

Internal Traces and Abstract Observations

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and categorical logic

Outline

- 1 Introduction
- 2 Stochastic Relations
- 3 TNCPM
- 4 Internal Traces
- 5 Properties of Internal Trace
- 6 Observations

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Summary

- Two interesting categories **SRel** (stochastic relations) and **TNCPM** (trace-nonincreasing completely positive maps).
- **SRel** is a natural category for describing probabilistic processes. **TNCPM** should be a natural category for quantum mechanics.
- Unfortunately, they do not have the traditional notion of trace.
- We propose a new construct that we call “internal traces.”

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Two Nice (?) Categories

Probabilistic Processes

SRel

Stochastic relations

Quantum Processes

TNCPM

Trace-nonincreasing
completely-positive maps

Both categories have monoidal structure, scalars ($Hom(I, I)$) and partially additive structure. They seem ideal for the analysis of probabilistic processes and quantum processes respectively.

But they do not have trace!

What is **SRel**? Part I

- **Mes**: Objects are measurable spaces (X, Σ_X) , X a set, Σ_X a σ -algebra on X . Morphisms $f : (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ are measurable functions.
- Define monad $\Pi : \mathbf{Mes} \rightarrow \mathbf{Mes}$ by

$$\Pi(X) = \{\nu \mid \nu : X \rightarrow [0, 1]\}$$

where ν is a (sub)probability measure.

- $\Pi(f)(\nu) = \nu \circ f^{-1}$.
- **SRel** is the Kleisli category of this monad.
- Π is a probabilistic powerset and **SRel** represents “stochastic relations.” [Lawvere 64, Giry 81]

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What is **SRel**? Part II

- An arrow, $h : X \rightarrow Y$, of **SRel** looks like $h : X \times \Sigma_Y \rightarrow [0, 1]$ where $h(x, \cdot)$ is a measure and $h(\cdot, A)$ is measurable.
- Composition is given by integration: $h : X \rightarrow Y$ and $k : Y \rightarrow Z$

$$(k \circ h)(x, C) = \int_Y k(y, C) h(x, dy).$$

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Why **SRel**?

- A categorical treatment of stochastic processes. A stochastic process is just a functor into **SRel**. The so-called Chapman-Kolmogorov equation is just functoriality and the notion of composition in this category.
- Played a significant role in the treatment of Labelled Markov Processes [Blute, Desharnais, Edalat, P., LICS97,98;IC 2002].
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Why no Trace in SRel?

- Consider $h : X \otimes U \rightarrow Y \otimes U$ in **SRel**.
- $h(x, u; B \times V)$, where $B \subseteq Y$ and $V \subseteq U$ are measurable subsets.
- The trace should be something like

$$\text{Tr}_U(h)(x, B) = \int_U h(x, u(?); B \otimes \text{“du”}).$$

- But this makes no sense! What “u” should we pick?
- Note: there *is* a “particle-style” trace.

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TNCPM

- **Objects:** Finite Dimensional Hilbert spaces.
- **Morphisms:** $f : \mathcal{H} \rightarrow \mathcal{K}$ linear, trace-nonincreasing, completely-positive map from $D(\mathcal{H}) \rightarrow D(\mathcal{K})$, where $D(\mathcal{H})$ stands for density matrices over \mathcal{H} .
- They can have trace less than 1.
- If we just had completely-positive maps then the category would be traced.
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Intuitions

- We would like to have a trace defined on “objects” that will allow us to compute a trace for global elements.
- We want $\text{tr} : X \rightarrow I$ so that for any element $x : I \rightarrow X$ we get a scalar $\text{tr} \circ x : I \rightarrow I$, we write this as $\text{tr}(x)$.
- We write tr_X for these internal traces as opposed to Tr for the usual trace.

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Axioms for internal traces

Defined on monoidal categories with zero morphisms and coproducts.

Morphisms $\text{tr}_X : X \rightarrow I$

Some obvious conditions:

$$\text{tr}_I = 1_I, \text{tr}_0 = 0,$$

$$\begin{array}{ccc} X & \xrightarrow{\text{tr}_X} & I \\ \lambda_X \downarrow & \nearrow \text{tr}_{I \otimes X} & \\ I \otimes X & & \end{array}$$

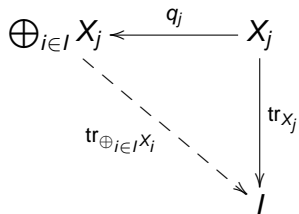
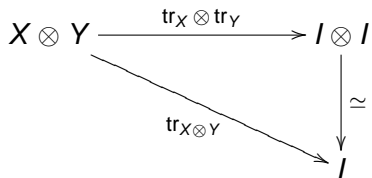
$$\begin{array}{ccc} X & \xrightarrow{\text{tr}_X} & I \\ \rho_X \downarrow & \nearrow \text{tr}_{X \otimes I} & \\ X \otimes I & & \end{array}$$

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\text{tr}_{X \otimes Y}} & I \\ \sigma_{XY} \downarrow & \nearrow \text{tr}_{Y \otimes X} & \\ Y \otimes X & & \end{array}$$

Coherence with associativity

$$\begin{array}{ccc}
 (X \otimes Y) \otimes Z & \xrightarrow{\text{tr}_{(X \otimes Y) \otimes Z}} & I \\
 \downarrow \alpha_{XYZ} & & \nearrow \text{tr}_{X \otimes (Y \otimes Z)} \\
 X \otimes (Y \otimes Z) & &
 \end{array}$$

Interaction with tensor and coproduct:



Internal trace for SRel

$$\text{tr}_X(\mathbf{x}, \{*\}) = 1 \quad \text{tr}_X(\mathbf{x}, \emptyset) = 0.$$

Given $h : I \rightarrow X$ (which is just a measure) by the definition of **SRel** composition

$$\text{tr}(h) = \int_X h(*, dx)$$

i.e. the total weight of h .

Internal trace for TNCPM

- Now we don't want to compute the trace of a completely-positive map,
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Something is missing!

- There is a missing condition.
- In **FDHilb** there is an internal trace

$$\sum_{i=1}^n \langle i |$$

is a trace.

- But this makes no sense! The internal trace should only be defined on “matrix like” things.
- Thanks Eric!

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Traces of global elements are abstract scalars:

$$\text{Tr}_X(u) : I \xrightarrow{u} X \xrightarrow{\text{tr}_X} I$$

Partial traces are defined from traces:

$$\text{tr}_X^{XY} : X \otimes Y \xrightarrow{\text{tr}_X \otimes 1} I \otimes Y \xrightarrow{\lambda^{-1}} Y$$

- These internal trace morphisms are applied by composition, and the result is an abstract scalar (in the example case of probabilities).
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- Partial traces correspond to to operation of discarding part of a state in the quantum case, and to taking marginal distributions in the probabilistic case.

Properties of internal traces 1

- $\text{tr}_X(s \cdot u) = s \cdot (\text{tr}_X u)$
where $s \in \text{Hom}(I, I)$, $x \in \text{Hom}(X, Y)$ and
 $s \cdot u \in \text{Hom}(X, Y)$ is given by

$$s \cdot u : X \xrightarrow{\cong} I \otimes X \xrightarrow{s \otimes x} I \otimes Y \xrightarrow{\cong} Y.$$

- $\text{tr}_X(u + v) = (\text{tr}_X u) + (\text{tr}_X v)$
- $\text{tr}_I(s) = s$
- Monoidal coherence isomorphisms preserve traces.

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Properties of internal traces 2

- $\text{tr}_{X \otimes Y}(u \boxtimes v) = (\text{tr}_X u) \cdot (\text{tr}_Y v)$
where, $u : I \rightarrow X$, $v : I \rightarrow Y$ and $(u \boxtimes v) : I \rightarrow X \otimes Y$ is

$$(u \boxtimes v) : I \xrightarrow{\cong} I \otimes I \xrightarrow{v \otimes v} X \otimes Y.$$

- For categories equipped with a partial additive structure:

$$\text{tr}_{X \otimes Y}(u \oplus v) = \text{tr}_X u \oplus \text{tr}_Y v.$$

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Observations

Observation on X :

- A state is a trace 1 global element.
- set of observation results M
- family of morphisms $f_m : X \rightarrow X$; how the state is modified by an observation producing the result m .
- For all global elements $u : I \rightarrow X$ we have

$$\sum_m P(m|u) = 1,$$

where

$$P(m|u) = \text{tr}_X f_m u.$$

- $P(m|u)$ is the probability of observing m in the global state u .

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- family of morphisms $f_m : X \rightarrow X$; how the state is modified by an observation producing the result m .
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where

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Family of observations \mathcal{O} such that

- \mathcal{O} contains all trivial observations (there is only outcome and the morphism is the identity),
- \mathcal{O} is closed under composition, conjugations by isomorphisms and extensions.
- If u, v are two states such that for all observations in \mathcal{O} , we have $P(m|u) = P(m|v)$, then $u = v$.

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