Cockett-Lack Restriction

Categories, Semigroups and Topologies

FMCS 2003, Ottawa

Ernie Manes

Department of Mathematics and Statistics

University of Massachusetts

Amherst, MA 01003 USA

manes@math.umass.edu
Many have written about categories of partial maps

- Di Paola and Heller, 1986
- Carboni, 1987
- Robinson and Rosolini, 1988
- Curien and Obtulowicz, 1989
- Jay, 1990
- Mulry, 1992
- Fiore, 1996
For $\mathcal{C}$ a category, $\mathcal{M}$ a stable system of monics, the **partial morphism category**

$$\text{Par}(\mathcal{C}, \mathcal{M})$$

has morphisms equivalence classes $[m, f]$ with $\xymatrix{X \ar[r]^m \ar[l]_m & Y}$ and $m \in \mathcal{M}$. Composition is via pullback as usual.

The “domain of definition” of $[m, f]$ can be modelled as the **restriction endomorphism** $[m, m]$. 
Cockett and Lack’s idea was to make restriction a primitive.


A restriction category is an “abstract category of partial morphisms”, being a category with a restriction operator

\[ f : X \to Y \mapsto \overline{f} : X \to X \]

satisfying the axioms \(R.1, \ldots, R.4\) on the board.

Note that a full subcategory of a restriction category again is one.
For a restriction category $\mathcal{C}$ denote by $\mathcal{R}(\mathcal{C})$ the set of restriction idempotents

$$\mathcal{R}(\mathcal{C}) = \{ x : x = \overline{x} \} = \{ \overline{f} : f \text{ a morphism} \}$$

The restriction idempotents $X \to X$ form a semilattice by $(D)$, $R.2$, $(A)$. 
A **split** restriction category has the property that all restriction idempotents split.

In that case, the set of all $m$ as above form a stable system of monics.
**Theorem** (Cockett & Lack) $Par(C, M)$ is a split restriction category. For every restriction category $C$ there exists $D$, $M$ for which $C$ is a full restriction category of $Par(D, M)$

**Proof Idea:** Let $E$ be the itempotent completion of $C$ splitting $R(C)$. $E$ is a restriction category: $e_1 \xrightarrow{f} e_2$ has restriction $\overline{f}e_1$.

In any restriction category, $f$ is **total** if $\overline{f} = id$.

Take $D$ to be the total morphisms of $E$. Take $M$ as the monics that arise in the splittings of restriction itempotents in $E$. (Though $\overline{f}$ is not total, all monics are total).

The embedding is

$$f \mapsto [X \xleftarrow{m} \xrightarrow{m} X \xrightarrow{f} Y]$$

where $m$ is the monic in the splitting of $\overline{f}$. 

7
Via the Yoneda embedding of the previous construction, one sees further that \( \mathcal{C} \) is a full restriction category of

\[
\text{Par}(\text{Set}^{\mathcal{D}^{\text{op}}}, \mathcal{N})
\]

for a suitably-chosen stable system of monics \( \mathcal{N} \).
Summary

- Restriction categories have captured partial morphism categories.

- There is no use of universal properties in the axioms. Any full subcategory continues to be a restriction category.
Earlier work by some theoretical programmers had a different emphasis.

- The logic is classical (Boolean)
- But programs can have nondeterministic behavior.

Edsger Dijkstra, *A Discipline of Programming*, Prentice-Hall, 1976:

“In this book –and that may turn out to be one of its distinctive features– I shall treat nondeterminancy as the rule and determinacy as the exception . . .”

Dijkstra’s guards are precisely restrictions.

**Boolean Categories**

(B.1) $X + Y$, initial 0

(B.2) Coproduct injections is stable system of monics

(B.3) Coproduct injections pull back binary coproducts

(B.4) Except for 0, coproduct injections in $X + X$ are different
Coproduct-injection subobjects are called **summands**.

**Theorem** The poset $\text{Summ}(X)$ of summands of $X$ forms a Boolean algebra.

The pullback of $X \xrightarrow{f} Y \xleftarrow{0}$ is the **kernel** of $f$, $\text{Ker}(f) \to X$.

$\text{Dom}(f) = (\text{Ker}(f))'$.

$f$ is **total** if $\text{Ker}(f) = 0$.

$\text{Dom}(f)$ is the largest summand restricted to which $f$ is total.

$f$ is **undefined** if $f$ factors through $0$. 
To define restrictions requires *canonical* undefined maps.

In a Boolean category, these are provided by “projection systems” which correspond bijectively to maximal Boolean subcategories with zero maps. Let us fix one of these so that

We now work in a Boolean category with a zero object.
Here’s how restrictions are defined in a Boolean category with zero:

\[ \text{Ker}(f) \xrightarrow{i'} X \xleftarrow{i} \text{Dom}(f) \]

\[ 0 \xrightarrow{\overline{f}} X \xleftarrow{i} \text{Dom}(f) \]

Fact: \( R.1, R.2, R.3 \) hold. Restriction idempotents split.

What is the situation with \( R.4 \)?
**Proposition**  Restrictions $X \to X$ form a Boolean algebra (with $a \land b = ab = ba$) isomorphic to $Summ(X)$.

**Proof Idea**

$$A \xrightarrow{i} X \xleftarrow{A'} \iff a = \overline{i}$$

$$a = \overline{a} : X \to X \iff A = eq(id_X, a)$$
In our Boolean category with zero, $f : X \to Y$ is deterministic if

\[
\begin{array}{ccc}
P & \longrightarrow & X & \longrightarrow & P' \\
\downarrow & & \downarrow f & & \downarrow \\
Q & \longrightarrow & Y & \longrightarrow & Q'
\end{array}
\]

$\forall Y = Q + Q' \ \exists X = P + P'$ and a commutative diagram as above.

Deterministic maps form a Boolean subcategory.
Toward an interpretation of Axiom R.4

**Theorem** In a Boolean category with zero, (R.4) holds for $f : X \to Y$, i.e. for all $g : Y \to Z$, $\overline{gf} = f\overline{gf}$ if and only if $f$ is deterministic.

Thus a Boolean category with zero is a restriction category if and only if all morphisms are deterministic.

Thus, for each Boolean category with zero, the determinisic morphisms constitute a restriction category.
Toward restrictions for semigroups

Semigroup theorists should be interested in restriction!

Let’s start with some basic semigroup stuff.
Let $S$ be a semigroup. $a \in S$ is **regular** if 
$\exists x \in S$ with $axa = a$.

$S$ is **regular** if all of its elements are regular.

An **inverse** of $a$ is $x$ with $axa = a$ and $xax = x$.

**Example** Let $S = A \times B$ with $(a, b)(c, d) = (a, d)$. Then $S$ is a semigroup in which each element is inverse to all elements.

Every regular element has an inverse.
An inverse semigroup is a semigroup in which each element \( a \) has a unique inverse \( a^{-1} \).

Inverse semigroups are equationally definable:

\[
\begin{align*}
x(yz) &= (xy)z \\
(x^{-1})^{-1} &= x \\
(xy)^{-1} &= y^{-1}x^{-1} \\
x^{-1}y^{-1} &= y^{-1}x^{-1}
\end{align*}
\]

Example Any group.

Example Injective partial functions \( X \to X \).

Proposition A semigroup is an inverse semigroup if and only if it is regular and any two idempotents commute.
**Vagner-Preston Theorem** If $S$ is an inverse semigroup then

$$S \xrightarrow{\lambda} \text{Pfn}(S, S), \quad a \mapsto \lambda_a$$

$$\lambda_a x = \begin{cases} 
ax & \text{if } x \in a^{-1}aS \\
\bot & \text{otherwise}
\end{cases}$$

is an injective semigroup homomorphism. Each $\lambda_a$ is an injective partial function.
Books on inverse semigroups

- Petrich, 1984 (674 pages)

- Lipscomb, 1996

- Lawson, 1998 “Self-similarities are examples of what we term partial symmetries...”.

There has been literature on semigroups with $x \mapsto x^*$ satisfying

\[
\begin{align*}
(xy)^* &= y^*x^* \\
x^{**} &= x \\
x x^* x &= x
\end{align*}
\]
Inverse semigroups are “abstract injective $\text{Pfn}(X, X)$”.

What plays the role of “abstract $\text{Pfn}(X, X)$”?

**Restriction algebras!**
A **restriction algebra** is a semigroup equipped with a unary operation \( x \mapsto \overline{x} \) which satisfies axioms \((R.1, \ldots, R.4)\)

Thus restriction algebras constitute an equationally definable class of universal algebras.

**Example** The endomorphisms of any object in a restriction category.
Example Let $\mathcal{C}$ be a restriction category. Let $S$ be the morphisms of $\mathcal{C}$ together with a new element 0. Then $S$ is a restriction algebra if

$$xy = \begin{cases} xy & \text{if } x \neq 0 \neq y, \ cod(y) = dom(x) \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{x} = \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
The Robinson-Rosolini $P$-categories / Cockett copy categories produce a restriction category by

$$\overline{f} = A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes 1} B \otimes A \xrightarrow{! \otimes 1} I \otimes A \cong A$$

whose endomorphism monoids are restriction algebras.

Not an example semigroup theorists would rush to.
Example Let $S$ be a semigroup, $a \in S$. Define $\overline{x} = a$. This is a restriction algebra if and only if $a$ is a unit for $S$.

Example Let $S$ be a left cancellative semigroup which is not a monoid. Then no restriction operator exists to make $S$ a restriction algebra. (Proof: $\overline{x} \overline{y} = \overline{x \overline{x} \overline{y}} \Rightarrow \overline{x} \overline{y} = \overline{y}$. As restriction itempotents commute, the same proof gives $\overline{x} \overline{y} = \overline{y}$. Now use the previous example.)

Example Every meet semilattice $xy = x \wedge y$ is a restriction algebra if $\overline{x} = x$. We say a restriction algebra “is a semilattice” if it is of this form.
Exercise for you: Show that the center

\[ Z(S) = \{ x \in S : \forall y \in S \ xy = yx \} \]

is a restriction subalgebra.

Hint: Use all four axioms.
Proposition (Cockett and Lack) Every inverse semigroup is a restriction algebra with $\overline{x} = x^{-1}x$. Inverse semigroups are a full coreflective subcategory of restriction algebras with the coreflection $I(S)$ of $S$ given by

$$\{x \in X : \exists a \in S \text{ with } xa = \overline{a}, ax = \overline{x}\}$$

$I(S)$ is analogous to the group of units of monoid.
By a **partially ordered semigroup** we mean a semigroup with a partial order such that

\[ x \leq y \implies \forall a \forall b \ axb \leq ayb \]

Every restriction algebra is a partially ordered semigroup if \( x \leq y \) means \( y \bar{x} = x \).

Restriction algebra homomorphisms are monotone.
“Vagner-Preston Theorem” for restriction algebras If $S$ is a restriction algebra then

$$S \xrightarrow{\lambda} \text{Pfn}(S, S), \quad a \mapsto \lambda_a$$

$$\lambda_a x = \begin{cases} 
ax & \text{if } \bar{a}x = x \\
\perp & \text{otherwise}
\end{cases}$$

is an injective restriction algebra homomorphism mapping $I(S)$ to injective partial functions.

This recaptures the classical theorem for inverse semigroups.

When $S$ is a monoid with $\bar{x} = 1$, get usual Cayley theorem.

Every meet semilattice can be embedded in a Boolean algebra.
Corollary Every small restriction category $\mathcal{C}$ is isomorphic to a restriction subcategory of $\text{Pfn}$.

Proof idea Cockett and Lack obtained this also. But the same constructions as “Vagner-Preston” give a more direct proof. Discover the details by regarding such a category as a restriction algebra as per earlier example.

Form itempotent completion $\hat{\mathcal{C}}$ of $\mathcal{C}$ so objects are restriction itempotents and maps $\alpha : e \rightarrow f$ satisfy $f\alpha e = \alpha$. Then

$$\hat{\mathcal{C}} \xrightarrow{\psi} \text{Pfn}, \quad \psi e = \{t : et = t\}$$

$$\psi(e \xrightarrow{\alpha} f)t = \begin{cases} \alpha t & \text{if } \alpha ft = t \\ \bot & \text{otherwise} \end{cases}$$
But, to paraphrase Marshall Stone,

One must topologize!
Let $\mathcal{T}$ be a topology of open sets on $X$.

For $A \subset X$ write the closure of $A$ as

$\overline{A}$  no, wait, that’s restriction.

$A^*$ no, wait, that’s the free monoid

$\overset{\_}{A}$

A function $f$ is continuous

$\Leftrightarrow \forall A f(\overset{\_}{A}) \subset (fA)^{\_}$

$\Leftrightarrow \forall A \forall B \overset{\_}{A} = \overset{\_}{B} \Rightarrow (fA)^{\_} = (fB)^{\_}$
A **pospace** is a topological space in which any intersection of open sets is open.

Let **PoSp** be the category of pospaces and continuous maps.

**Proposition** (Lorrain, 1969) The category **PreO** of sets with reflexive and transitive relation and monotone maps is isomorphic over **Set** to **PoSp**.

\[ x \leq y \iff y \in \{x\}^\wedge \text{ (specialization order)} \]

open set = lower set

closed set = upper set

\[ A^\wedge = \uparrow A \]
A right topological semigroup is \((X, \cdot, T)\) with \((X, \cdot)\) a semigroup and \((X, T)\) a topological space such that

\[
\forall x \in X \; \rho_{xy} y = yx \text{ is continuous}
\]

- Use \textit{rts} for right topological semigroup
- Use \textit{rtm} for right topological monoid
The forgetful functor from monoids to semigroups has a left adjoint $S \mapsto S^1$.

Here $S^1 = S + \{1\}$ with $x1 = x = 1x$.

The same is true for $rtm$ and $rts$ (let 1 be an isolated point).
Proposition Let $X$ be rts. Let $C$ be the family of closed subsets of $X$. Then

$$S = X^1 \times C$$

is a restriction algebra if

$$(x, C')(y, D) = (xy, (Cy) \cup D)$$

$$\overline{(x, C')} = (1, C')$$

Call this the full restriction algebra of $X$.

Observation Every semigroup $X$ is a subsemigroup of a restriction algebra $S$ whose restriction itempotents form a Boolean algebra.

For let $S$ be the full restriction algebra of $X$ where $X$ has the discrete topology. Use the embedding $x \mapsto (x, X)$
Let $X$ be any semigroup. \textbf{Green’s left order} is

$$x \leq_L y \iff x \in X^{1}y$$

Being reflexive and transitive, this induces the pospace

$$\hat{A} = \uparrow A = \{y : \exists x \in S^{1} xy \in A\}$$

and $S$ is \textit{rts} because $x = zy \Rightarrow xa = z(ya)$, i.e., right translations are monotone.
Predecessors in Semigroup theory:

- Scheiblich, 1973
- Munn, 1974
- Schein, 1975

**Theorem** (Cockett and Lack) The free re-
striction algebra generated by a semigroup $X$
is the sub-restriction algebra of the full one $X^1 \times C$, $C =$ closed sets of the $L$-topology, of all

$$(x, \{a_1, \ldots, a_n\}), \ x \neq 1 \Rightarrow x \in \{a_1, \ldots, a_n\}$$

The inclusion of the generators is

$$x \mapsto (x, \{x\})$$
The literature on topological semigroups is primarily about the Hausdorff case, often compact Hausdorff.

Here’s a rich supply of compact Hausdorff topological restriction algebras.

Start with a compact Hausdorff monoid $M$. Let $\mathcal{C}$ be the “hyperspace” of closed subsets of $M$ with the “finite topology” (see Vietoris 1923, and Michael, 1951). Then $\mathcal{C}$ is compact Hausdorff.

The full restriction algebra $M \times \mathcal{C}$ is then a compact Hausdorff restriction algebra.
Ai yai yai, another new structure

A **band with restriction** is a semigroup $xy$ with unary $x \mapsto \overline{x}$ satisfying $(R.1), (R.2), (R.3)$ as well as axiom $(\alpha)$ on the board.

**Extremal example** A semilattice $xy = x \land y$ with $\overline{x} = x$. This is the only example if a unit exists – consider axiom $(\alpha)$ with $x = 1$.

**Extremal example** A left zero semigroup $xy = x$ with $\overline{x} = e$ any fixed $e$

**Observation** A restriction algebra satisfies $(\alpha)$ if and only if $\overline{x} = x$, in which case it is a semi-lattice.
We are interested in bands with restriction because there is a forgetful functor over \textbf{Set} from restriction algebras to bands with restriction. Given a restriction algebra \( S \) with multiplication \( xy \) and restriction \( \overline{x} \),

\[ x \ast y = xy \]

with the same restriction gives a band with restriction.

The example of partial functions \( X \rightarrow X \) shows that a great deal of information is lost.
A **band** is a semigroup in which each element is idempotent.

Three extremal cases are

- Left zero semigroup: $xy = x$

- Right zero semigroup: $xy = y$

- Rectangular band: $axa = a$

The varieties of left zero and right zero semigroups are isomorphic to **Set**, the algebras of the identity monad. Rectangular bands are the algebras of **id** × **id**.
Let $\mathcal{V}_0$ be the class of all semigroups for which $x$ exists yielding a band with restriction. Let $\mathcal{V}$ be the variety generated by $\mathcal{V}_0$

**Theorem** $\mathcal{V}$ is the variety of all left normal bands:

\[
x^2 = x \\
axy = ayx
\]
Proof Idea C. F. Fennemore, 1971 classified all varieties of bands. Consider the band with restriction \( \{0, \alpha, a\} \) with

\[
\begin{align*}
0x &= x &= 0 \\
\alpha\alpha\alpha &= \alpha\alpha &= a \\
\alpha\alpha &= \alpha\alpha &= \alpha
\end{align*}
\]

and with restriction

\[
\overline{0} = 0, \quad \overline{\alpha} = \overline{a} = a
\]
Example The free left normal band generated by \( \{a, b\} \) has multiplication table

\[
\begin{array}{c|cccc}
 & a & b & ab & ba \\
\hline
a & a & ab & ab & ab \\
ba & ba & b & ba & ba \\
ab & ab & ab & ab & ab \\
ba & ba & ba & ba & ba \\
\end{array}
\]

No \( x \mapsto \overline{x} \) exists making this a band with restriction.
Let $\mathcal{W}$ be any class of semigroups. A semigroup $S$ is a **semilattice of type** $\mathcal{W}$ if there exists a semilattice $L$ and a surjective semigroup homomorphism $\psi : S \to L$ such that each $\psi^{-1}(e)$ (obviously a subsemigroup of $S$ is in $\mathcal{W}$.

Thus $S$ is partitioned into subsemigroups $S_e = \psi^{-1}(e)$ with $S_e S_f \subset S_{ef}$.

**Example** every semilattice is a semilattice of groups.
**Theorem** (Clifford 1941, McLean 1954) Every band is a semilattice of rectangular bands.
The following strengthening is due to Clifford:

Let $L$ be a meet semilattice and let

$$F : (L, \leq)^{op} \rightarrow \text{Semigroups}$$

be a functor. Let

$$S = \bigsqcup_{e \in L} F_e$$

Then $S$ is a semigroup with multiplication

$$x \in F_e, y \in F_f \mapsto xy = F_{e,ef}(x)F_{f,ef}(y)$$

a product in the semigroup $F(ef)$.

Such $S$ is a **strong semilattice** of the semigroups $F_e$. 
A band is **normal** if $axya = ayxa$. Note that every rectangular band is normal.

Left normal (recall $axy = ayx$) is **stronger** than normal.

**Theorem** (Yamada and Kimura 1958) The normal bands are precisely the strong semilattices of rectangular bands.

**Corollary** The left normal bands are precisely the strong semilattices of left zero semigroups.
We can now characterize $\mathcal{V}_0$, the class of semigroups of bands with restriction.

A semilattice of semigroups $\psi : S \rightarrow L$ is **split** if $\psi$ is split epic in the category of semigroups.

**Theorem** A semigroup has the structure of a band with restriction if and only if it is a split strong semilattice of left zero semigroups.
A band with restriction is a partially ordered semigroup via

\[ x \leq y \text{ if } yx = x \]

Notice that the order on a restriction algebra is exactly this order on its underlying band with restriction.

Homomorphisms of bands with restriction are monotone.
**Theorem** The category of restriction algebras and monotone maps is cartesian closed. The category of bands with restriction and monotone maps is cartesian closed.

**Theorem** (Linton 1966) The variety of bands with restriction is a symmetric monoidal closed category.
For any partially ordered semigroup $S$, the **negative cone** $N(S)$ is defined by

$$N(S) = \{ x \in S : \forall y \ xy \leq y, \ yx \leq y \}$$

For any semigroup $S$, let its center be denoted $Z(S)$. If $S$ is a restriction algebra or a band with restriction, let the set of restriction idempotents of form $\overline{x}$ be denoted $R(S)$.

**Proposition** For bands with restriction,

$$N(S) = Z(S) \subset R(S)$$

For restriction algebras,

$$N(S) = R(S)$$
You did it!

You got through

56 slides!

Quiz tonight at 3 AM