Internal Traces and Abstract Observations

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Outline

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Summary

- Two interesting categories **SRel** (stochastic relations) and **TNCPM** (trace-nonincreasing completely positive maps).
- **SRel** is a natural category for describing probabilistic processes. **TNCPM** should be a natural category for quantum mechanics.
- Unfortunately, they do not have the traditional notion of trace.
- We propose a new construct that we call "internal traces."

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Two Nice (?) Categories

Probabilistic Processes	**Quantum Processes**
SRel	TNCPM
Stochastic relations	Trace-nonincreasing completely-positive maps

Both categories have monoidal structure, scalars ($Hom(I, I)$) and partially additive structure. They seem ideal for the analysis of probabilistic processes and quantum processes respectively.

But they do not have trace!

What is **SRel**? Part I

- **Mes**: Objects are measurable spaces (X, Σ_X), X a set, Σ_X a σ-algebra on X. Morphisms $f : (X, \Sigma_X) \longrightarrow (Y, \Sigma_Y)$ are measurable functions.
- Define monad Π : **Mes** $\longrightarrow$ **Mes** by

$$\Pi(X) = \{\nu | \nu : X \longrightarrow [0, 1]\}$$

  where ν is a (sub)probability measure.
- $\Pi(f)(\nu) = \nu \circ f^{-1}$.
- **SRel** is the Kleisli category of this monad.
- Π is a probabilistic powerset and **SRel** represents "stochastic relations." [Lawvere 64, Giry 81]

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What is **SRel**? Part II

- An arrow, $h : X \longrightarrow Y$, of **SRel** looks like $h : X \times \Sigma_Y \longrightarrow [0, 1]$ where $h(x, \cdot)$ is a measure and $h(\cdot, A)$ is measurable.
- Composition is given by integration: $h : X \longrightarrow Y$ and $k : Y \longrightarrow Z$

$$(k \circ h)(x, C) = \int_Y k(y, C) h(x, dy).$$

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Why **SRel**?

- A categorical treatment of stochastic processes. A stochastic process is just a functor into **SRel**. The so-called Chapman-Kolmogorov equation is just functoriality and the notion of composition in this category.
- Played a significant role in the treatment of Labelled Markov Processes [Blute, Desharnais, Edalat, P., LICS97,98;IC 2002].
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Why no Trace in **SRel**?

- Consider $h : X \otimes U \longrightarrow Y \otimes U$ in **SRel**.
- $h(x, u; B \times V)$, where $B \subseteq Y$ and $V \subseteq U$ are measurable subsets.
- The trace should be something like

$$\mathrm{Tr}_U(h)(x, B) = \int_U h(x, u(?); B \otimes \text{“du”}).$$

- But this makes no sense! What “u” should we pick?
- Note: there *is* a “particle-style” trace.

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TNCPM

- Objects: Finite Dimensional Hilbert spaces.
- Morphisms: $f : \mathcal{H} \longrightarrow \mathcal{K}$ linear, trace-nonincreasing, completely-positive map from $\mathrm{D}(\mathcal{H}) \longrightarrow \mathrm{D}(\mathcal{K})$, where $\mathrm{D}(\mathcal{H})$ stands for density matrices over $\mathcal{H}$.
- They can have trace less than 1.
- If we just had completely-positive maps then the category would be traced.
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Intuitions

- We would like to have a trace defined on “objects” that will allow us to compute a trace for global elements.
- We want $\mathrm{tr} : X \longrightarrow I$ so that for any element $x : I \longrightarrow X$ we get a scalar $\mathrm{tr} \circ x : I \longrightarrow I$, we write this as $\mathrm{tr}(x)$.
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Axioms for internal traces

Defined on monoidal categories with zero morphisms and coproducts.
Morphisms $\mathrm{tr}_X : X \longrightarrow I$
Some obvious conditions:

$$\mathrm{tr}_I = 1_I, \ \mathrm{tr}_0 = 0,$$

$$\begin{tikzcd} X \arrow[r, "\mathrm{tr}_X"] \arrow[d, "\lambda_X"'] & I \\ I \otimes X \arrow[ur, "\mathrm{tr}_{I\otimes X}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{tr}_X"] \arrow[d, "\rho_X"'] & I \\ X \otimes I \arrow[ur, "\mathrm{tr}_{X\otimes I}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "\mathrm{tr}_{X\otimes Y}"] \arrow[d, "\sigma_{XY}"'] & I \\ Y \otimes X \arrow[ur, "\mathrm{tr}_{Y\otimes X}"'] & \end{tikzcd}$$

Coherence with associativity

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[rr, "\mathrm{tr}_{(X \otimes Y) \otimes Z}"] \arrow[d, "\alpha_{XYZ}"'] & & I \\
X \otimes (Y \otimes Z) \arrow[urr, "\mathrm{tr}_{X \otimes (Y \otimes Z)}"'] & &
\end{tikzcd}$$

Interaction with tensor and coproduct:

$$\begin{tikzcd} X \otimes Y \arrow[r, "\mathrm{tr}_X \otimes \mathrm{tr}_Y"] \arrow[dr, "\mathrm{tr}_{X \otimes Y}"'] & I \otimes I \arrow[d, "\simeq"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} \bigoplus_{i \in I} X_j \arrow[dr, dashed, "\mathrm{tr}_{\oplus_{i \in I} X_i}"'] & X_j \arrow[l, "q_j"'] \arrow[d, "\mathrm{tr}_{X_j}"] \\ & I \end{tikzcd}$$

Internal trace for SRel

$$\mathrm{tr}_X(x, \{*\}) = 1 \quad \mathrm{tr}_X(x, \emptyset) = 0.$$

Given $h : I \longrightarrow X$ (which is just a measure) by the definition of **SRel** composition

$$\mathrm{tr}(h) = \int_X h(*, \mathrm{d}x)$$

i.e. the total weight of h.

Internal trace for TNCPM

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- But this is just the plain freshman linear algebra trace!

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Something is missing!

- There is a missing condition.
- In **FDHilb** there is an internal trace

$$\sum_{i=1}^{n} \langle i |$$

is a trace.

- But this makes no sense! The internal trace should only be defined on "matrix like" things.
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The Fix (?)

- We can fix the problem in some cases by requiring invariance of tr_X under unitaries.
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Traces of global elements are abstract scalars:

$$Tr_X(u) : I \xrightarrow{u} X \xrightarrow{\mathrm{tr}_X} I$$

Partial traces are defined from traces:

$$\mathrm{tr}_X^{XY} : X \otimes Y \xrightarrow{\mathrm{tr}_X \otimes 1} I \otimes Y \xrightarrow{\lambda^{-1}} Y$$

- These internal trace morphisms are applied by composition, and the result is an abstract scalar (in the example case of probabilities).
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Properties of internal traces 1

- $\mathrm{tr}_X(s \cdot u) = s \cdot (\mathrm{tr}_X\, u)$
  where $s \in Hom(I, I), x \in Hom(X, Y)$ and
  $s \cdot u \in Hom(X, Y)$ is given by

$$s \cdot u : X \xrightarrow{\simeq} I \otimes X \xrightarrow{s \otimes x} I \otimes Y \xrightarrow{\simeq} Y.$$

- $\mathrm{tr}_X(u + v) = (\mathrm{tr}_X\, u) + (\mathrm{tr}_X\, v)$
- $\mathrm{tr}_I(s) = s$
- Monoidal coherence isomorphisms preserve traces.

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Properties of internal traces 2

- $\mathrm{tr}_{X \otimes Y}(u \boxtimes v) = (\mathrm{tr}_X\, u) \cdot (\mathrm{tr}_Y\, v)$
  where, $u : I \longrightarrow X, v''I \longrightarrow Y$ and $(u \boxtimes v) : I \longrightarrow X \otimes Y$ is

$$(u \boxtimes v) : I \xrightarrow{\simeq} I \otimes I \xrightarrow{v \otimes v} X \otimes Y .$$

- For categories equipped with a partial additive structure:

$$\mathrm{tr}_{X \otimes Y}(u \oplus v) = \mathrm{tr}_X\, u \oplus \mathrm{tr}_Y\, v .$$

- Both **SRel** and **TNCPM** have partial additive structure.

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Properties of internal traces 3

- $\mathrm{tr}_Y^{XY}(u) = (\sigma_{XY}^{-1}\,\mathrm{tr}_X^{XY}\,\sigma_{XY})u$
- $\mathrm{tr}_Y^{XY}(s \cdot u) = s \cdot \mathrm{tr}_Y^{XY}\,u$
- $\mathrm{tr}_Y^{XY}(u \boxtimes v) = (\mathrm{tr}_X\,u) \cdot v$
- If we have partial additive structure then
- $\mathrm{tr}_Y^{XY}(u + v) = \mathrm{tr}_Y^{XY}\,u + \mathrm{tr}_Y^{XY}\,v$

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Observations

Observation on X:

- A state is a trace 1 global element.
- set of observation results M
- family of morphisms $f_m : X \longrightarrow X$; how the state is modified by an observation producing the result m.
- For all global elements $u : I \longrightarrow X$ we have

$$\sum_m P(m|u) = 1,$$

where

$$P(m|u) = \mathrm{tr}_X\, f_m u.$$

- $P(m|u)$ is the probability of observing m in the global state u.

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Observation on X:

- A state is a trace 1 global element.
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- family of morphisms $f_m : X \longrightarrow X$; how the state is modified by an observation producing the result m.
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Family of observations $\mathcal{O}$ such that

- $\mathcal{O}$ contains all trivial observations (there is only outcome and the morphism is the identity),
- $\mathcal{O}$ is closed under composition, conjugations by isomorphisms and extensions.
- If u, v are two states such that for all observations in $\mathcal{O}$, we have $P(m|u) = P(m|v)$, then $u = v$.

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