tau^i_j(s) = u$. Then,

$$\begin{aligned}
f_j^{-1}(u/\mathcal{D}) &= f_j^{-1}(\tau^i_j(s)/\mathcal{D}) \\
&= f_j^{-1}(\tau_{j-1}\dots\tau_i(s)/\mathcal{D}) \\
&= f_j^{-1}(t_{j-1}\dots t_i(s/\mathcal{D})) \\
&= f_j^{-1}(f_j\ell_{j-1}f_{j-1}^{-1}f_{j-1}\dots f_{i+1}^{-1}f_{i+1}\ell_i f_i^{-1}(s/\mathcal{D})) \\
&= \ell_{j-1}\dots\ell_{i+1}\ell_i f_i^{-1}(s/\mathcal{D}) \\
&= f_i^{-1}(s/\mathcal{D}).
\end{aligned}$$

In all cases, the claim is proved.

We are now ready to prove F is well defined in $\mathcal{D}$ classes. Suppose now $s \in S_i$ and $u \in S_j$ such that $[s]/\mathcal{D} = [u]/\mathcal{D}$. That $[s]$ and $[u]$ are in the same $\mathcal{D}$ class means there exists $v \in S_k$ for some k such that

$$[s^{-1}s] = [v^{-1}v],\ [vv^{-1}] = [u^{-1}u].$$

We thus have

$$\begin{aligned}
F([s]/\mathcal{D}) &= f_i^{-1}(s/\mathcal{D}) && \text{(by definition)} \\
&= f_i^{-1}(s^{-1}s/\mathcal{D}) && \text{(same $\mathcal{D}$-class in S_i)} \\
&= f_k^{-1}(v^{-1}v/\mathcal{D}) && \text{(by above observation)} \\
&= f_k^{-1}(vv^{-1}/\mathcal{D}) && \text{(same $\mathcal{D}$-class in S_k)} \\
&= f_j^{-1}(u^{-1}u/\mathcal{D}) && \text{(by above observation)} \\
&= f_j^{-1}(u/\mathcal{D}) && \text{(same $\mathcal{D}$-class in S_j)} \\
&= F([u]/\mathcal{D}) && \text{(by definition).}
\end{aligned}$$

Next, we prove injectivity of F. Suppose $s \in S_i$, $u \in S_j$, and $F([s]/\mathcal{D}) = F([u]/\mathcal{D})$, so $f_i^{-1}(s/\mathcal{D}) = f_j^{-1}(u/\mathcal{D})$. We assume without loss of generality that $i \leq j$. We have that

$$e_j \cdot s = \tau_j^i(e_j)\tau_j^i(s) = e_j \cdot \tau_j^i(s),$$

so $[s] = [\tau_j^i(s)]$. We compute

$$\begin{aligned}
f_j^{-1}(\tau_j^i(s)/\mathcal{D}) &= f_j^{-1}(\tau_{j-1} \dots \tau_{i+1}\tau_i(s)/\mathcal{D}) \\
&= f_j^{-1}(t_{j-1} \dots t_i(s/\mathcal{D})) \\
&= f_j^{-1} f_j \ell_{j-1} f_{j-1}^{-1} \dots \ell_i f_i^{-1}(s/\mathcal{D}) \\
&= f_i^{-1}(s/\mathcal{D}) \\
&= f_j^{-1}(u/\mathcal{D}).
\end{aligned}$$

Since f_j^{-1} is by hypotheses an isomorphism, it is, in particular, injective, so $\tau_j^i(s)/\mathcal{D} = u/\mathcal{D}$. Thus,

$$[s]/\mathcal{D} = [\tau_j^i(s)]/\mathcal{D} = [u]/\mathcal{D}.$$

For surjectivity of F, let $a \in A$. Then let i be the smallest integer such that $a \in A_i$. Consider $f_i(a) \in S_i/\mathcal{D}$. Choose an element $s \in S_i$ which is in the $\mathcal{D}$-class $f_i(a)$. We have

$$F([s]/\mathcal{D}) = f_i^{-1}(s/\mathcal{D}) = f_i^{-1}(f_i(a)) = a.$$

That F is an MV algebra homomorphism follows directly from the fact that all the f_i are MV algebra homomorphisms. Thus, we have explicitly constructed a map giving us $S_\infty/\mathcal{D} \cong A$, and so A is coordinatized as stated. $\square$

The converse of this theorem is also true, as given below.

Theorem 4.2 (Decomposition Theorem II)**.** *Suppose A is an MV algebra coordinatized by the directed colimit of*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots .$$

Then, A has a sequence of subalgebras forming a chain of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^\infty A_i$, and each A_i is coordinatized by S_i.

Proof. Denote $S_\infty = \lim_\to(S_0 \to S_1 \to \dots)$ as usual. As A is coordinatized by S_∞, we choose an explicit MV isomorphism $F \colon S_\infty/\mathcal{D} \to A$. We put

$$A_i := \{F([a]/\mathcal{D}) \mid a \in S_i\}.$$

To show each A_i is a MV subalgebra of A, it suffices to show $0_A \in A_i$, and that A_i is closed under $\oplus$ and $\neg$.

Recall that the zero of S_∞, considered as a boolean inverse monoid, is the $\equiv$-class containing the zeroes of every S_i. In particular, as each S_i is boolean by hypothesis, we know, denoting z_i as the zero of S_i, that z_i is in 0_{S_∞}. From [LS17, Proposition 2.9 & Theorem 2.10], the $\mathcal{D}$-class of 0_{S_∞} is also the zero of $S_\infty/\mathcal{D}$ considered as an MV algebra. As MV maps preserve zeroes, we thus have for each i that $F^{-1}([z_i]/\mathcal{D}) = 0_A$, so $0_A \in A_i$.

Next we show closure under negation. Let $a \in S_i$ (without loss of generality we may take a to be an idempotent; replace it with $a^{-1}a$ if necessary, which is in the same $\mathcal{D}$-class). From [LS17, Theorem 2.10], in $S_i/\mathcal{D}$, $\neg(a/\mathcal{D}) = (\overline{a}/\mathcal{D})$, where $\overline{(-)}$ denotes boolean complementation. Since the restriction of the operation in $S_\infty/\mathcal{D}$ to $\mathcal{D}$-classes of S_∞-classes of elements of S_i coincides with the operation in S_i, we have in $S_\infty/\mathcal{D}$ that $\neg([a]/\mathcal{D}) = ([\overline{a}]/\mathcal{D})$. Thus, if $x \in A_i$, then $x = F([a]/\mathcal{D})$ for some $a \in S_i$. Since F preserves negation and $\overline{a} \in S_i$, we have

$$F([\overline{a}/\mathcal{D}]) = F(\neg([a]/\mathcal{D})) = \neg F([a]/\mathcal{D}) = \neg x \in A_i.$$

For closure under $\oplus$, let $x, y \in A_i$. So $x = F([a]/\mathcal{D})$ and $y = F([b]/\mathcal{D})$ for some $a, b \in S_i$. In $S_i/\mathcal{D}$, we know $a/\mathcal{D} \oplus b/\mathcal{D} = c/\mathcal{D}$ for some $c \in S_i$, so by the same remark about restriction as before, we have that

$$x \oplus y = F([a]/\mathcal{D}) \oplus F([b]/\mathcal{D}) = F([a]/\mathcal{D} \oplus [b]/\mathcal{D}) = F([c]/\mathcal{D}),$$

whence $x \oplus y \in A_i$. Thus, we see that each A_i is indeed an MV-subalgebra of A.

We now argue that $A_i \subseteq A_{i+1}$. Let $x \in A_i$. So $x = F([a]/\mathcal{D}$ for some $a \in S_i$. As argued in Theorem 4.1, $[a] = [\tau_i(a)]$, so

$$x = F([a]/\mathcal{D}) = F([\tau_i(a)]/\mathcal{D}) \in A_{i+1}.$$

It is clear that $\bigcup_{i=1}^\infty A_i \subseteq A$. For the reverse inclusion, suppose $x \in A$. Then, $F^{-1}(x) = [a]/\mathcal{D}$, where $a \in S_j$ for some j. Then,

$$F([a]/\mathcal{D}) = FF^{-1}(x) = x,$$

so $x \in A_j \subseteq \bigcup_{i=1}^\infty A_i$.

Finally, we must show $A_i \cong S_i/\mathcal{D}$ for each i. Define $f_i \colon A_i \to S_i/\mathcal{D}$ as follows. For $x \in A_i$, we know $x = F([a]/\mathcal{D})$ for some $a \in S_i$. Define $f_i(x) = a/\mathcal{D}$ – this is well defined because F is an isomorphism on $S_\infty/\mathcal{D}$. That the f_i are MV maps follows directly from the fact that F^{-1} is one.

For injectivity of f_i, if $f_i(x) = a/\mathcal{D} = b/\mathcal{D} = f_i(y)$, then $[a]/\mathcal{D} = [b]/\mathcal{D}$, and $x = F([a]/\mathcal{D}) = F([b]/\mathcal{D}) = y$. For surjectivity, let $a/\mathcal{D} \in S_i/\mathcal{D}$. Then, $F([a]/\mathcal{D}) \in A_i$, and $f_i(F[a]/\mathcal{D}) = a/\mathcal{D}$. □

5 The coordinatization of Chang's MV algebra

We now give our first example of the coordinatization of an MV algebra which does not embed into $[0, 1]$ (which will be evident by its order type).

Example 5.1. Let $\mathcal{C}$ be the set consisting of the formal symbols

$$0, c, c + c, c + c + c, \ldots,$$

$$1, 1 - c, 1 - c - c, 1 - c - c - c, \ldots.$$

For short, we write (for $n \in \mathbb{N}^+$), $0 \cdot c = 0$, $n \cdot c = \underbrace{c + c + \ldots + c}_{n \text{ factors of } c}$, $1 - 0 \cdot c = 1$, and $1 - n \cdot c = 1 - \underbrace{c - c - \ldots - c}_{n \text{ factors of } c}$.

We define addition as follows:

- If $x = n \cdot c$ and $y = m \cdot c$, then $x \oplus y = (n + m) \cdot c$.
- If $x = 1 - n \cdot c$ and $y = 1 - m \cdot c$, then $x \oplus y = 1$.
- If $x = n \cdot c$ and $y = 1 - m \cdot c$, or if $x = 1 - m \cdot c$ and $y = n \cdot c$, then

$$x \oplus y = \begin{cases} 1 - (m - n) \cdot c & \text{if } n < m, \\ 1 & \text{if } m \leq n. \end{cases}$$

We define complementation by

$$\neg x = \begin{cases} 1 - n \cdot c & \text{if } x = n \cdot c, \\ n \cdot c & \text{if } x = 1 - n \cdot c. \end{cases}$$

Then, $(\mathcal{C}, \oplus, \neg, 0)$ is a MV algebra known as *Chang's MV algebra*, which was first introduced in [Cha58].

We will give the coordinatization of $\mathcal{C}$, the details of which are quite straightforward once we come up with the right idea, but we highlight how the various theorems seen so far led to this idea and emphasize their usefulness.

A priori, we knew the coordinatization existed, but it could have been any AF inverse monoid, and the colimit of any number of inverse semigroups. However, it is readily seen that any nontrivial subalgebra of $\mathcal{C}$ (e.g. generated by $n \cdot c$ for some $n \in \mathbb{N}^+$) is isomorphic to $\mathcal{C}$ itself. As such, there is no meaningful way to write $\mathcal{C}$ as a union of successive subalgebras, whence the contrapositive of Theorem 4.2 tells us that we are in fact looking for a *single* inverse semigroup.

Next, the Preston-Wagner Theorem ([Law98, p.36]) tells us that every inverse semigroup is in fact a subsemigroup of some symmetric inverse monoid. As $\mathcal{C}$ is a countably infinite MV algebra, the natural place to start is $\mathcal{I}(\mathbb{N})$. We now need to figure out what $\mathcal{D}$-classes of partial bijections correspond to the elements of $\mathcal{C}$.

The obvious thing is that 0 should be the class of the empty function, 1 should be the class of total bijections, and $n \cdot c$ should be the class of partial bijections with domain size n. If we now look at the (linear) ordering of $\mathcal{C}$, we see that it looks like a copy of $\mathbb{N}$ at the bottom, with $\mathbb{N}^{op}$ above it, such that the two chains never meet. So, if the element $1 - n \cdot c$ is to be the complement to $n \cdot c$, then the complement to the $\mathcal{D}$-class of bijections on n elements should be the class of bijections on all but n elements. For brevity, we will henceforth in this section say "the co-size of X" to mean "the size of $\mathbb{N} \backslash X$".

There is, however, one additional condition — we only include the partial bijections with cofinite domain of the same co-size as the image. We will say a partial bijection that satisfies this property has *balanced cofinite* domain. Thus, we define $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ to be the subset of $\mathcal{I}(\mathbb{N})$ to be those partial bijections on $\mathbb{N}$ whose domain are either finite or balanced cofinite.

Lemma 5.2. *$\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a sub-inverse semigroup of $\mathcal{I}(\mathbb{N})$.*

Proof. We need to show closure under composition and pseudoinverses. It is clear the pseudoinverse of a partial bijection with finite domain also has finite domain, and the pseudoinverse of a function with cofinite domain also has cofinite domain.

Let $f, g \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$. Recall that $g \circ f$ has domain $f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$. If either or both of f, g have finite domain/image, then $\operatorname{dom} g \cap \operatorname{im} f$ is finite, and so is its inverse image under f, so $g \circ f \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$.

On the other hand, if both f, g have cofinite domain/image, then $\operatorname{dom} g \cap \operatorname{im} f$ is an intersection of two cofinite sets and is again cofinite. Furthermore, it is a subset of $\operatorname{im} f$ consisting of all but finitely many elements of $\operatorname{im} f$, so its inverse image under f is a cofinite set with finitely elements removed, and remains cofinite, so $g \circ f$ has cofinite domain.

Note that if we take a partial bijection on a balanced cofinite domain, and further restrict to a cofinite subset of its domain, the restricted map will also have balanced cofinite domain. Thus, as f and g are by hypothesis balanced, we have

$$\begin{aligned} |\mathbb{N}\backslash(\operatorname{dom} g \circ f)| &= |\mathbb{N}\backslash f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash g(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash(\operatorname{im} g \circ f)|, \end{aligned}$$

whence $g \circ f \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$. □

Lemma 5.3. *The $\mathcal{D}$-classes of $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ are precisely as follows. For $f, g \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, $f\mathcal{D}g$ if and only if either f, g have finite domains of the same size or f, g have balanced cofinite domains with the same co-size.*

Proof. Suppose $f\mathcal{D}g$, and suppose f has finite domain $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{N}$. So there is $h \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ such that

$$\mathrm{id}_X = f^{-1}f = h^{-1}h,\ hh^{-1} = g^{-1}g.$$

Then,

$$|\operatorname{dom} g| = |\operatorname{dom} g^{-1}g| = |\operatorname{dom} hh^{-1}| = |\operatorname{dom} h^{-1}h| = |\operatorname{dom} f^{-1}f| = n.$$

Now suppose f has balanced cofinite domain $\mathbb{N}\backslash X$ with $X = \{x_1, \ldots, x_n\}$. Then denote its image as $\mathbb{N}\backslash Y = \{y_1, \ldots, y_n\}$. If $f\mathcal{D}g$, then we have $h \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ such that

$$\mathrm{id}_{\mathbb{N}\backslash X} = f^{-1}f = h^{-1}h,\ hh^{-1} = g^{-1}g.$$

This means h also has domain with co-size n. Since it is in $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, it is balanced and has image with co-size n. Thus h^{-1} also has domain and image with co-size n, hence the same is true of $g^{-1}g$, and finally, of g.

Now suppose f has domain $X = \{x_1, \ldots, x_n\}$ and g has domain $Y = \{y_1, \ldots, y_n\}$. Define the partial bijection h by $x_i \mapsto y_i$ for $1 \leq i \leq n$. We see that $h^{-1}h$ is given by $x_i \mapsto y_i \mapsto x_i$, i.e. $h^{-1}h = \mathrm{id}_X = f^{-1}f$, and that hh^{-1} is given by $y_i \mapsto x_i \mapsto y_i$, i.e. $hh^{-1} = \mathrm{id}_Y = g^{-1}g$. So $f\mathcal{D}g$.

Now if X and Y as defined above are instead the complements of $\operatorname{dom} f$ and $\operatorname{dom} g$. We write $\operatorname{dom} f = \mathbb{N}\backslash X = \{x_{n+1}, x_{n+2}, \ldots\}$, where we enumerate all the elements of $\mathbb{N}\backslash X$ in the reader's favorite order, and similarly $\operatorname{dom} g = \mathbb{N}\backslash Y = \{y_{n+1}, y_{n+2}, \ldots\}$. Define the partial bijection h by $x_{n+i} \mapsto y_{n+i}$ for $i \in \mathbb{N}^+$. We see that $f\mathcal{D}g$ via h. □

Remark 5.4. We would like to thank Mark Lawson for the following observation. Initially, we defined $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ to contain all the partial bijections with cofinite domain, without the additional clause of being balanced. He noted that the successor function $s\colon \mathbb{N} \to \mathbb{N}^{+}$, $n \mapsto n+1$, would be in our inverse semigroup and that it clearly cannot be extended to a total bijection. Hence with this definition of $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, one obtains an inverse semigroup which is not factorizable and so not a Foulis monoid.

Moreover, denoting the pseudoinverse of s by $t\colon \mathbb{N}^{+} \to \mathbb{N}$, $n \mapsto n-1$, observe that

$$\mathrm{id}_{\mathbb{N}^{+}} = st = ss^{-1},\ s^{-1}s = ts = \mathrm{id}_{\mathbb{N}},$$

so in fact this gives a $\mathcal{D}$-relation between the total identity on $\mathbb{N}$ (with domain co-size 0), and the identity on $\mathbb{N}^{+}$ (with domain co-size 1), completely destroying our attempt to mirror the behaviour of complementary elements in $\mathcal{C}$!

Theorem 5.5. *$\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ coordinatizes $\mathcal{C}$.*

Proof. We first must prove that $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a Foulis monoid. All symmetric inverse monoids are boolean, and in particular $\mathcal{I}(\mathbb{N})$ is boolean. We begin by showing $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is a boolean subalgebra of $E(\mathcal{I}(\mathbb{N}))$.

For a partial identity $\mathrm{id}_X \in E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$, either X is finite or cofinite (note that all partial identities on cofinite domains are balanced), and its boolean complement is $\mathrm{id}_{\mathbb{N}\setminus X}$. In either case, the complement is again defined on a finite or cofinite domain, so $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is closed under complementation.

Next, we check closure under $\wedge$. Recall for partial bijections f, g, that $f \wedge g$ has domain $\{x \in X \mid f(x) = g(x)\}$ and $(f \wedge g)(x) = f(x) = g(x)$. So suppose $\mathrm{id}_X, \mathrm{id}_Y \in E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$. Since they are partial identities, the domain of $\mathrm{id}_X \wedge \mathrm{id}_Y$ is simply $X \cap Y$ and $\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X\cap Y}$. If at least one of X or Y is finite, then so is $X \cap Y$. On the other hand, if both are cofinite, then there are only finitely many elements of $\mathbb{N}$ missing from each of X and Y, hence only finitely many elements missing from either X or Y, and so $X \cap Y$ is cofinite.

Closure under $\vee$ now follows from DeMorgan's Law. Thus, $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is indeed a boolean algebra, from which conditions (DIM 1)–(DIM 3) are immediate. Finally, we need to show $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is factorizable. The group of units are the total bijections. Any partial bijection with a finite domain can be extended to a total bijection by enumerating $\mathbb{N}\setminus \operatorname{dom} f = \{x_1, \ldots, x_n, \ldots\}$ and $\mathbb{N}\setminus \operatorname{im} f = \{y_1, \ldots, y_n, \ldots\}$ and sending $x_i \mapsto y_i$. Similarly, we can do this for partial bijections with balanced cofinite domain — the condition of being balanced being precisely what makes this possible. Thus, $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a Foulis monoid.

Now define

$$\begin{aligned} H\colon \mathcal{C} &\to \mathcal{I}(\mathbb{N})_{\mathbf{fc}}/\mathcal{D},\\ 0 &\mapsto !_{\emptyset}/\mathcal{D},\\ n\cdot c &\mapsto \mathrm{id}_{\{0,1,\ldots,n-1\}}/\mathcal{D},\\ 1 - n\cdot c &\mapsto \mathrm{id}_{\{n,n+1,n+2,\ldots\}}/\mathcal{D},\\ 1 &\mapsto \mathrm{id}_{\mathbb{N}}/\mathcal{D}. \end{aligned}$$

Bijectivity of H is immediate from the preceding lemmas. It is also immediately obvious that H preserves zero and negation. We will check that it preserves $\oplus$.

Recall how the operation of $\mathcal{C}$ is defined in Example 5.1 — we will follow the same breakdown of cases. Recall also how the MV and MV-effect algebra operations are defined from each other (Theorem 2.6) and how the operation on $\mathcal{D}$-classes is defined (Proposition 2.19).

Let $x, y \in \mathcal{C}$.

- Case 1: $x = n \cdot c$ and $y = m \cdot c$

  We have that

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(m \cdot c) \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \oplus \mathrm{id}_{\{n,n+1,\dots,n+m-1\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{n,n+1,\dots\}} / \mathcal{D} \wedge \mathrm{id}_{\{n,n+1,\dots,n+m-1\}} / \mathcal{D}) \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{n,n+1,\dots,n+m-1\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \vee \mathrm{id}_{\{n,n+1,\dots,n+m-1\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\dots,n+m-1\}} / \mathcal{D} \\
&= H((n+m) \cdot c) \\
&= H((n \cdot c) \oplus (m \cdot c)) \\
&= H(x \oplus y).
\end{aligned}$$

- Case 2: $x = 1 - n \cdot c$ and $y = 1 - m \cdot c$

  We have that

$$\begin{aligned}
H(x) \oplus H(y) &= H(1 - n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{n,n+1,\dots\}} / \mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\dots\}} / \mathcal{D}) \\
&= \mathrm{id}_{\{n,n+1,\dots\}} / \mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \\
&= \mathrm{id}_{\{n,n+1,\dots\}} / \mathcal{D} \vee \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \\
&= \mathrm{id}_{\mathbb{N}} / \mathcal{D} \\
&= H(1) \\
&= H((1 - n \cdot c) \oplus (1 - m \cdot c)) \\
&= H(x \oplus y).
\end{aligned}$$

- Case 3: $x = n \cdot c$ and $y = 1 - m \cdot c$

  First suppose $n < m$. Then,

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{n,n+1,\dots\}} / \mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\dots\}} / \mathcal{D}) \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{m,m+1,\dots\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\dots,n-1\}} / \mathcal{D} \vee \mathrm{id}_{\{m,m+1,\dots\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\dots,n-1,m,m+1,\dots\}} / \mathcal{D} \\
&= \mathrm{id}_{\{m-n-1,m-n,\dots,m-1,m,m+1,\dots\}} / \mathcal{D} \\
&= H(1 - (m-n) \cdot c) \\
&= H(x \oplus y).
\end{aligned}$$

On the other hand, if $n \geq m$, we have

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} / \mathcal{D} \mathbin{\widetilde{\oplus}} (\mathrm{id}_{\{n,n+1,\ldots\}} / \mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\ldots\}} / \mathcal{D}) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} / \mathcal{D} \mathbin{\widetilde{\oplus}} \mathrm{id}_{\{n,n+1,\ldots\}} / \mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} / \mathcal{D} \vee \mathrm{id}_{\{n,n+1,\ldots\}} / \mathcal{D} \\
&= \mathrm{id}_{\mathbb{N}} / \mathcal{D} \\
&= H(1) \\
&= H(x \oplus y).
\end{aligned}$$

□

6 Future directions

The following table by Daniele Mundici lists a number of concrete MV algebras, along with their corresponding AF C* algebras, as developed in [Mun86, Mun93]. These make challenging candidates for coordinatization.

Countable MV Algebra	Its AF C*-algebraic correspondent
$\{0, 1\}$	$\mathbb{C}$, the complex numbers;
finite	finite dimensional;
boolean	commutative;
dyadic rationals in the unit interval	CAR algebra of a Fermi gas;
algebra generated by an irrational $\rho \in [0, 1]$	Effros-Shen algebra $\mathfrak{F}_\rho$;
real algebraic numbers in [0, 1]	Blackadar algebra B ;
Chang algebra	Behncke-Leptin algebra $A_{0,1}$;
atomless boolean	$C(2^\omega)$;
rational simple algebra	Uniformly hyperfinite (Glimm);
$\mathbb{Q} \cap [0, 1]$	Glimm's universal UHF algebra;
totally ordered	With Murray-von Neumann comparability of projections;
free with countably many generators	The universal AF C*-algebra $\mathfrak{M}$;
free with one generator	The "Farey" AF C*-algebra $\mathfrak{M}_1$ (Mundici (1998), Boca (2008));

It is believed that the coordinatization of MV algebras is functorial, but this needs to be explicitly written out. Furthermore, it would be desirable to reconcile the coordinatization theorem of Wehrung [Weh15] with that of Lawson & Scott [LS17]. In particular, it would be useful to have a simpler, direct proof of Wehrung's coordinatization theorem for uncountable MV algebras along the lines of [LS17].

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