Traces in Functional Analysis and Operator Algebras

Matthias Neufang
Carleton University, Ottawa

I. Traces in Banach space theory

1. The projective tensor product

Let E be a Banach space. Then the dual pair (E^*, E) determines the **trace** Tr on $E^* \otimes E$ via

$$\mathrm{Tr}(\phi \otimes x) = \langle \phi, x \rangle.$$

Definition. *Let E, F be Banach spaces. The* ***projective tensor product*** *$E \otimes_\gamma F$ is the completion of $E \otimes F$ w.r.t. the norm*

$$\|z\|_\gamma = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| \mid n \in \mathbb{N}, z = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

Alternatively: $(E \otimes_\gamma F)^* = \mathcal{B}(E, F^*)$

Putting on operator glasses, we have a natural map

$$j : E^* \otimes E \to \mathcal{B}(E)$$

given by $j(\phi \otimes x)(y) = \langle \phi, y \rangle x$. Consider its extension

$$J : E^* \otimes_\gamma E \to \mathcal{B}(E).$$

Grothendieck defined the image $E^* \otimes_\gamma E/Ker(J)$ with the quotient norm, as the space of **nuclear operators** $\mathcal{N}(E)$.

Since $|\langle \phi, x \rangle| \leq \|\phi\| \, \|x\|$, the trace Tr extends to a functional on $E \otimes_\gamma E^*$.

If J is injective, then the trace is defined for all nuclear operators. When is this the case?

Theorem. *J is injective iff E has the approximation property (AP).*

Definition. *E has the AP if for any compact set $K \subseteq E$ and $\varepsilon > 0$ there is a finite-rank map $S : E \to E$ such that $\|S(x) - x\| < \varepsilon$ for all $x \in K$.*

$\rightsquigarrow$ All "usual" spaces – such as c_0, $C(K)$, L_p – have the AP; but NOT $\mathcal{B}(\mathcal{H})$!!

2. The Schatten ideals

The nuclear operators on a Hilbert space $\mathcal{H}$ are usually called **trace class operators** $\mathcal{T}(\mathcal{H})$. By the above,

$$\mathcal{T}(\mathcal{H}) = \mathcal{H} \otimes_\gamma \mathcal{H}.$$

Alternative description: Every $S \in \mathcal{K}(\mathcal{H})$ has the form

$$S = \sum_{i=1}^{\infty} s_i \langle \cdot, \xi_i \rangle \eta_i$$

for appropriate ONS (ξ_i), (η_i). Here, $s_i(S) \geq 0$ are the eigenvalues of $|S| = (S^* S)^{1/2}$ (= singular values).

Now, $S \in \mathcal{K}(\mathcal{H})$ is trace class iff $(s_i(S)) \in \ell_1$. Then $\|S\|_\gamma = \|(s_i(S))\|_1$.

Example: Operator on ℓ_2 of multiplication with ℓ_1-function

The trace $\mathrm{Tr}(\rho)$ for $\rho \in \mathcal{T}(\mathcal{H})$ can be expressed as

$$\mathrm{Tr}(\rho) = \sum_{i=1}^{\infty} \langle \rho \xi_i, \xi_i \rangle$$

with any ONB of $\mathcal{H}$.

One defines the **Schatten class** $S_p(\mathcal{H})$ as the collection of $S \in \mathcal{K}(\mathcal{H})$ with $(s_i(S)) \in \ell_p$ ($1 < p < \infty$), with the norm $\|S\|_p := \|(s_i(S))\|_p$. One sets $S_\infty(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

- $S_1(\mathcal{H}) \subseteq S_p(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$
- $S_p(\mathcal{H})$ form 2-sided ideals in $\mathcal{B}(\mathcal{H})$
- $S_p(\mathcal{H})^* = S_q(\mathcal{H})$ where $\frac{1}{p} + \frac{1}{q} = 1$
- the trace gives the dualities $\mathcal{K}(\mathcal{H})^* = \mathcal{T}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$
- $\mathcal{B}(\mathcal{H})^* = \mathcal{T}(\mathcal{H}) \oplus_1 \mathcal{K}(\mathcal{H})^\perp$

3. Convolution of trace class operators

commutative world: $L_1(\mathcal{G})$	noncommutative world: $\mathcal{T}(L_2(\mathcal{G}))$
pointwise product in $(\ell_1, \cdot)$	composition in $(\mathcal{T}(\ell_2(\mathcal{G})), \cdot)$
convolution in $(L_1(\mathcal{G}), *)$	???

Let $\mathcal{G}$ be a locally compact group. Then

$$L_\infty(\mathcal{G}) \hookrightarrow \mathcal{B}(L_2(\mathcal{G}))$$

as multiplication operators. The pre-adjoint of this embedding gives a canonical quotient map

$$\pi : \mathcal{T}(L_2(\mathcal{G})) = L_2(\mathcal{G}) \otimes_\gamma L_2(\mathcal{G}) \to L_1(\mathcal{G}).$$

We have $\pi(\xi \otimes \eta) = \xi\overline{\eta}$.

Definition. [N.] *For trace class operators* $\rho, \tau \in \mathcal{T}(L_2(\mathcal{G}))$ *we define*

$$\rho * \tau = \int_{\mathcal{G}} L_x \rho L_{x^{-1}} \pi(\tau)(x) d\lambda(x).$$

Then we obtain a ***new associative product*** *on the space* $\mathcal{T}(L_2(\mathcal{G}))$*. This defines a* ***convolution of trace class operators****!*

Your (legitimate) objection: Does this indeed extend classical convolution to the non-commutative context?

Let's convolve matrices!

If $A = [A_{i,j}]$ and $B = [B_{k,l}]$ are matrices, then their convolution product $C = A * B$ is given by

$$C_{i,j} = \sum_{t \in \mathcal{G}} A_{t^{-1}i, t^{-1}j} B_{t,t}.$$

Thus, in case A and B are just functions (i.e., diagonal matrices), C as well is a diagonal matrix, namely

$$C_i = \sum_{t \in \mathcal{G}} A_{t^{-1}i} B_t.$$

So $(\ell_1, *)$ becomes a **subalgebra** of $(\mathcal{T}(\ell_2), *)$.

Some "amuse-gueules" for the study of

The Non-Commutative Convolution Algebra
$\boldsymbol{\mathcal{S}_1(\mathcal{G}) = (\mathcal{T}(L_2(\mathcal{G})), *)}$

- If $\mathcal{S}_1(\mathcal{G}) \cong \mathcal{S}_1(\mathcal{G}')$, then $\mathcal{G} \cong \mathcal{G}'$.
- $\mathcal{G}$ **amenable** $\Longleftrightarrow$ $\mathcal{S}_1(\mathcal{G})$ right amenable (à la Lau)
- The dual space $\mathcal{S}_\infty(\mathcal{G}) = \mathcal{S}_1(\mathcal{G})^*$ becomes a **Hopf–von Neumann algebra**; its comultiplication extends the one of $L_\infty(\mathcal{G}) = L_1(\mathcal{G})^*$.
  Thus, $\mathcal{S}_1(\mathcal{G})$ is an **operator convolution algebra** in the sense of Effros–Ruan.

$$\begin{aligned} \int (f * g)\, d\lambda &= \int f d\lambda \int g d\lambda \\ \mathrm{Tr}(\rho * \tau) &= \mathrm{Tr}(\rho)\, \mathrm{Tr}(\tau) \end{aligned}$$

- $\mathcal{S}_1(\mathcal{G})$ yields a Banach algebra **extension** of $L_1(\mathcal{G})$:

$$0 \to \mathcal{ML}_\infty(\mathcal{G})_\perp \xrightarrow{\iota} (\mathcal{T}(L_2(\mathcal{G})), *) \xrightarrow{\pi} (L_1(\mathcal{G}), *) \to 0$$

  The ideal $I := \mathcal{ML}_\infty(\mathcal{G})_\perp$ satisfies $I^2 = (0)$.
  In case $\mathcal{G}$ is discrete, $(\mathcal{S}_1(\mathcal{G}), \pi, \iota)$ is a **singular extension** of the group algebra $(\ell_1(\mathcal{G}), *)$.

Some more homology theory

Talk at **International Conference on Banach Algebras 2001** in Odense, Denmark:
Presentation of my $\mathcal{S}_1(\mathcal{G})$ to the Banach algebra community

$$\text{homological properties of } \boldsymbol{\mathcal{S}_1(\mathcal{G})} \longleftrightarrow \text{properties of } \boldsymbol{\mathcal{G}}$$

The following results are due in part to Prof. Pirkovskii – a former student of Prof. Helemskii and active member of his "Moscow school" – and in part to myself.

$\mathcal{G}$ is **discrete**	$\Leftrightarrow$	the above extension **splits**
	$\Leftrightarrow$	$L_1(\mathcal{G})$ is **projective** in $\text{mod-}\mathcal{S}_1(\mathcal{G})$
$\mathcal{G}$ is **compact**	$\Leftrightarrow$	$\mathbb{C}$ is **projective** in $\text{mod-}\mathcal{S}_1(\mathcal{G})$
$\mathcal{G}$ is **finite**	$\Leftrightarrow$	$\mathcal{S}_1(\mathcal{G})$ is **biprojective**
$\mathcal{G}$ is **amenable**	$\Leftrightarrow$	$\mathcal{S}_1(\mathcal{G})$ is **biflat**
	$\Leftrightarrow$	$\mathbb{C}$ is **flat** in $\text{mod-}\mathcal{S}_1(\mathcal{G})$

Alternative view through Hopf algebra glasses

Definition. *A* *Hopf–von Neumann algebra* *is a pair* (M, Γ) *where*

- M *is a von Neumann algebra;*
- $\Gamma : M \longrightarrow M\overline{\otimes}M$ *is a co-multiplication, i.e., a normal, unital, isometric* $*$*-homomorphism satisfying* $(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma$ *(co-associativity).*

Extend **comultiplication** of the Hopf–vN (even Kac) algebra $L_\infty(\mathcal{G})$ to $\mathcal{B}(L_2(\mathcal{G}))$ by setting

$$\Delta(x) = W(1 \otimes x)W^* \quad (x \in \mathcal{B}(L_2(\mathcal{G})))$$

where W is the **fundamental unitary** for $L_\infty(\mathcal{G})$: $W\xi(s,t) = \xi(s,st)$. Then our product in $\mathcal{S}_1(G) := (\mathcal{T}(L_2(\mathcal{G})), *)$ is precisely Δ_*.

Consider **dual** Kac algebra $VN(G) = \{\lambda(x) \mid x \in \mathcal{G}\}'' =$ group vN algebra. Its predual is the **Fourier algebra** $A(\mathcal{G}) = \{\langle \lambda(\cdot)\xi, \eta \rangle \mid \xi, \eta \in L_2(\mathcal{G})\}$.

Then using $\widehat{W}$ yields a "pointwise" product on $\mathcal{T}(L_2(\mathcal{G}))$.

We recover $(L_1(G), *)$ and $(A(G), \cdot)$ as complete quotient **algebras** of $(\mathcal{T}(L_2), *)$ resp. $(\mathcal{T}(L_2), \bullet)$:

$$(\mathcal{T}(L_2), *) \twoheadrightarrow (L_1(G), *)$$

$$(\mathcal{T}(L_2), \bullet) \twoheadrightarrow (A(G), \cdot)$$

• $\mathcal{T}(L_2(\mathcal{G}))$ carries 2 dual structures: a "**convolution**" and a "**pointwise**" product!

• We have: $\mathrm{Tr}(\rho * \tau) = \mathrm{Tr}(\rho \bullet \tau) = \mathrm{Tr}(\rho)\mathrm{Tr}(\tau)$

$\rightsquigarrow$ This can even be done over **locally compact quantum groups** (Junge–N.–Ruan)!

II. Traces in operator space theory

1. What are operator spaces?

Modern answer:

www.math.uni-sb.de/~ag-wittstock/projekt2001.html

Short answer:

Definition. *A concrete operator space is a subspace X of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Simple observation: X inherits the structure of $\mathcal{B}(\mathcal{H})$ on each **matrix level**, i.e., we have isometrically

$$M_n(X) \subseteq M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \ldots \oplus \mathcal{H}) = \mathcal{B}(\mathcal{H}^n)$$

for every $n \in \mathbb{N}$.

The norms $\|\cdot\|_n$ obtained on $M_n(X)$ satisfy **Ruan's axioms**:

(R 1) $\|a \cdot x \cdot b\|_n \leq \|a\|_{M_n} \, \|x\|_n \, \|b\|_{M_n}$

for all $x \in M_n(E)$, $a, b \in M_n$

(R 2) $\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$

for all $x \in M_n(E), y \in M_m(E)$

Definition. [Ruan '88] *An abstract operator space is a Banach space which carries a sequence of (matrix) norms satisfying (R1) and (R2).*

Theorem. [Ruan '88] *Every abstract operator space is a concrete one.*

For any Banach space E, we have an isometric embedding

$$\iota : \; E \hookrightarrow \mathrm{C}(\Omega)$$

where $\Omega = BALL(E^*)$. Here,

$$(\iota(x))(\xi) = \langle \xi, x \rangle .$$

But $\mathrm{C}(\Omega)$ is the **prototype of a commutative C^*-algebra**. Hence we obtain the following scheme:

Banach spaces	**Operator spaces**
$E \hookrightarrow \mathcal{A}$ C^*-algebra	$E \hookrightarrow \mathcal{A}$ C^*-algebra
$\mathcal{A}$ **commutative**	$\mathcal{A}$ **non-commutative**

Another nice way of distinguishing both concepts is the following.

E Banach space	$\longleftrightarrow$	norm $\|\cdot\|_E$
$E \subseteq \mathcal{B}(\mathcal{H})$ operator space	$\longleftrightarrow$	sequence of matrix norms $(\|\cdot\|_{M_n(E)})$

We now define, for each $x \in \bigcup_n M_n(E)$:

$$\begin{aligned} \|x\|_\infty &:= \lim_n \|x\|_{M_n(E)} \\ K(E) &:= \overline{\bigcup_n M_n(E)}^{\|\cdot\|_\infty}. \end{aligned}$$

It is crucial to note that even if dim$E < \infty$, we have

$$\sup_n \text{ dim } M_n(E) = \infty, \text{ i.e., dim } K(E) = \infty.$$

Thus, the **"non-commutative" unit ball** is **NON compact**. The reason for this is that we replace the scalars $\mathbb{C}$ by the $n \times n$-matrices (i.e., the "scalars" of quatum mechanics) – or, equivalently, by the elements of

$$\mathcal{K} = K(\mathbb{C}) = \overline{\bigcup_n M_n(\mathbb{C})}^{\|\cdot\|_\infty}.$$

Summarizing we obtain the following picture:

E Banach space $\longleftrightarrow$ $\|\sum \lambda_i x_i\|_E$
where $\lambda_i \in \mathbb{C}$, $x_i \in E$

$E \subseteq \mathcal{B}(\mathcal{H})$ operator space $\longleftrightarrow$ $\|\sum \lambda_i \otimes x_i\|_{K(E)}$
where $\lambda_i \in \mathcal{K}$, $x_i \in E$

Philosophy: Operator space theory studies the many ways a Banach space can "sit" in $\mathcal{B}(\mathcal{H})$.

We saw that every Banach space sits isometrically in some (commutative) C^*-algebra $\mathrm{C}(\Omega)$, and hence, in particular, in some $\mathcal{B}(\mathcal{H})$.

Hence, every Banach space can be "realized" as an operator space in at least one manner. Thus, in the **category of operator spaces**, the Banach spaces appear as **objects**. But the **morphisms** are different; these are the

completely bounded operators

instead of merely bounded (linear) operators. – They are exactly the operators on the space E which take into account all the matrix levels $M_n(E)$.

More precisely, let E and F be operator spaces, and $\Phi : E \longrightarrow F$ a bounded (linear) operator. Then we define the **n-th amplification**

$$\begin{array}{cccc} \Phi^{(n)} : & M_n(E) & \longrightarrow & M_n(F) \\ & [a_{ij}] & \mapsto & [\Phi(a_{ij})] \ . \end{array}$$

We say that Φ is **completely bounded** if

$$\|\Phi\|_{\mathrm{cb}} := \sup_n \|\Phi^{(n)}\| < \infty.$$

It is readily checked that $\| \cdot \|_{\mathrm{cb}}$ is a norm, and it is thus just a functional analytic reflex to consider the Banach space

$$\mathcal{CB}(E, F) := \{\Phi : E \longrightarrow F \mid \Phi \text{ completely bounded}\}.$$

By the way – in case $E = F$, this is of course a Banach algebra.

Examples

First we discuss the morphisms.

Completely bounded maps

- Let E be an operator space, and $\mathcal{A}$ a **commutative** C^*-algebra. Then every bounded linear operator $\Phi : X \longrightarrow \mathcal{A}$ is CB with $\|\Phi\|_{\rm cb} = \|\Phi\|$.
- For the **transposition map** $\tau_n : M_n \longrightarrow M_n$, we have $\|\tau_n\|_{\rm cb} = n$. Hence, the transposition map τ on $\mathcal{K}$ is **NOT** completely bounded.
- Consider two operator spaces $E \subseteq \mathcal{B}(\mathcal{H}_1)$ and $F \subseteq \mathcal{B}(\mathcal{H}_2)$. Let $\Phi : E \longrightarrow F$ have the form

  $$\Phi(x) = a\pi(x)b$$

  where $\pi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H})$ is a C^*-representation, and $a : \mathcal{H} \to \mathcal{H}_2$ and $b : \mathcal{H}_2 \to \mathcal{H}$ are bounded operators. Then Φ is CB with $\|\Phi\|_{\rm cb} \le \|a\| \, \|b\|$.

In fact, the last example is the **prototype** of a CB map.

Now let's give examples for the objects of our category!

Operator spaces

- **Row Hilbert space** $R := \overline{\mathrm{lin}}\{e_{1j} \mid j = 1, 2, \ldots\}$.
  Finite dimensional version:
  $R_n := \overline{\mathrm{lin}}\{e_{1j} \mid j = 1, 2, \ldots, n\}$
- **Column Hilbert space** $C := \overline{\mathrm{lin}}\{e_{i1} \mid i = 1, 2, \ldots\}$.
  Finite dimensional version:
  $C_n := \overline{\mathrm{lin}}\{e_{i1} \mid i = 1, 2, \ldots, n\}$
  We have $M_1(R) = M_1(C) = \ell_2$ as Banach spaces – but $R \not\simeq C$ as operator spaces!
- Blecher–Paulsen have introduced the operator space structures $MIN(B)$ and $MAX(B)$, for every Banach space B.
  Here, $MIN(B)$ is the (boring) operator space structure stemming from the "commutative" embedding $B \hookrightarrow \mathrm{C}(\Omega)$, where $\Omega = BALL(B^*)$.
  On the contrary, $MAX(B)$ is precisely the other "extremal" structure, in the following sense: For every operator space E being isometric to B as a Banach space, we have canonical completely contractive "inclusions"

$$MAX(B) \subseteq E \subseteq MIN(B).$$

In other words: The set of norms α on $\mathcal{K}(B)$ satisfying Ruan's axioms admits a minimal element $\alpha_{\min}$ and a maximal element $\alpha_{\max}$, and

$$\alpha_{\min} \leq \alpha \leq \alpha_{\max}.$$

Paulsen showed that if at least $\dim(B) \geq 5$, then $MIN(B)$ and $MAX(B)$ are not completely isometrically isomorphic.

Summarizing, we have 4 different operator space structures

$$R, C, MIN(\ell_2), MAX(\ell_2)$$

which as Banach spaces are all isometric to the Hilbert space ℓ_2.

Baby quantized functional analysis

- **A New Duality:** One defines the **operator space projective tensor product** $\hat{\otimes}$ similarly to the Banach space setting. We know that $M_n^* = T_n$ (trace duality!!). For an operator space E, one then defines the **operator space dual** by norming

$$M_n(E^*) := T_n(E)^* := (T_n \hat{\otimes} E)^*.$$

Alternatively: $E^* := \mathcal{CB}(E, \mathbb{C})$

For example, we have

$$R^* = C \quad \text{and} \quad C^* = R$$

and, for every Banach space B,

$$MIN(B)^* = MAX(B^*) \quad \text{and} \quad MAX(B)^* = MIN(B^*).$$

- **Quotients** are formed using the identification

$$M_n(E_1/E_2) = M_n(E_1)/M_n(E_2).$$

- **Complex Interpolation:** For $0 < \theta < 1$, the interpolated space $E_\theta = (E_0, E_1)_\theta$ becomes an operator space by setting

$$M_n(E_\theta) = (M_n(E_0), M_n(E_1))_\theta.$$

- **The Operator Hilbert Space:** Thanks to Pisier, we know that there is a **unique** operator space which is isometric to ℓ_2 (as a Banach space) and **self-dual**. It is denoted by OH. "Explicitly", we have

$$OH = (R, C)_{\frac{1}{2}} = (MIN(\ell_2), MAX(\ell_2))_{\frac{1}{2}}.$$

- **Special feature: Haagerup tensor product**

$$R \otimes_h C = \mathcal{T}(\mathcal{H}) \text{ but } C \otimes_h R = \mathcal{K}(\mathcal{H})$$

2. A glance at quantum information theory

Let H be a finite-dimensional Hilbert space.

The entropy of a positive density $d \in S_1(H)$ is defined by

$$S(d) := -\mathrm{Tr}(d \ln d).$$

Let ι_p denote the canonical inclusion of $S_1(H)$ into $S_p(H)$.

The minimal entropy is given by

$$\begin{aligned} S_{min}(\Phi) &:= \inf\{S(\Phi(d)) : d \in S_1(H)^+ \,, tr(d) = 1\} \\ &= -\frac{d}{dp}\|\iota_p \circ \Phi : S_1(H) \to S_p(H)\|\Big|_{p=1} \end{aligned}$$

for a quantum channel, i.e., a trace preserving completely positive map Φ.

The completely bounded minimal entropy (in short, cb-entropy) of a c.b. map $\Phi : S_1(H) \to S_1(H)$ is defined as

$$S_{min,cb}(\Phi) := -\frac{d}{dp}\|\iota_p \circ \Phi : S_1(H) \to S_p(H)\|_{cb}\Big|_{p=1}.$$

Theorem. [Devetak–Junge–King–Ruskai '06]
$S_{min,cb}(\Phi \otimes \Psi) = S_{min,cb}(\Phi) + S_{min,cb}(\Psi)$

• New examples of quantum channels from harmonic analysis with explicitly determined cb-entropy (Junge–N.–Ruan '07)

III. Amplification of CB Maps and Parametrized Traces (Slice Maps)

1. Algebraic amplifications

Consider two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. If $u : \mathcal{H} \to \mathcal{H}$ is a bounded operator, there is a unique operator

$$u \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{H} \otimes_2 \mathcal{K} \to \mathcal{H} \otimes_2 \mathcal{K}$$

such that for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ we have:

$$(u \otimes \mathrm{id}_{\mathcal{K}})(\xi \otimes \eta) = u(\xi) \otimes \eta.$$

Let's climb one level now: We consider a bounded operator $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Here, the existence of an **algebraic amplification**, i.e., of a bounded operator

$$\Phi \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})} : \underbrace{\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})}_{\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})} \to \underbrace{\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})}_{\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})}$$

such that for all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{K})$:

$$(\Phi \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})(S \otimes T) = \Phi(S) \otimes T,$$

of course forces our original map Φ to be CB. For we have:

$$\|\Phi \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})}\| = \sup_{n \in \mathbb{N}} \|\Phi \otimes \mathrm{id}_{M_n}\| =: \|\Phi\|_{\mathrm{cb}}.$$

Indeed, such an amplification always exists and is given by

$$\left(\Phi^{(\infty)}\right)([a_{ij}]) = [\Phi(a_{ij})].$$

Note that this apparently simple formula hides a subtle point – and it is exactly this fact which is at the heart of our later considerations: The latter equality makes sense even though the infinite matrix $[a_{ij}]$ really is a w^*-limit and our map Φ is not at all supposed to respect this topology. In other words, we do **not** assume that Φ is $\mathbf{w}^*$-$\mathbf{w}^*$-**continuous** (**normal**)!

Consider now a more general situation. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be two von Neumann algebras, and let $\Phi : \mathcal{M} \to \mathcal{M}$ be a CB map. We wish to "construct" an amplification

$$\Phi \otimes \mathrm{id}_{\mathcal{N}} : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{M}\overline{\otimes}\mathcal{N}.$$

Using the operator Hahn-Banach Theorem due to Haagerup–Paulsen–Wittstock, we obtain an extension $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of Φ which preserves the cb-norm. We then get an amplification $\widetilde{\Phi}^{(\infty)}$ of the latter on the level of $\mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K})$.
Finally we restrict to the sub-von Neumann algebra $\mathcal{M}\overline{\otimes}\mathcal{N}$:

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K}) & \xrightarrow{\widetilde{\Phi}^{(\infty)}} & \mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K}) \\ \uparrow & & \uparrow \\ \mathcal{M}\overline{\otimes}\mathcal{N} & \longrightarrow & \mathcal{M}\overline{\otimes}\mathcal{N} \end{array}$$

Here, the only non-trivial step is obviously to verify the inclusion:

$$\widetilde{\Phi}^{(\infty)}\left(\mathcal{M}\overline{\otimes}\mathcal{N}\right) \subseteq \mathcal{M}\overline{\otimes}\mathcal{N}.$$

The latter is easily seen by using a classical theorem of Tomiyama – which leads us to his **slice maps**.

2. Tomiyama's slice maps

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be two von Neumann algebras – or more generally two dual operator spaces with their corresponding w^*-embeddings. Let's begin with the left slice.
For every $\tau \in \mathcal{N}_*$ there is a unique normal map $L_\tau : \mathcal{M}\overline{\otimes}\mathcal{N} \longrightarrow \mathcal{M}$ such that for all $S \in \mathcal{M}$ and $T \in \mathcal{N}$:

$$L_\tau(S \otimes T) = \langle \tau, T \rangle \; S.$$

In an analogous fashion, for every $\rho \in \mathcal{M}_*$ there is a unique normal map $R_\rho : \mathcal{M}\overline{\otimes}\mathcal{N} \longrightarrow \mathcal{N}$ such that for all $S \in \mathcal{M}$ and $T \in \mathcal{N}$:

$$R_\rho(S \otimes T) = \langle \rho, S \rangle \; T.$$

The **Fubini product** $\mathcal{F}(\mathcal{M}, \mathcal{N}, \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ is then defined to be the space

$$\{u \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K}) \quad | \quad L_\tau(u) \in \mathcal{M} \text{ and } R_\rho(u) \in \mathcal{N} \\ \forall \tau \in \mathcal{T}(\mathcal{K}) \; \forall \rho \in \mathcal{T}(\mathcal{H})\}.$$

It turns out that this space actually does not depend on the particular choice of embeddings so that we denote it just by $\mathcal{F}(\mathcal{M}, \mathcal{N})$.
We recall the fundamental Slice Map Theorem which allows us to deduce the desired inclusion mentioned above.

Theorem. [Tomiyama] *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras. Then we have:*

$$\mathcal{F}(\mathcal{M}, \mathcal{N}) = \mathcal{M} \overline{\otimes} \mathcal{N}.$$

A serious drawback of the procedure described above is that it is highly non-constructive. Our first aim is to present a simple, **explicit** formula for an amplification – which even applies in a far more general situation.

3. The construction

We place ourselves in the setting of arbitrary (dual) operator spaces. The following is a crucial notion which has its natural motivation in the above-mentioned theorem of Tomiyama.

Definition. [Kraus] *A dual operator space $\mathcal{M}$ is said to have* ***property*** S_σ *if the equality*

$$\mathcal{F}(\mathcal{M},\mathcal{N}) = \mathcal{M}\overline{\otimes}\mathcal{N}$$

holds true for all dual operator spaces $\mathcal{N}$ (here, $\overline{\otimes}$ denotes the normal spatial tensor product).

At this point we recall the following important well-known facts:

- There are even separably acting factors without property S_σ; but every injective von Neumann algebra has property S_σ [Kraus].
- $\mathcal{M}$ has property S_σ if and only if $\mathcal{M}_*$ has the OAP [Effros–Ruan–Kraus].

Before presenting our explicit construction of an amplification, we shall make precise which topological properties the latter should meet beyond the obvious algebraic one.

Definition. [N.] *Let* $(\mathcal{M}, \mathcal{N})$ *be an* $\boldsymbol{admissible\ pair}$, *i.e.,* $\mathcal{M}$ *and* $\mathcal{N}$ *are either von Neumann algebras or dual operator spaces with at least one of them having property* S_σ.

A linear map $\chi : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})$ *satisfying the algebraic amplification condition*

$$\chi(\Phi)(S\otimes T) = \Phi(S)\otimes T$$

for all $\Phi \in \mathcal{CB}(\mathcal{M})$, $S \in \mathcal{M}$ *and* $T \in \mathcal{N}$, *will be called an* $\boldsymbol{amplification}$ *if in addition it enjoys the following properties:*

(i) χ *is a complete isometry;*
(ii) χ *is multiplicative;*
(iii) χ *is* w^*-w^**-continuous;*
(iv) $\chi(\mathcal{CB}^\sigma(\mathcal{M})) \subseteq \mathcal{CB}^\sigma(\mathcal{M}\overline{\otimes}\mathcal{N})$.

We now give a simple formula of an amplification for every admissible pair.

Theorem. [N.] *Let* $(\mathcal{M}, \mathcal{N})$ *be an admissible pair. Then an amplification is explicitly given by*

$$\langle \chi_{\mathcal{N}}(\Phi)(u), \rho \otimes \tau \rangle = \langle \Phi(L_\tau(u)), \rho \rangle,$$

where $\Phi \in \mathcal{CB}(\mathcal{M})$, $u \in \mathcal{M}\overline{\otimes}\mathcal{N}$, $\rho \in \mathcal{M}_*$, $\tau \in \mathcal{N}_*$.

We omit the proof and restrict ourselves to very roughly sketch the **IDEA** in the following diagram – which describes the situation on the predual level:

$$\begin{array}{ccc} \underbrace{(\mathcal{M}\overline{\otimes}\mathcal{N})_*}_{\mathcal{M}_*\widehat{\otimes}\mathcal{N}_*}\widehat{\otimes}(\mathcal{M}\overline{\otimes}\mathcal{N}) & \stackrel{\chi_{\mathcal{N}*}}{\longrightarrow} & \mathcal{M}_*\widehat{\otimes}\mathcal{M} \\ \rho\otimes\tau\otimes u & \mapsto & \rho\otimes L_\tau(u) \end{array}$$

4. Basic properties

We first note a natural compatibility property of our amplification with respect to different spaces $\mathcal{N}$.

Proposition. [N.] *Let ($\mathcal{M}$, $\mathcal{N}$) be an admissible pair, and let further $\mathcal{N}_0 \subseteq \mathcal{N}$. Then we have for all $\Phi \in \mathcal{CB}(\mathcal{M})$:*

$$\chi_{\mathcal{N}}(\Phi)|_{\mathcal{M}\overline{\otimes}\mathcal{N}_0} = \chi_{\mathcal{N}_0}(\Phi).$$

Going back to our original "amplifying reflex", we remark:

Proposition. [N.] *Let ($\mathcal{M}$, $\mathcal{N}$) be an admissible pair, where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$. Let further $\Phi \in \mathcal{CB}(\mathcal{M})$. Then for an arbitrary Hahn-Banach extension $\widetilde{\Phi} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ obtained as above, we have:*

$$\widetilde{\Phi}^{(\infty)}|_{\mathcal{M}\overline{\otimes}\mathcal{N}} = \chi_{\mathcal{N}}(\Phi).$$

This shows by the way that

• in our first method, any Hahn-Banach extension chosen produces the same – namely, our – amplification;

• the amplification $\Phi^{(\infty)}$ does not depend on the particular choice of basis for the second Hilbert space $\mathcal{K}$.

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be CB. For $u \in \mathcal{A}\overline{\otimes}\mathcal{N}$, $\rho \in \mathcal{B}_*$, $\tau \in \mathcal{N}_*$ set

$$\langle \chi_{\mathcal{N}}(\Phi)(u), \rho \otimes \tau \rangle = \langle \Phi(L_\tau(u)), \rho \rangle.$$

This gives of course an amplification $\chi_{\mathcal{N}}(\Phi) : \mathcal{A}\overline{\otimes}\mathcal{N} \to \mathcal{B}\overline{\otimes}\mathcal{N}$.
Let $\Psi : \mathcal{A}\overline{\otimes}\mathcal{N} \to \mathcal{B}\overline{\otimes}\mathcal{N}$ be CB. For $\tau \in \mathcal{N}_*$ consider $\mathrm{Tr}_\tau(\Psi) : \mathcal{A} \to \mathcal{B}$ given by

$$\langle \mathrm{Tr}_\tau(\Psi)(a), \rho \rangle = \langle \Psi(a \otimes 1), \rho \otimes \tau \rangle$$

($a \in \mathcal{A}$, $\rho \in \mathcal{B}_*$).
<u>Remark:</u> We have $\mathrm{Tr}_\tau \circ \chi_{\mathcal{N}} = \mathrm{id}_{\mathcal{CB}(\mathcal{A},\mathcal{B})}$ for all $\tau \in \mathcal{N}_*$ with $\langle 1, \tau \rangle = 1$.
Why? – Fix $\Phi : \mathcal{A} \to \mathcal{B}$. Then, for all $a \in \mathcal{A}$ and $\rho \in \mathcal{B}_*$:

$$\begin{aligned} \langle \mathrm{Tr}_\tau(\chi_{\mathcal{N}}(\Phi))(a), \rho \rangle &= \langle (\chi_{\mathcal{N}}(\Phi))(a \otimes 1), \rho \otimes \tau \rangle \\ &= \langle \Phi(a) \otimes 1, \rho \otimes \tau \rangle \\ &= \langle \Phi(a), \rho \rangle. \end{aligned}$$

5. Uniqueness

Looking just at **algebraic** amplifications, of course we can by no means hope for uniqueness even in the case of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ (despite the fact that this is claimed at several places in the literature). – Namely, for any non-zero functional $\varphi \in (\mathcal{M}\overline{\otimes}\mathcal{N})^*$ which vanishes on $\mathcal{M}\overset{\vee}{\otimes}\mathcal{N}$, and any non-zero vector $v \in \mathcal{M}\overline{\otimes}\mathcal{N}$,

$$\chi_{\mathcal{N}}^{\varphi;v}(\Phi) := \chi_{\mathcal{N}}(\Phi) - \langle \varphi, \chi_{\mathcal{N}}(\Phi)(\cdot) \rangle \; v$$

trivially defines an algebraic amplification.

Nevertheless, we briefly note the following **positive** result.

Proposition. *Let $\mathcal{M}$ be an injective factor and $\mathcal{N}$ any dual operator space. Then an amplification is uniquely determined by properties (iii) and (iv).*

6. Various applications

6.1 A generalization of the Ge–Kadison Lemma

In 1996, Ge and Kadison proved the following fundamental result which solved the famous **splitting problem** for factors.

Theorem. *Let $\mathcal{M}$ be a factor and $\mathcal{S}$ be a von Neumann algebra. Suppose further that $\mathcal{B}$ is a von Neumann algebra such that*

$$\mathcal{M}\overline{\otimes}\mathbf{C}1 \subseteq \mathcal{B} \subseteq \mathcal{M}\overline{\otimes}\mathcal{S}.$$

Then $\mathcal{B} = \mathcal{M}\overline{\otimes}\mathcal{T}$ for some von Neumann subalgebra $\mathcal{T}$ in $\mathcal{S}$.

In order to prove this theorem, they first establish a result on amplifications of normal, completely positive maps on von Neumann algebras. Instead of stating the latter, we generalize it!

We have the following uniqueness result.

Proposition. [N.] *Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair, and $\Phi \in \mathcal{CB}(\mathcal{M})$. Suppose $\Theta : \mathcal{M}\overline{\otimes}\mathcal{N} \longrightarrow \mathcal{M}\overline{\otimes}\mathcal{N}$ is* **any** *map which satisfies, for some $0 \neq n \in \mathcal{N}$:*

(i) Θ commutes with the slice maps $\mathrm{id}_{\mathcal{M}} \otimes \tau n$ ($\tau \in \mathcal{N}_*$)

(ii) Θ coincides with $\Phi \otimes \mathrm{id}_{\mathcal{N}}$ on $\mathcal{M} \otimes \mathbf{C}n$.

Then we must have $\Theta = \Phi \otimes \mathrm{id}_{\mathcal{N}}$.

This generalizes the corresponding result of Ge–Kadison who proved the above assuming that Φ is **normal** and **completely positive** and $\mathcal{M}$ and $\mathcal{N}$ are **von Neumann algebras**. Remark: Our amplification result may be useful in order to

- deal with **singular** conditional expectations;
- attack the splitting problem for dual operator spaces with property S_σ.

6.2 An algebraic characterization of normality

Theorem. [N.] *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with $\mathcal{N}$ properly infinite. Then for an arbitrary $\Phi \in \mathcal{CB}(\mathcal{M})$, TFAE:*
(i) $(\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi) = (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}) \forall \Psi \in \mathcal{CB}(\mathcal{N})$
(ii) Φ *is normal.*

Here, (ii) $\Rightarrow$ (i) holds for any admissible pair.

Our Theorem suggests considering **two Arens type tensor products**! the product on a Banach algebra $\mathcal{A}$ can be extended in two natural ways to its bidual, giving rise to the **two Arens products** on $\mathcal{A}^{**}$. One defines the **topological centre**

$$Z_t := \{m \in \mathcal{A}^{**} \mid m \odot_1 n = m \odot_2 n \ \forall n \in \mathcal{A}^{**}\}.$$

In our context, setting

$$\Phi \otimes_1 \Psi := (\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi)$$

and

$$\Phi \otimes_2 \Psi := (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}),$$

we have constructed **two natural "tensor products"** – instead of multiplications! – which in general are different. It is then natural to introduce the **topological tensor centre**

$$Z_t^{\otimes} := \{\Phi \in \mathcal{CB}(\mathcal{M}) \mid \Phi \otimes_1 \Psi = \Phi \otimes_2 \Psi \ \forall \Psi \in \mathcal{CB}(\mathcal{N})\}.$$

The above Theorem may now be **equivalently** rephrased as

$$Z_t^{\otimes} = \mathcal{CB}^{\sigma}(\mathcal{M}).$$

This is exactly what one expects – since the (topological) centre should correspond to the nice, i.e., **normal** part in the Tomiyama-Takesaki decomposition

$$\mathcal{CB}(\mathcal{M}) = \mathcal{CB}^{\sigma}(\mathcal{M}) \oplus \mathcal{CB}^{s}(\mathcal{M}).$$

6.3 Completely bounded module homomorphisms

A result of May–Neuhardt–Wittstock (whose proof is rather involved) implies that whenever $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$, then the amplification $\Phi^{(\infty)}$ is automatically a $1 \otimes \mathcal{B}(\mathcal{K})$-bimodule homomorphism on $\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})$.

Using our **explicit** formula, we obtain a **simpler** proof of the following even **more general** result:

Proposition. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and let $\Phi \in \mathcal{CB}(\mathcal{M})$. Then $\chi_{\mathcal{N}}(\Phi)$ is a $1 \otimes \mathcal{N}$-bimodule homomorphism on $\mathcal{M} \overline{\otimes} \mathcal{N}$.*

The proof uses nothing more than the following elementary property of slice maps:

$$L_\tau((1 \otimes a)u(1 \otimes b)) = L_{b \cdot \tau \cdot a}(u),$$

where $\langle b \cdot \tau \cdot a, u \rangle = \langle \tau, aub \rangle$.

IV. Traces in von Neumann algebras

1. Characterizations of vN algebras through traces

Definition. *A positive linear functional φ on $\mathcal{M}$ is called a trace if*

$$\varphi(ab) = \varphi(ba)$$

for all $a, b \in \mathcal{M}$.

Note: Equivalently, $\varphi(aa^*) = \varphi(a^*a)$ for all $a \in \mathcal{M}$.

Extension beyond finiteness:

Definition. *A weight on $\mathcal{M}$ is an additive map $\varphi : \mathcal{M}^+ \to [0, \infty]$ such that $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \in \mathbb{R}^+$ and $x \in \mathcal{M}^+$. If, in addition $\varphi(x^*x) = \varphi(xx^*)$ for all $x \in \mathcal{M}$, then φ is called a trace.*

Set

$$\mathcal{M}_\varphi^+ := \{x \in \mathcal{M}^+ \mid \varphi(x) < \infty\}, \quad \mathcal{M}_\varphi := \mathrm{lin}\mathcal{M}_\varphi^+$$

and

$$\mathcal{N}_\varphi := \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}.$$

Then φ extends to a linear map on $\mathcal{M}_\varphi$, and $\mathcal{N}_\varphi$ is a left ideal of $\mathcal{M}$.

We say that

- φ *is* <u>*normal*</u> *if* $\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha)$ *for each bounded, increasing net* (x_α) *in* $\mathcal{M}^+$*;*
- φ *is* <u>*semifinite*</u> *if* $\mathcal{M}_\varphi$ *is* w^**-dense in* $\mathcal{M}$*;*
- φ *is* <u>*faithful*</u> *if* $\varphi(x) = 0$ *for* $x \in \mathcal{M}^+$ *implies* $x = 0$*.*

Given an n.s.f. weight φ on $\mathcal{M}$, the left ideal $\mathcal{N}_\varphi$, equipped with the scalar product $(x, y) := \varphi(y^*x)$, is a pre-Hilbert space. We denote by $L_2(\mathcal{M}, \varphi)$ its completion. Then $\mathcal{M}$ can be identified as a subalgebra of $\mathcal{B}(L_2(\mathcal{M}, \varphi))$ – its **standard form**.

Theorem. *There is a unique decomposition*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$$

where

- $\mathcal{M}_1$ *is finite* $\Leftrightarrow$ *there is a faithful normal tracial state on* $\mathcal{M}_1$
- $\mathcal{M}_2$ *is properly infinite but semifinite:*

* *properly infinite* $\Leftrightarrow$ *there is NO normal tracial state on* $\mathcal{M}_2$
* *semifinite* $\Leftrightarrow$ *there is a faithful semifinite normal trace on* $\mathcal{M}_2^+$

- $\mathcal{M}_3$ *is purely infinite* $\Leftrightarrow$ *there is NO (non-zero) semifinite normal trace on* $\mathcal{M}_3^+$

$\boxed{\text{Refinement}}$

Theorem. *There is a unique decomposition*

$$\mathcal{M} = \mathcal{M}_{I_n} \oplus \mathcal{M}_{I_\infty} \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{II_\infty} \oplus \mathcal{M}_{III}$$

where

- $\mathcal{M}_{I_n}$ *and* $\mathcal{M}_{II_1}$ *are finite*
- $\mathcal{M}_{I_\infty}$ *and* $\mathcal{M}_{II_\infty}$ *are properly infinite and semifinite*
- $\mathcal{M}_{III}$ *is purely infinite*

Examples:

- baby example $\mathcal{M} = M_n(\mathbb{C})$ [type I_n] from the beginning: non-normalized trace $\mathrm{Tr}(a_{ij}) = \sum_{i=1}^n a_{ii}$; normalized trace $\mathrm{Tr}^n = \frac{1}{n}\sum_{i=1}^n a_{ii}$
- $\mathcal{M} = L_\infty(\mathcal{G})$ for a compact group $\mathcal{G}$ [type I]: $\mathrm{Tr} = \int_{\mathcal{G}} \cdot \, d\lambda$, where $\lambda =$ (normalized) Haar measure
- $\mathcal{M} = VN(\mathcal{G})$ for an ICC group $\mathcal{G}$ [type II_1]: $\mathrm{Tr} = \langle \cdot \, \delta_e, \delta_e \rangle$

Note: Given $\mathcal{M}$ with an n.s.f. trace Tr one associates to $\mathcal{M}$ the **non-commutative** L_p **spaces** $L_p(\mathcal{M}, \mathrm{Tr})$.

For Tr finite, $L_p(\mathcal{M}, \mathrm{Tr})$ = completion of $\mathcal{M}$ w.r.t. the norm $\|x\|_p = (\mathrm{Tr}(|x|^p))^{1/p}$.

We have $L_1(\mathcal{M}, \mathrm{Tr}) = \mathcal{M}_*$ and $L_\infty(\mathcal{M}, \mathrm{Tr}) = \mathcal{M}$.

One can obtain $L_p(\mathcal{M}, \mathrm{Tr})$ by **complex interpolation** between $\mathcal{M}_*$ and $\mathcal{M}$. This yields a natural **operator space structure** on $L_p(\mathcal{M}, \mathrm{Tr})$.

Examples:

- $L_p(\mathcal{B}(\mathcal{H}), \mathrm{Tr}) = S_p(\mathcal{H})$
- $L_p(L_\infty(\Omega), \mu) = L_p(\Omega, \mu)$

2. En guise d'épilogue

• The **centre-valued trace** on a finite von Neumann algebra $\mathcal{M}$ is a faithful normal projection of norm 1 (=conditional expectation) from $\mathcal{M}$ onto $Z = \mathcal{M}' \cap \mathcal{M}$, such that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for all $a, b \in \mathcal{M}$.

• **Caution:** There are non-normal traces on $\mathcal{B}(\mathcal{H})$ (Dixmier '66)! The so-called **Dixmier traces** vanish on $\mathcal{K}(\mathcal{H})$ – in particular on $\mathcal{T}(\mathcal{H})$.

* closely linked to invariant means on ℓ_∞ (they vanish on c_0)
* useful in Noncommutative Geometry in calculations modulo finite rank operators

• .....