Functorial boxes in string diagrams

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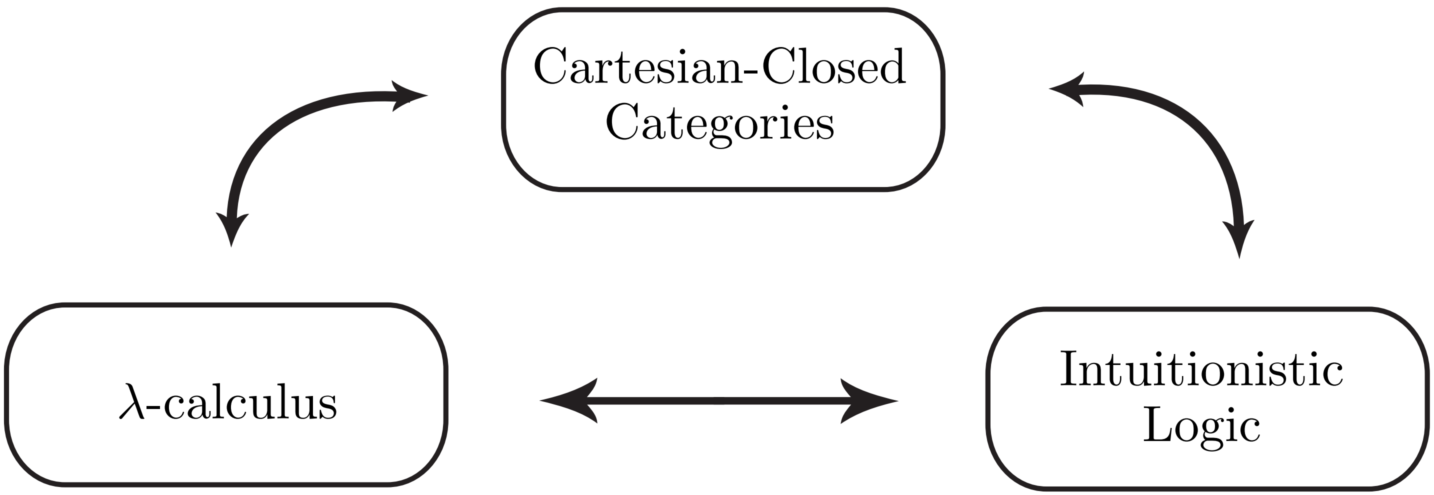
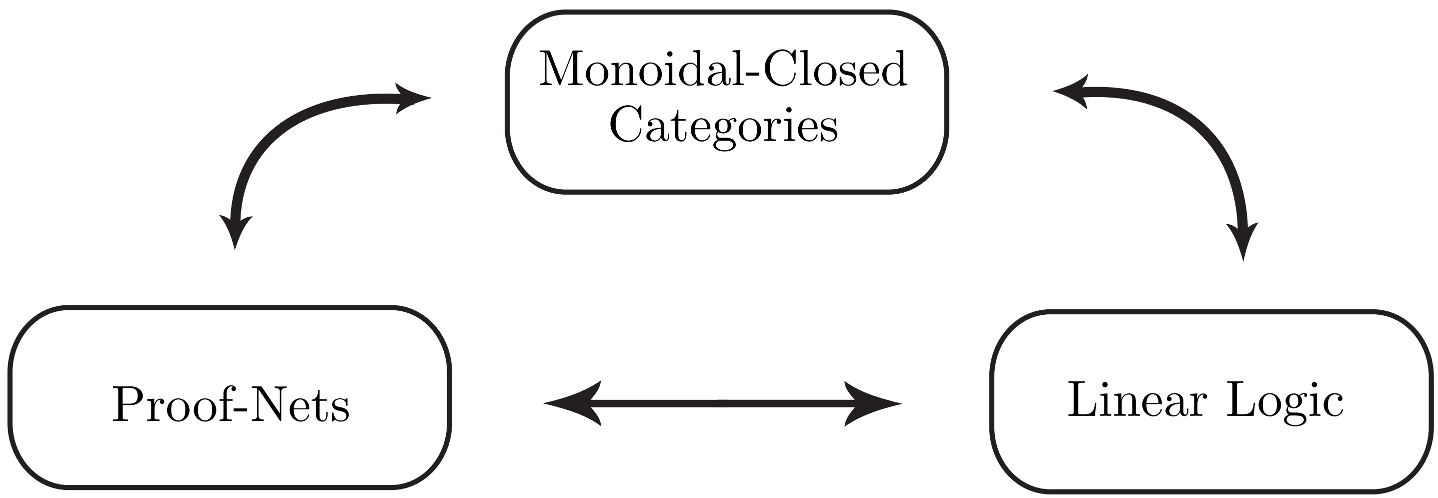
Paper available at www.pps.jussieu.fr/∼mellies/

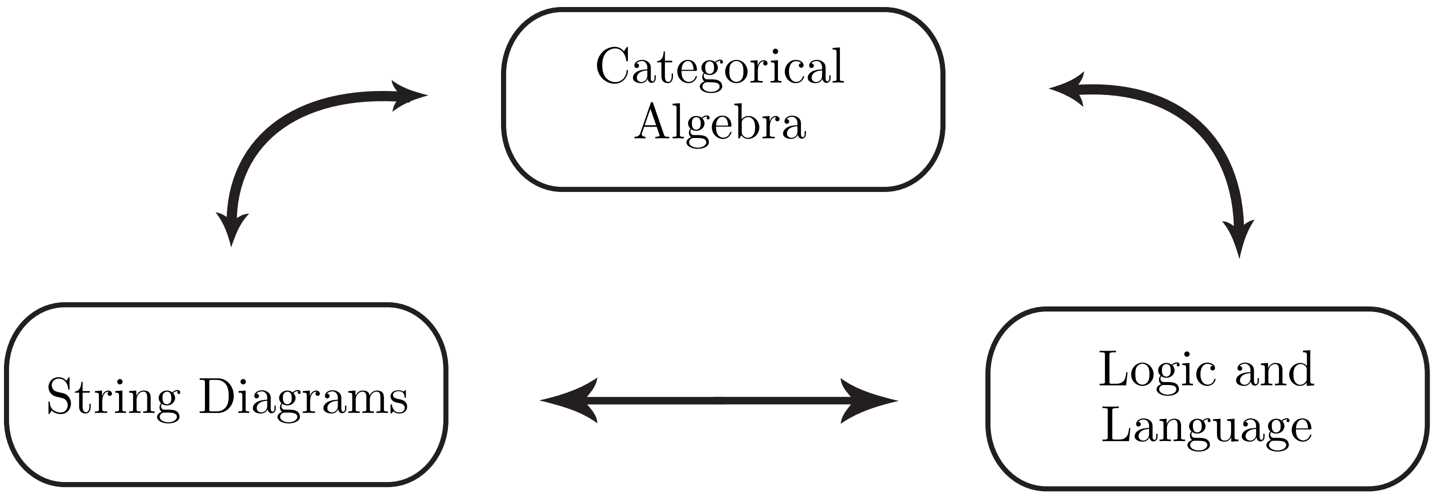
Denotational Semantics in the 1970s



Linear Logic in the 1990s

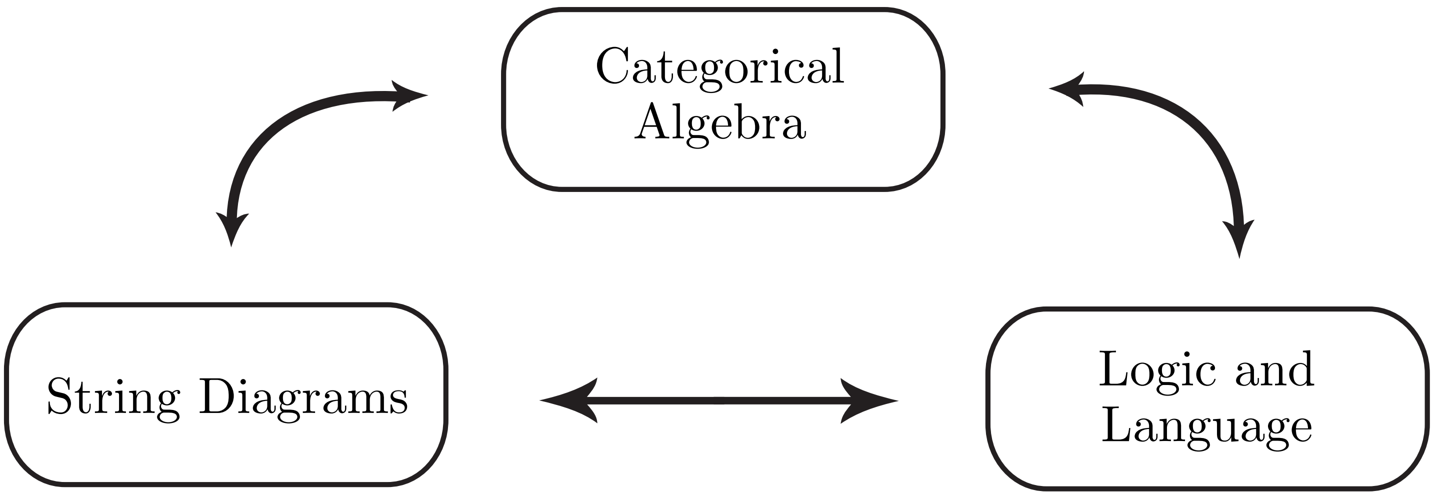


String Diagrams



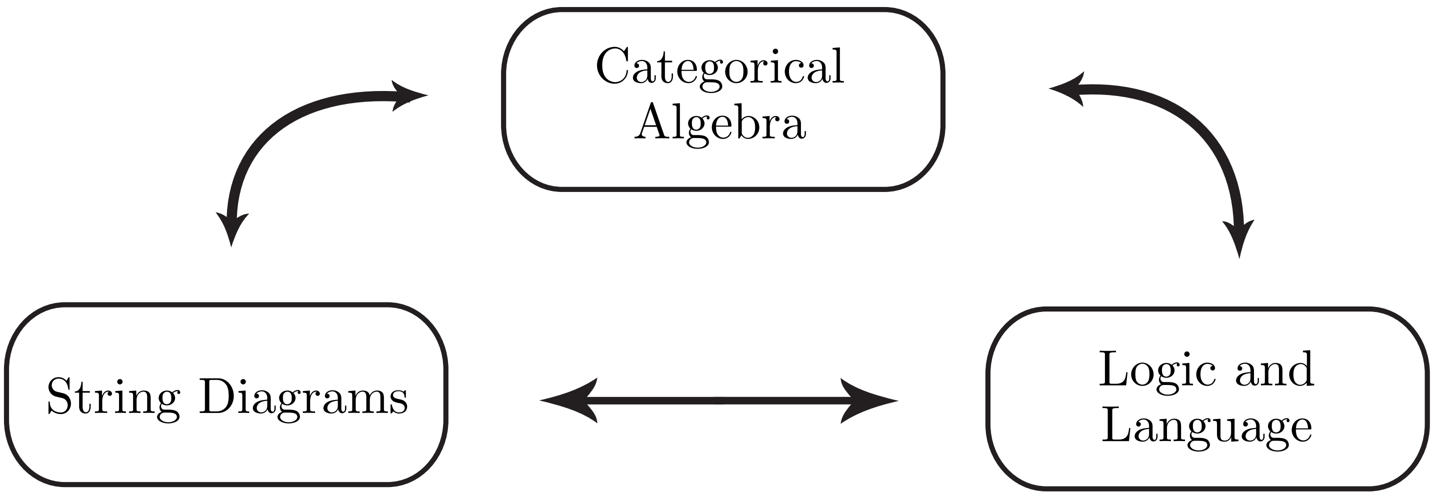
An algebraic investigation of logic

String Diagrams



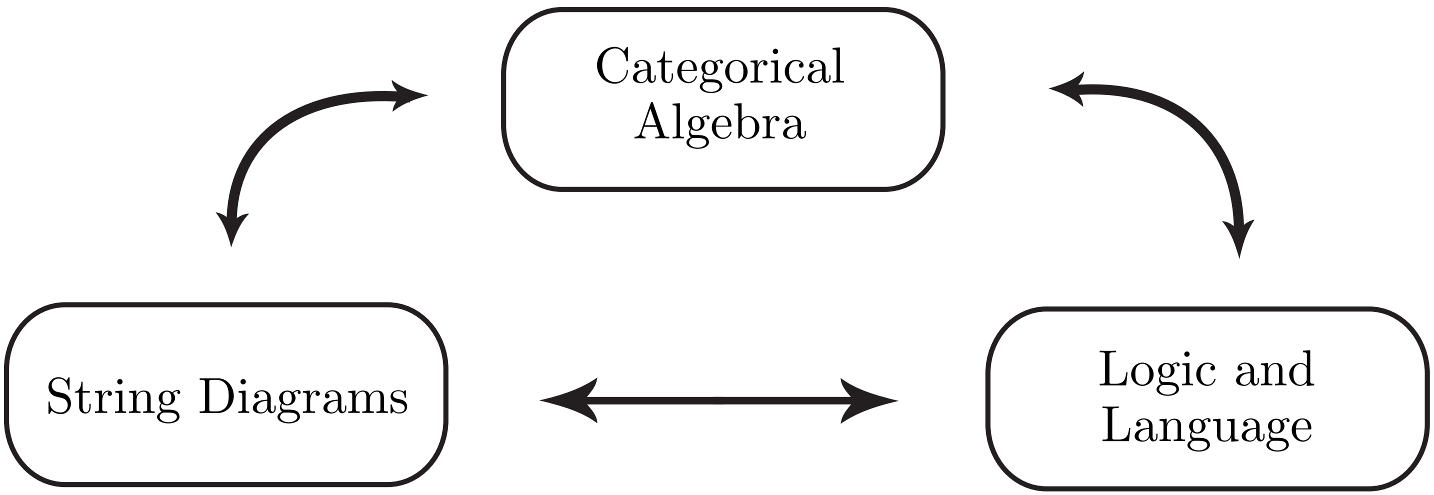
A logical investigation of algebra

String Diagrams



Connections to physics and n-dimensional algebra

String Diagrams



Extending the methodology of linear logic to other effects

String Diagrams

An idea by Roger Penrose (1970)

Monoidal Categories

A **monoidal category** is a category $\mathbb{C}$ equipped with a functor:

$$\otimes \;:\; \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

an object:

$$I$$

and three natural transformations:

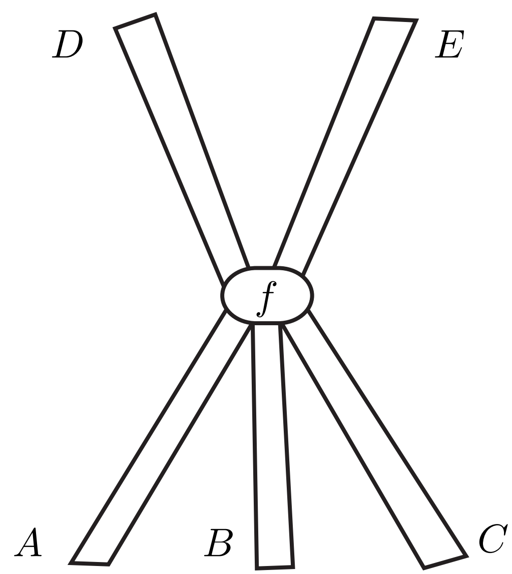
$$(A \otimes B) \otimes C \quad \xrightarrow{\alpha} \quad A \otimes (B \otimes C)$$

$$I \otimes A \quad \xrightarrow{\lambda} \quad A \qquad\qquad\qquad A \otimes I \quad \xrightarrow{\rho} \quad A$$

satisfying a series of coherence properties.

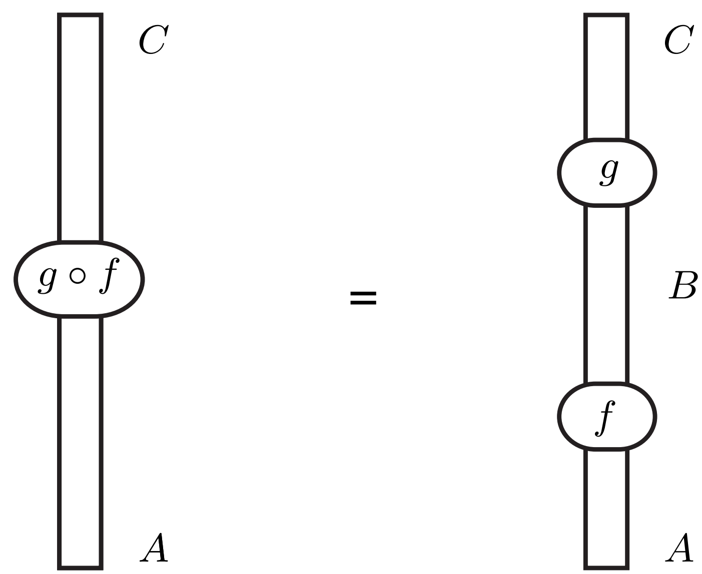
String Diagrams

A morphism $f : A \otimes B \otimes C \longrightarrow D \otimes E$ is depicted as:



Composition

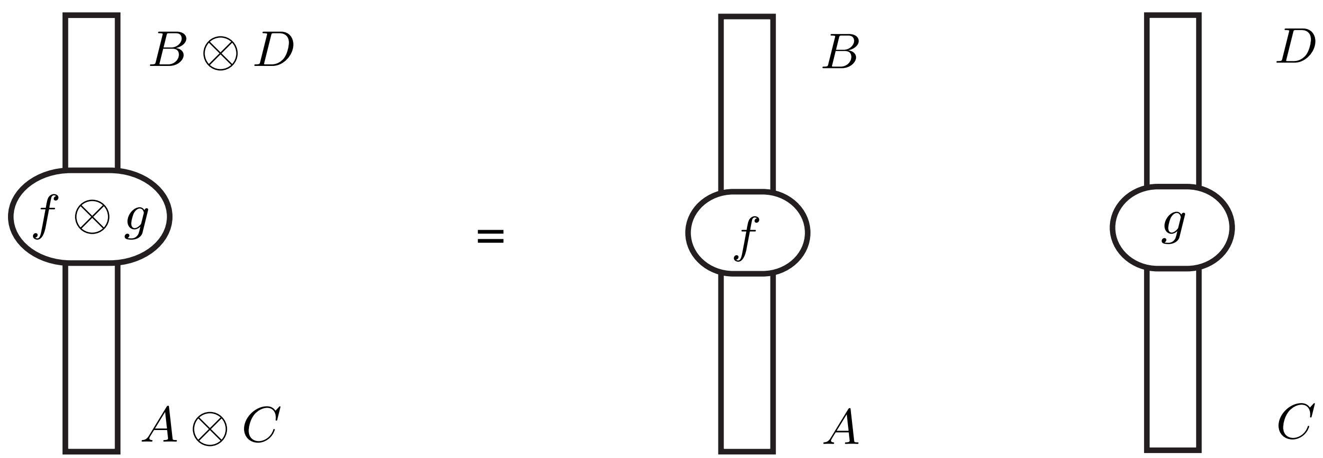
The morphism $A \xrightarrow{f} B \xrightarrow{g} C$ is depicted as



Vertical composition

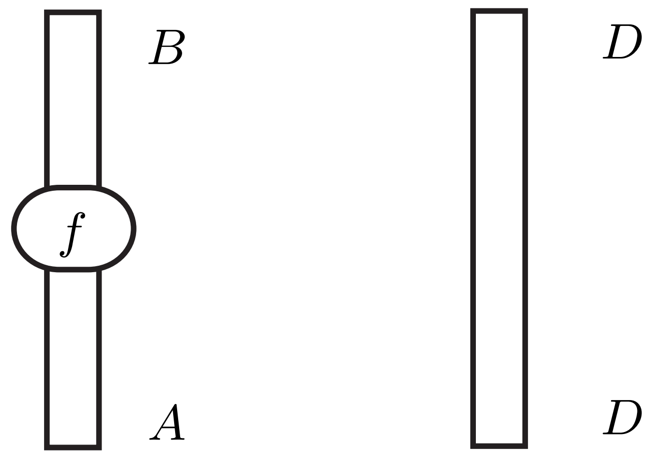
Tensor product

The morphism $(A \xrightarrow{f} B) \otimes (C \xrightarrow{g} D)$ is depicted as



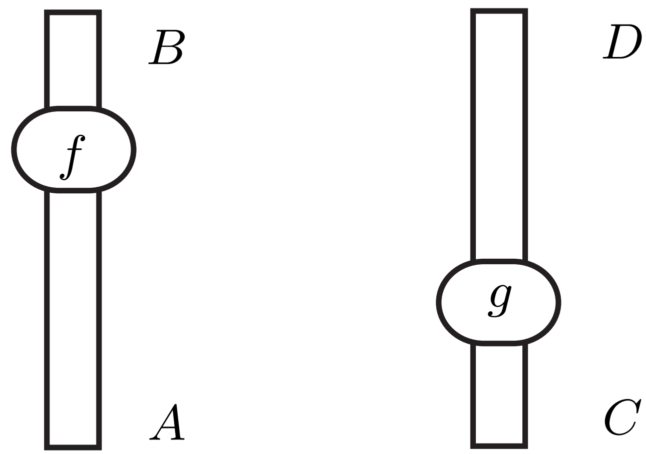
$\boxed{\text{Horizontal tensor product}}$

Example



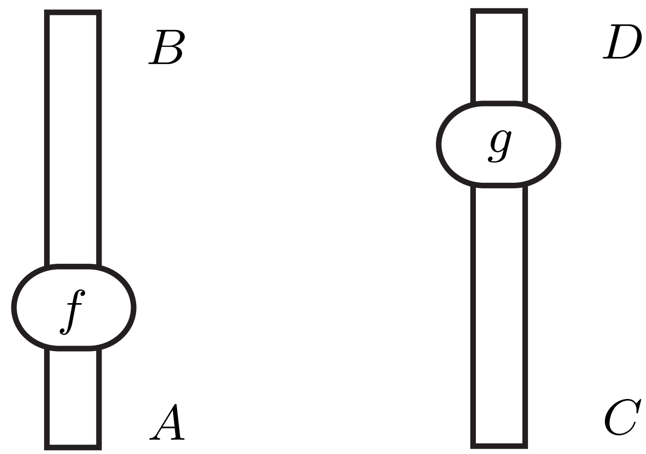
$$f \otimes id_D$$

Example



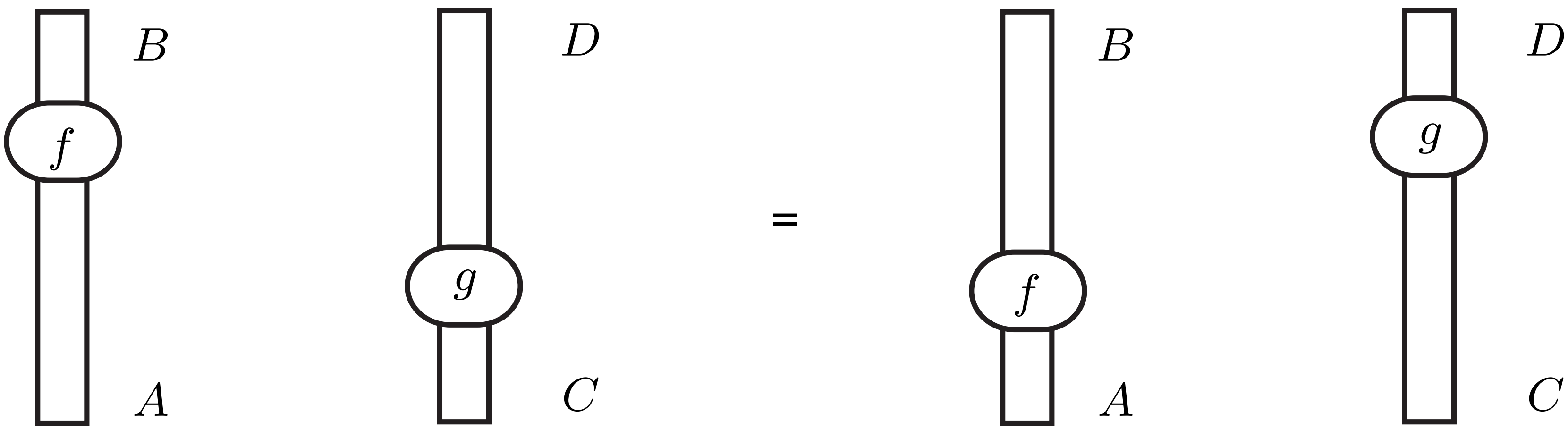
$$(f \otimes id_D) \circ (id_A \otimes g)$$

Example



$$(id_B \otimes g) \circ (f \otimes id_C)$$

Meaning preserved by deformation

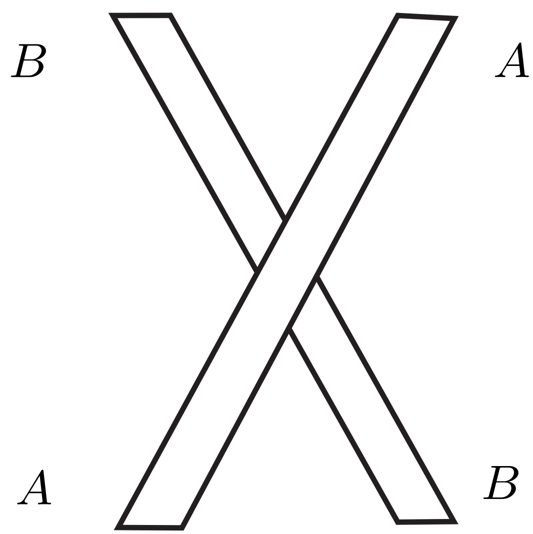


$$(f \otimes id_D) \circ (id_A \otimes g) \quad = \quad (id_B \otimes g) \circ (f \otimes id_C)$$

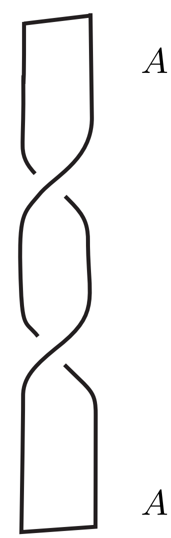
Balanced categories (Joyal, Street 1993)

A **balanced category** is a monoidal category equipped with

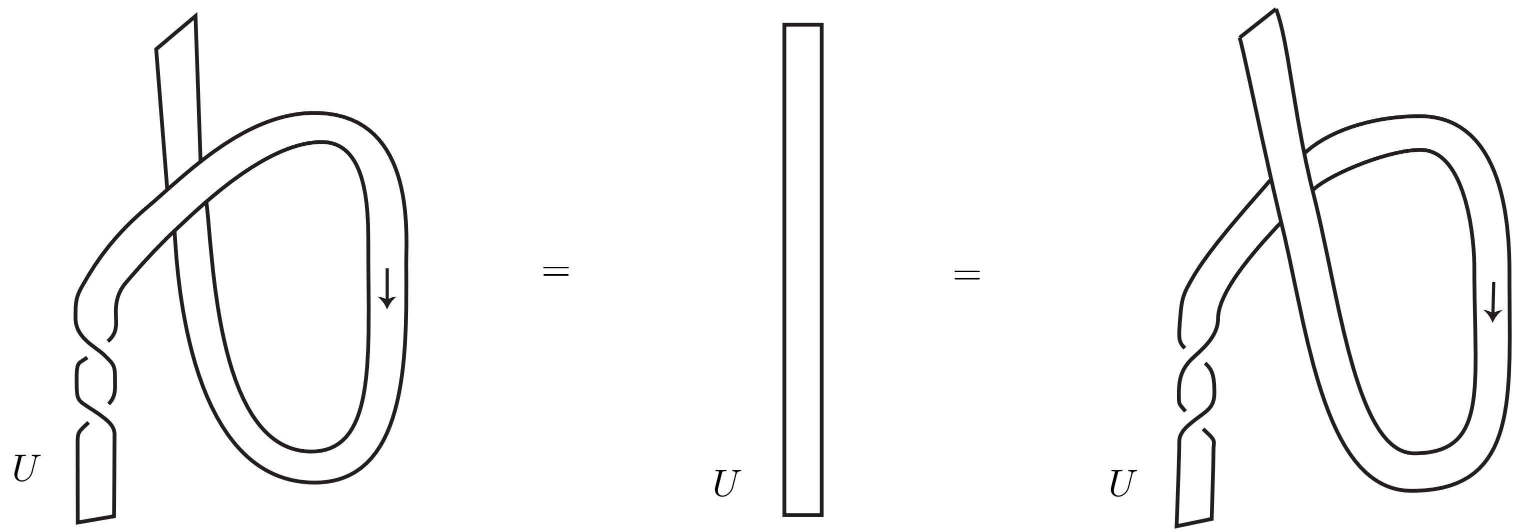
Braid maps $A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A$



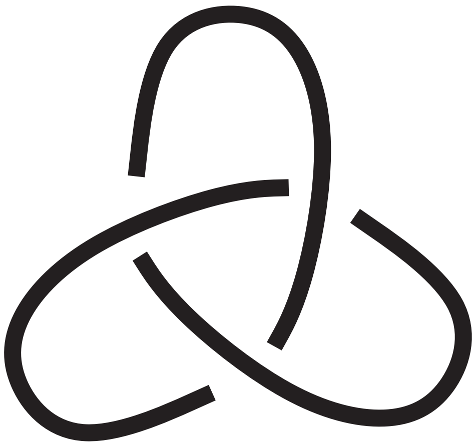
Twist maps $A \xrightarrow{\theta_A} A$



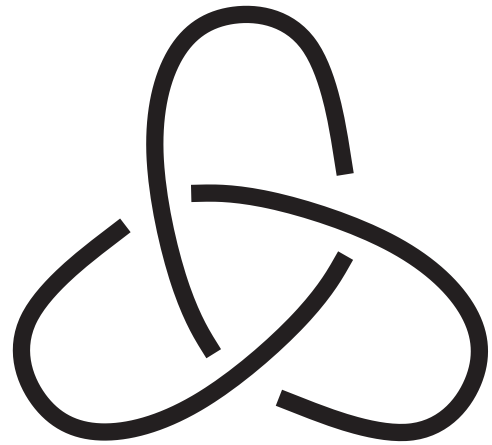
Low dimensional topology



Low dimensional topology



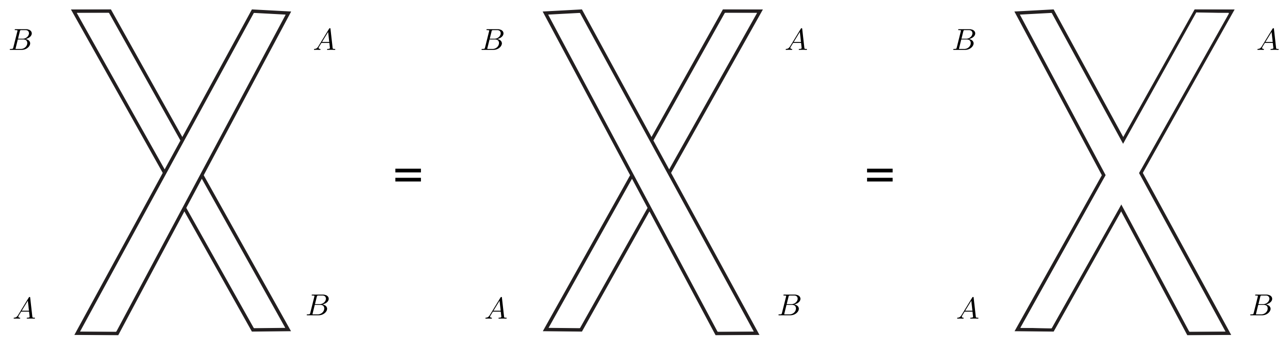
$\not\sim$

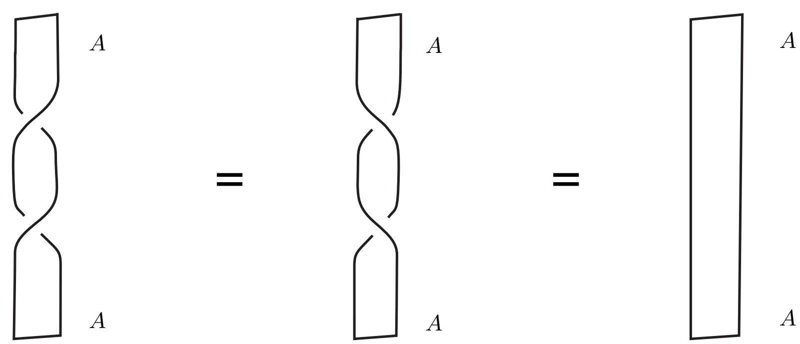


$$2x^{-2} - x^{-4} + x^{-2}y^{-2} \qquad\qquad 2x^{2} - x^{4} + x^{2}y^{2}$$

Jones polynomial = a semantics of knots

Logic is symmetric — not braided





Leads to a ribbon variant of Linear Logic

Proof nets

An idea by Jean-Yves Girard (1986)

Sequent calculus

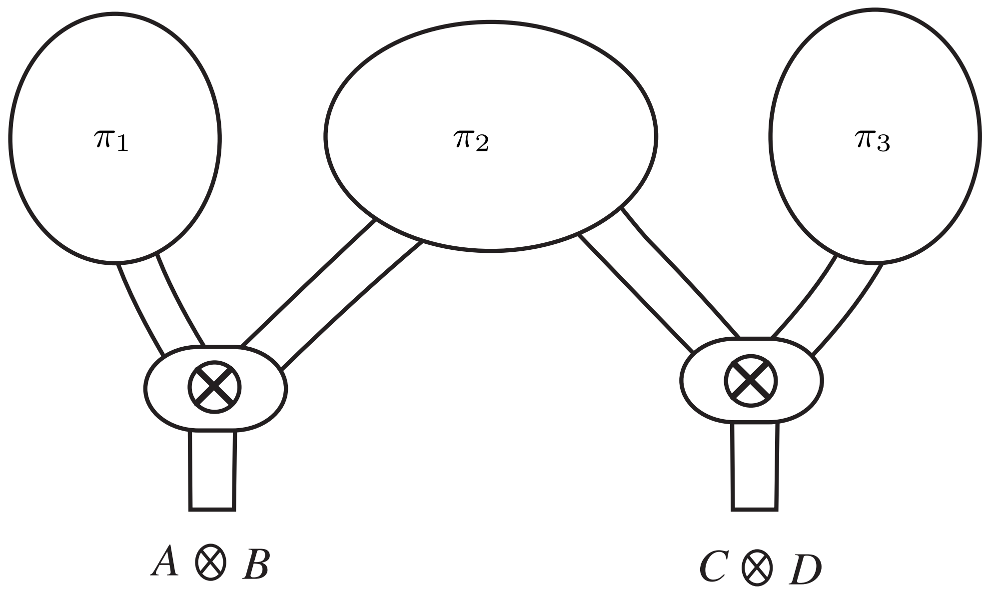
The two equivalent proofs:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \vdash A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \hline \vdash B, C\end{array}}{\vdash A \otimes B, C} \quad \begin{array}{c}\pi_3\\ \vdots\\ \hline \vdash D\end{array}}{\vdash A \otimes B, C \otimes D} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \vdash A\end{array} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \vdash B, C\end{array} \quad \begin{array}{c}\pi_3\\ \vdots\\ \hline \vdash D\end{array}}{\vdash B, C \otimes D}}{\vdash A \otimes B, C \otimes D}$$

A permutation equivalence

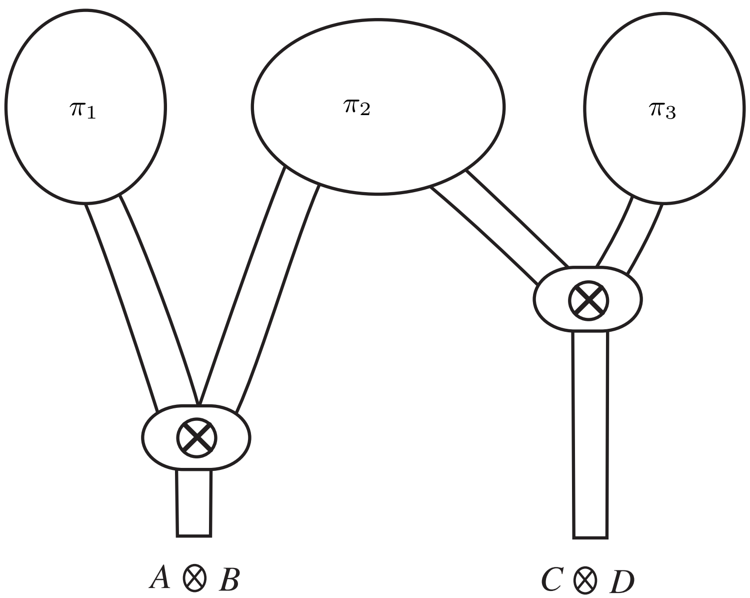
Proof nets

are interpreted by the same proof net:



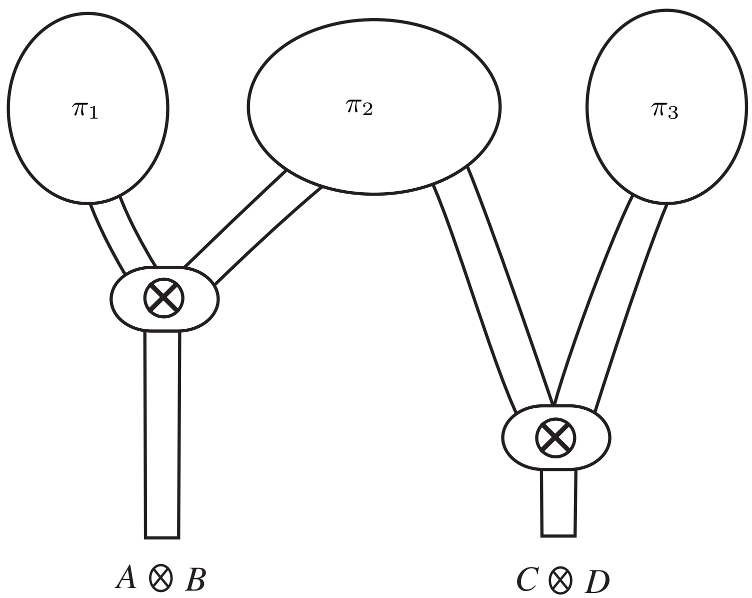
A geometric notation

Sequentialization by deformation



$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \hline \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \hline \vdash B, C\end{matrix} \quad \begin{matrix}\pi_3\\ \vdots\\ \hline \vdash D\end{matrix}}{\vdash B, C \otimes D}}{\vdash A \otimes B, C \otimes D}$$

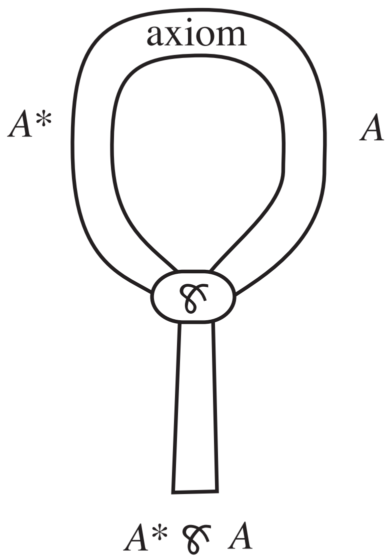
Sequentialization by deformation

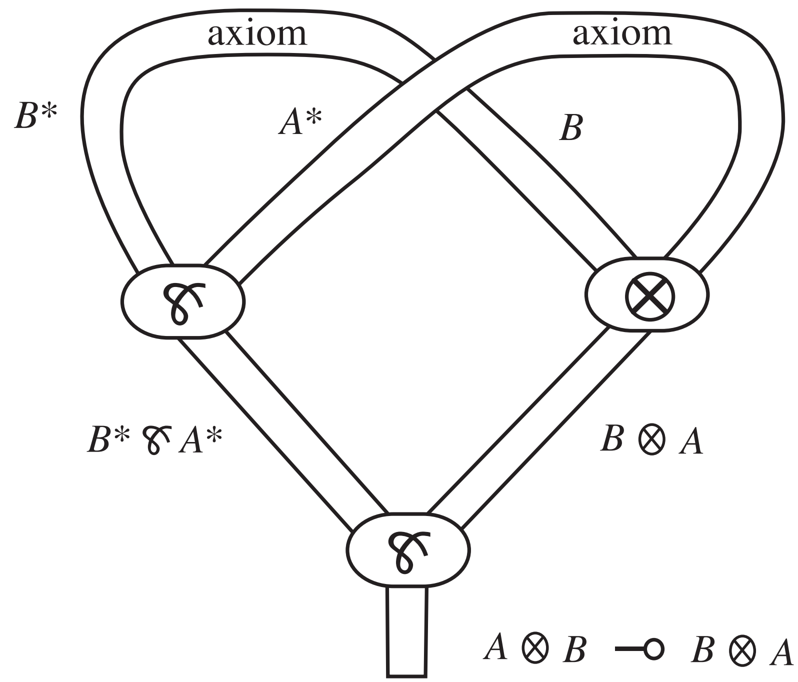


$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdots}}{\vdash A} \quad \dfrac{\dfrac{\pi_2}{\vdots}}{\vdash B, C}}{\vdash A \otimes B, C} \quad \dfrac{\dfrac{\pi_3}{\vdots}}{\vdash D}$$

$$\vdash A \otimes B, C \otimes D$$

Multiplicative proof nets





Multiplicative proof nets are string diagrams!

Question

Can one extend string diagrams with boxes?

Functorial boxes

Rediscovery of an idea by Robin Cockett and Robert Seely (1996)

The categorical semantics of linear logic (Nick Benton — CSL'94)

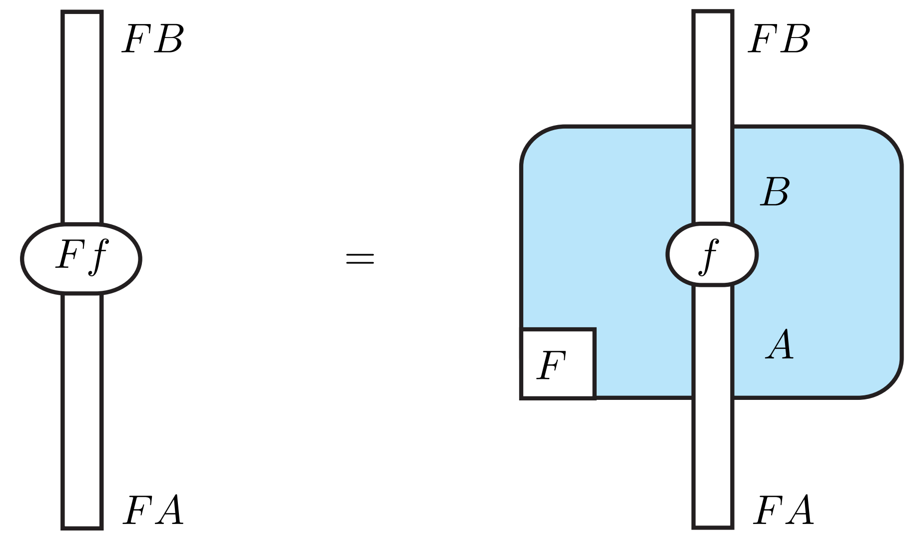
A **symmetric monoidal adjunction**

$$\begin{tikzcd} \mathbb{M} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{L} \arrow[ll, bend left, "M"] \end{tikzcd}$$

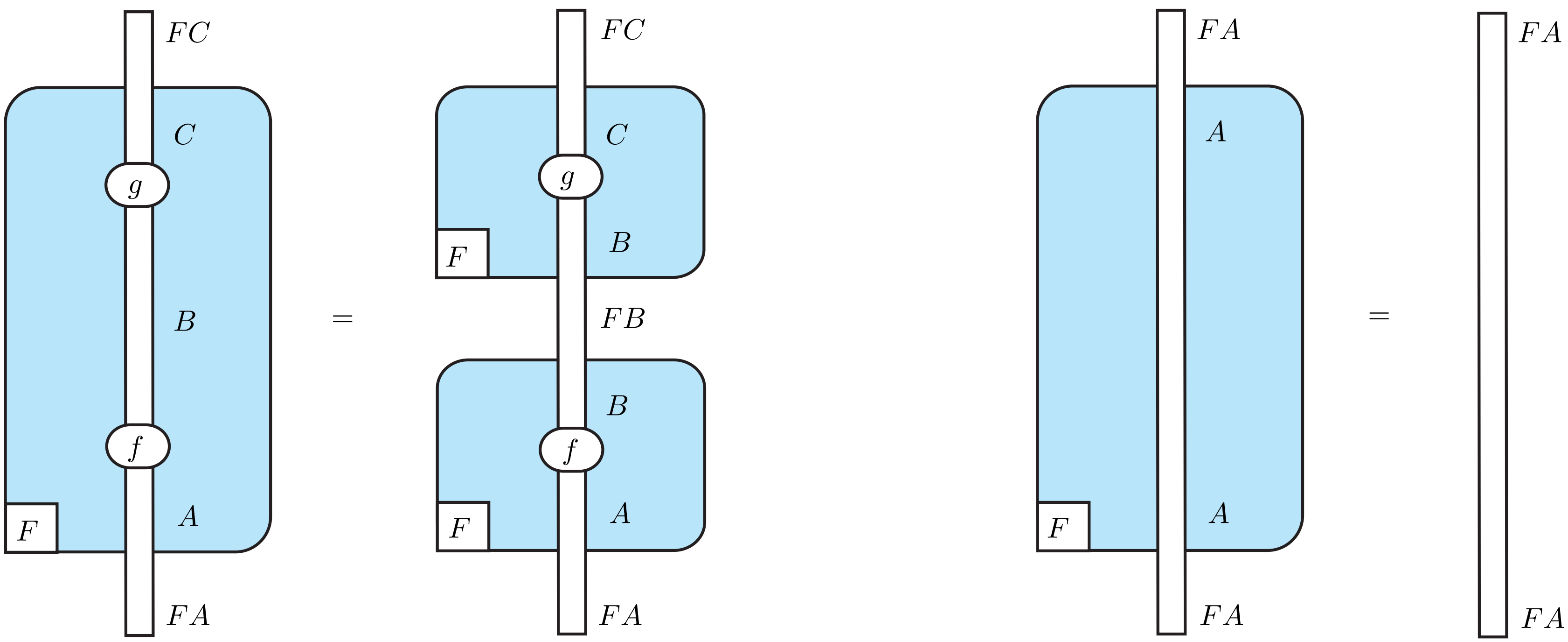
$\mathbb{M}$ cartesian $\qquad\qquad$ $\mathbb{L}$ symmetric monoidal closed

$$! \quad = \quad L \circ M$$

Functorial boxes in string diagrams



Functorial equalities



Lax monoidal functor

A **lax monoidal functor** is a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ equipped with morphisms

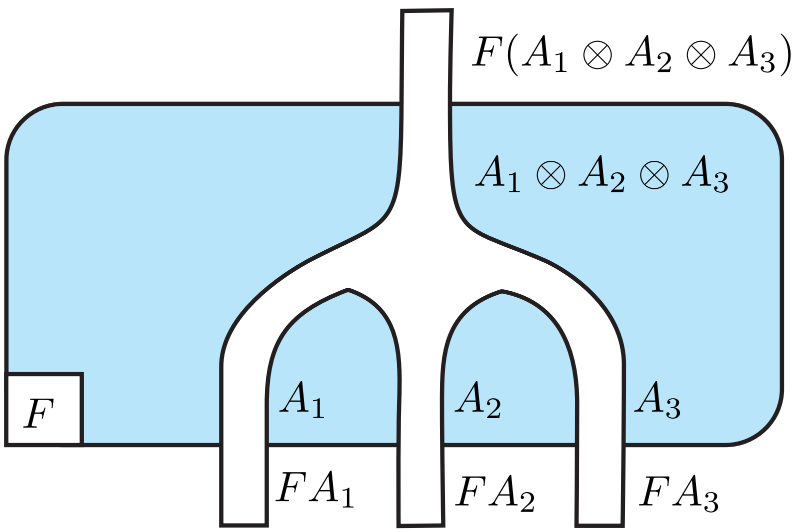
$$m_{[A,B]} \ : \ FA \otimes FB \ \longrightarrow \ F(A \otimes B)$$

$$m_{[-]} \ : \ I \longrightarrow FI$$

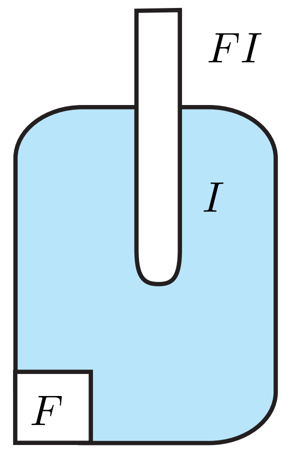
satisfying a series of coherence relations.

A **strong monoidal functor** is lax monoidal with **invertible** coercions.

The purpose of coercions



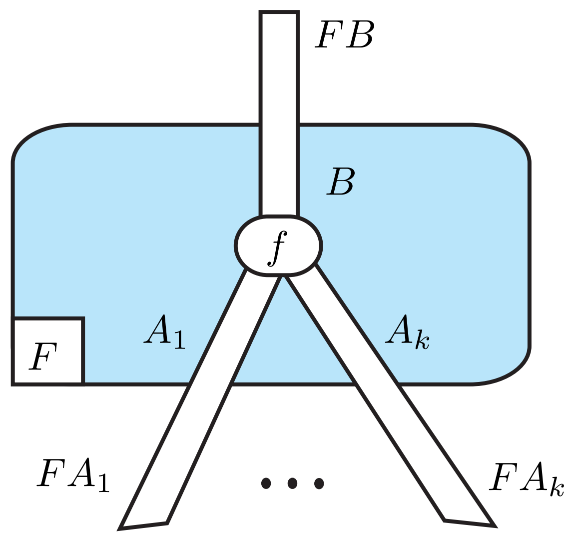
$m_{[A_1,A_2,A_3]}$



$m_{[-]}$

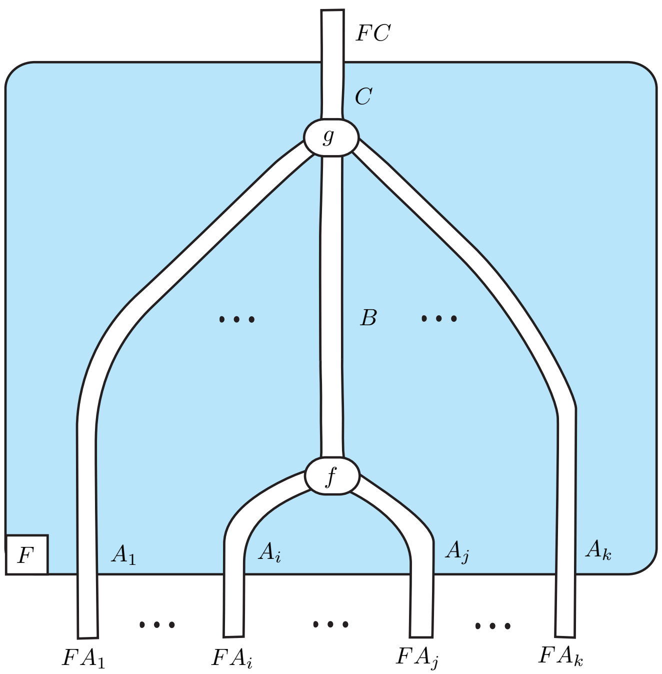
Lax monoidal functor

A **lax monoidal functor** is a box with **many inputs - one output**.



$$F(f) \circ m_{[A_1,\cdots,A_k]} \ : \ FA_1 \otimes \cdots \otimes FA_k \longrightarrow FB$$

Functorial equalities (on lax functors)



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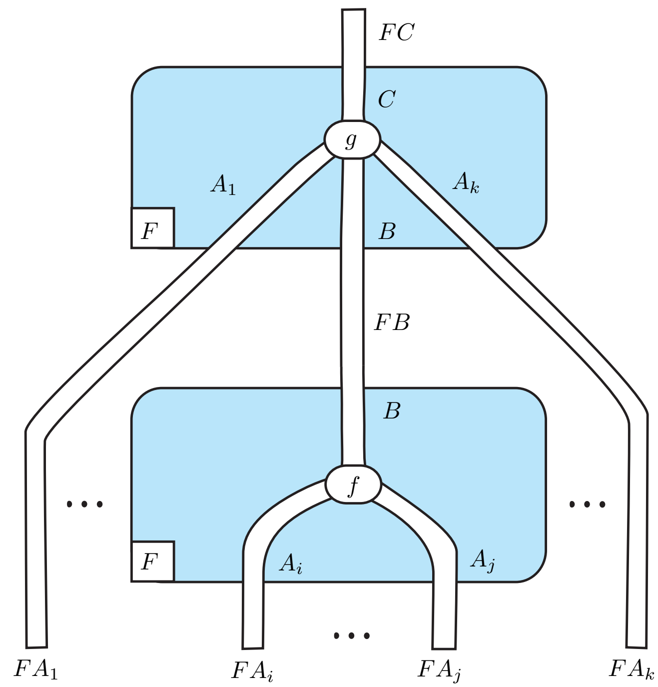
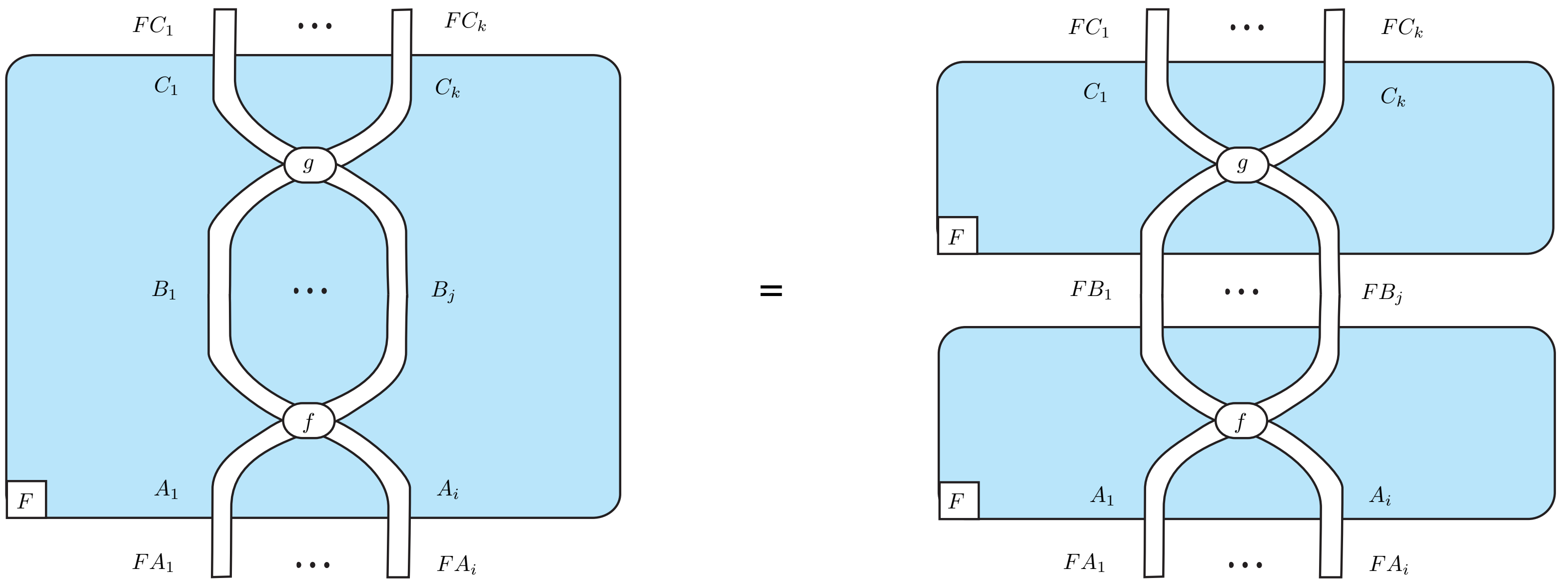


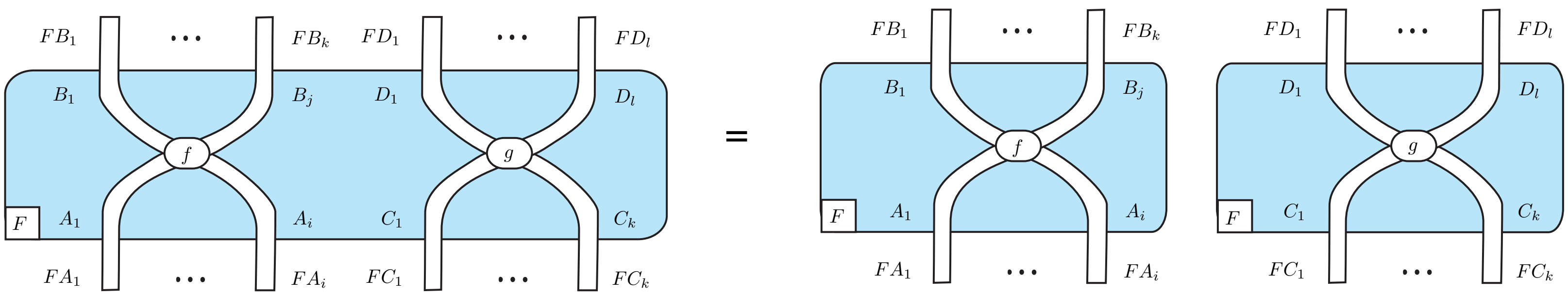
Strong monoidal functors

A **strong monoidal functor** is a box with **many inputs - many outputs**

Functorial equalities (on strong functors)



Functorial equalities (on strong functors)

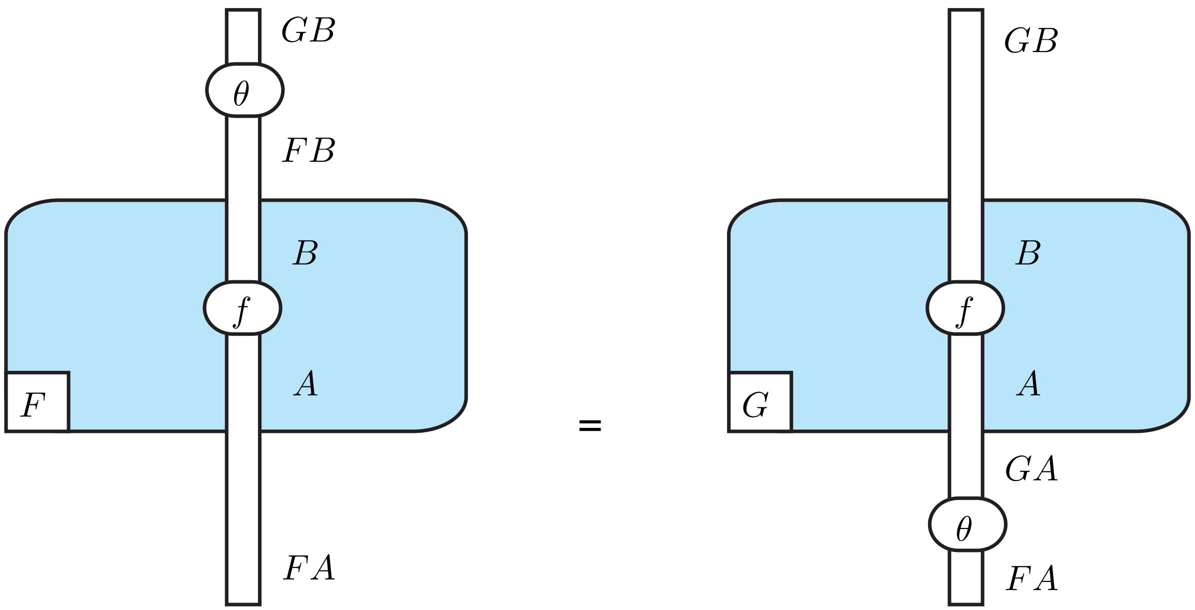


Natural transformations

A natural transformation

$$\theta \quad : \quad F \longrightarrow G \quad : \quad \mathbb{C} \longrightarrow \mathbb{D}$$

satisfies the pictorial equality:

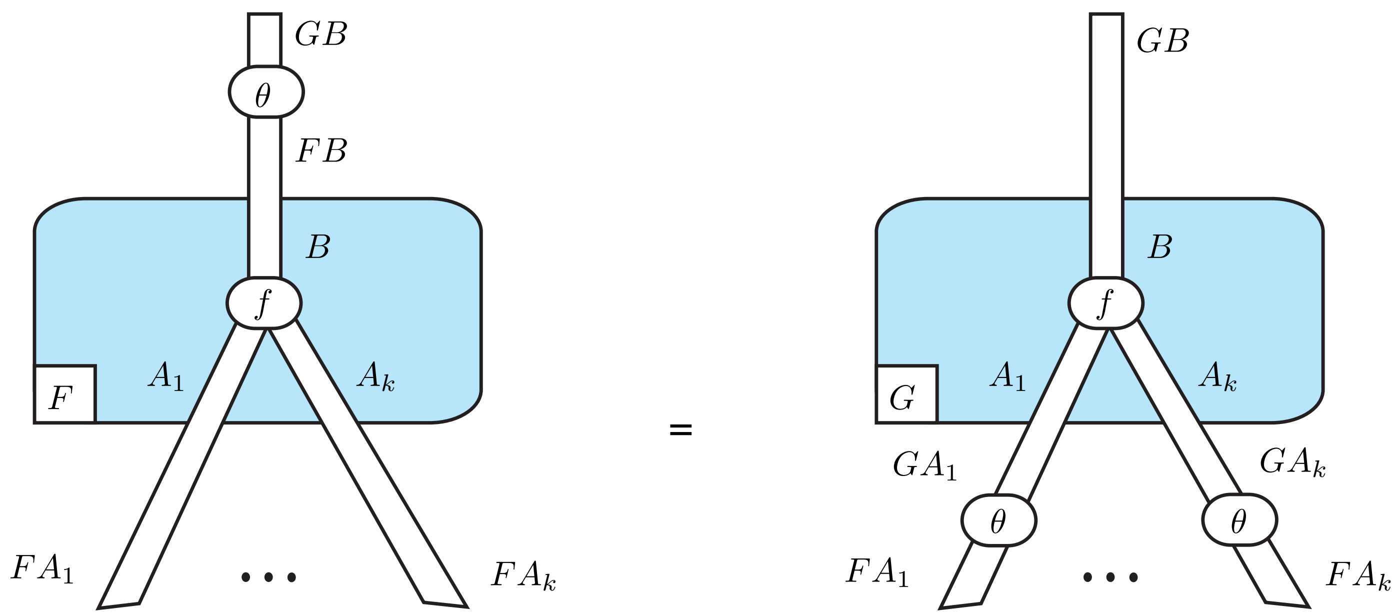


Monoidal natural transformations

A **monoidal** natural transformation

$$\theta \quad : \quad F \longrightarrow G \quad : \quad \mathbb{C} \longrightarrow \mathbb{D}$$

satisfies the pictorial equality:



Exercise 1

Transport of trace

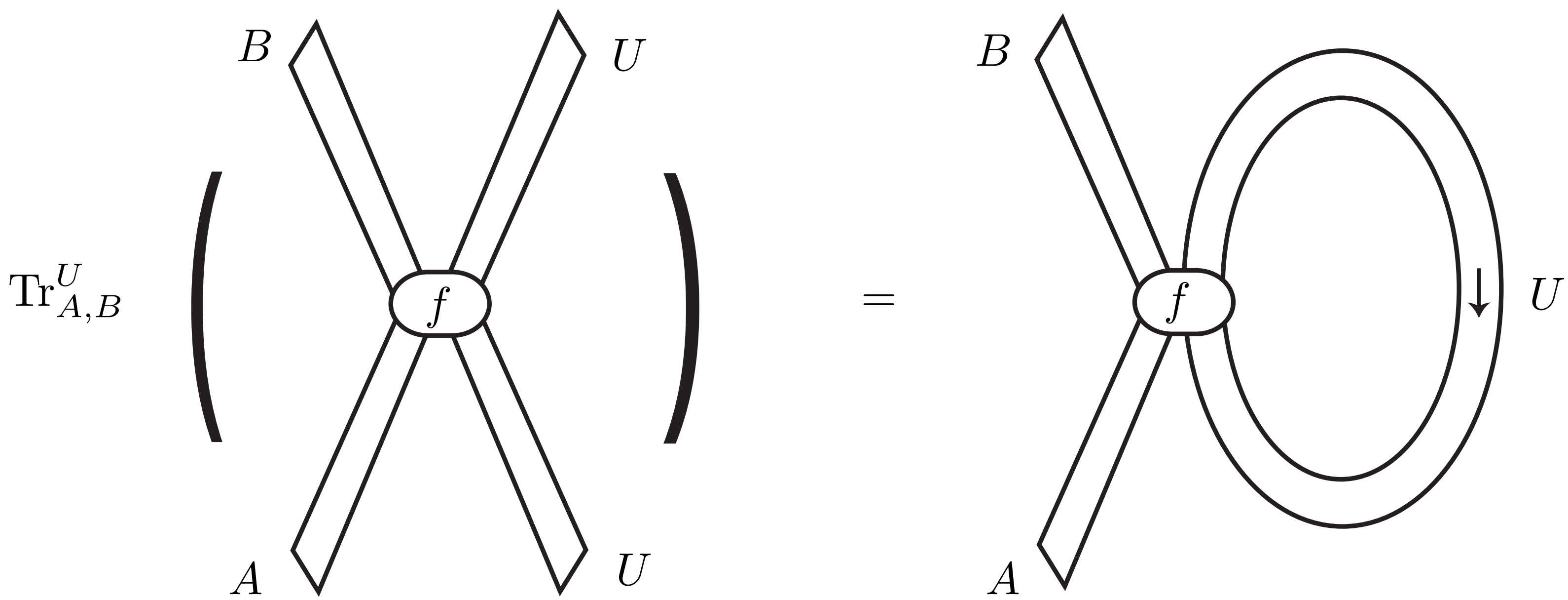
Trace operator (Joyal - Street - Verity 1996)

A **trace** in a balanced category $\mathbb{C}$ is an operator

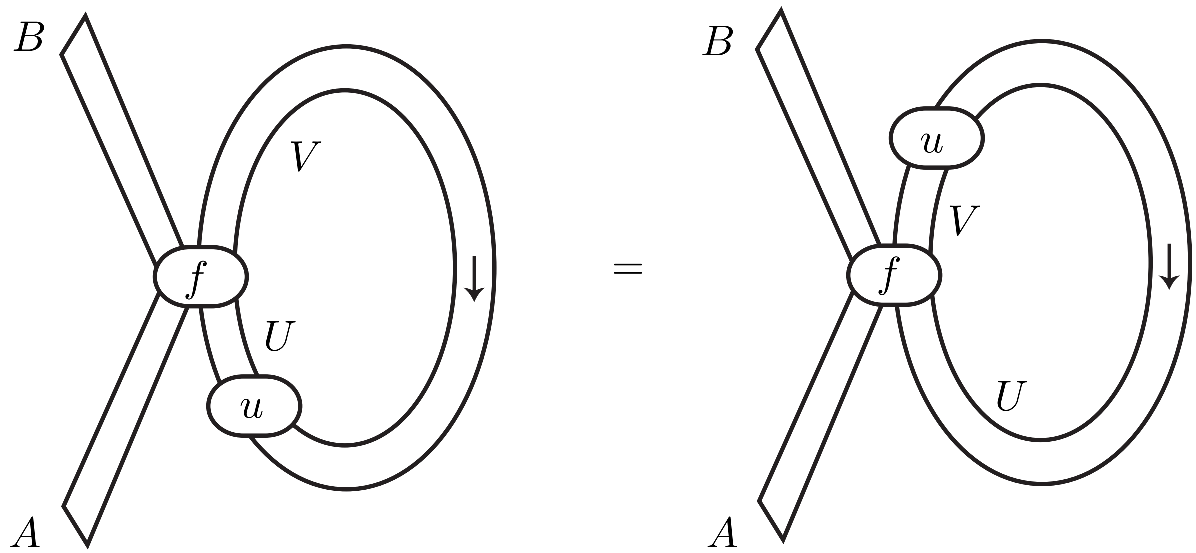
$$\mathsf{Tr}^{U}_{A,B} \qquad \frac{A \otimes U \longrightarrow B \otimes U}{A \longrightarrow B}$$

depicted as **feedback** in string diagrams:

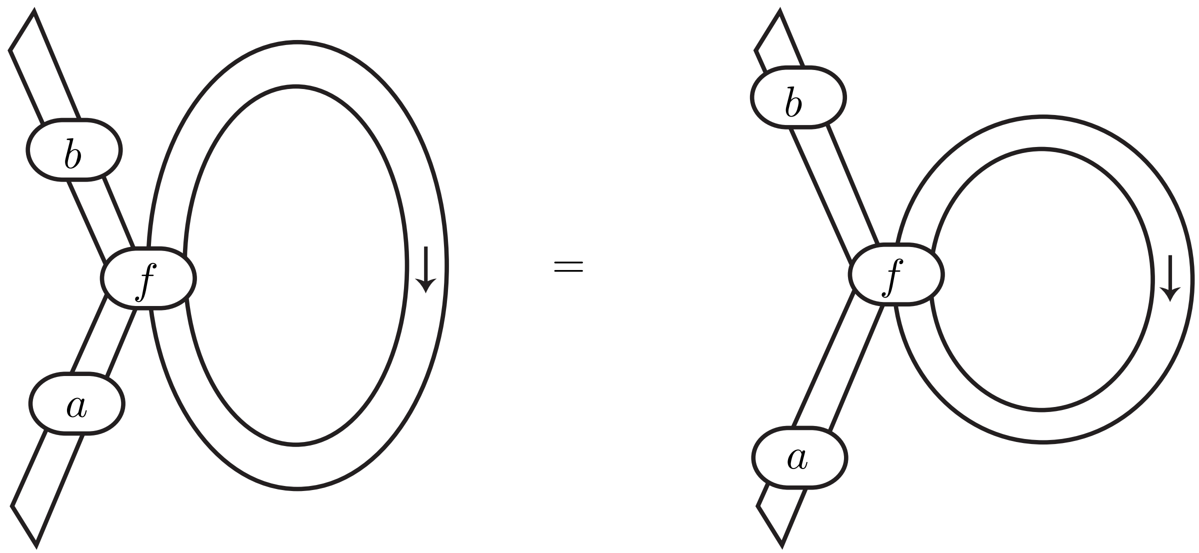
Trace operator



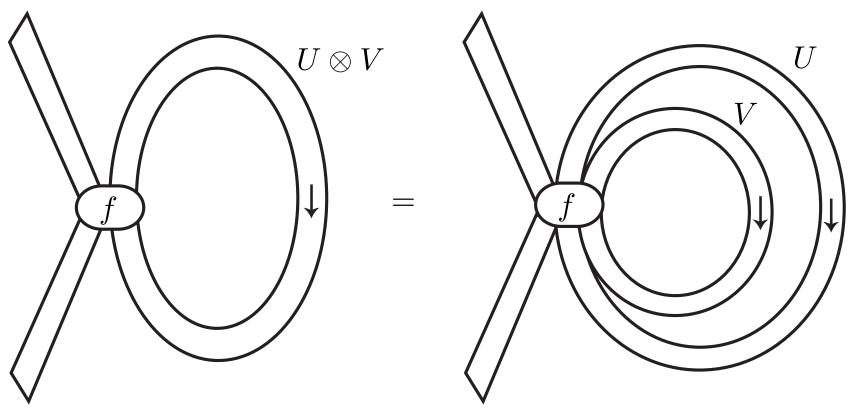
Sliding (naturality in U)

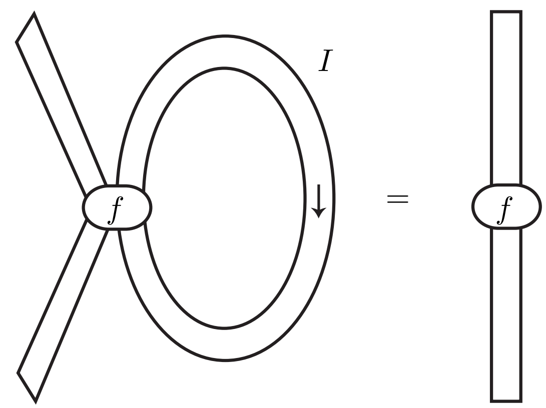


Tightening (naturality in A, B)

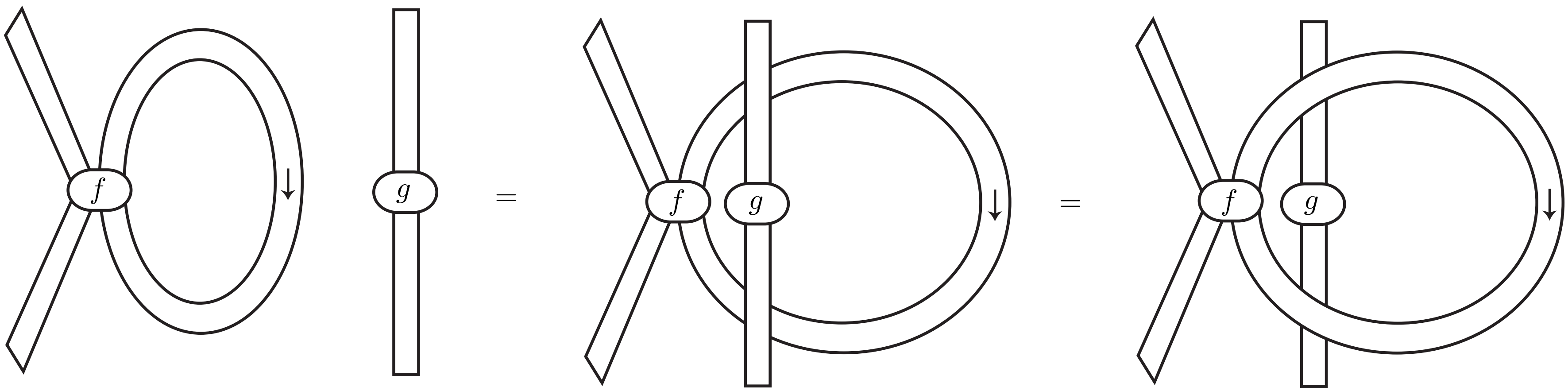


Vanishing (monoidality in U)

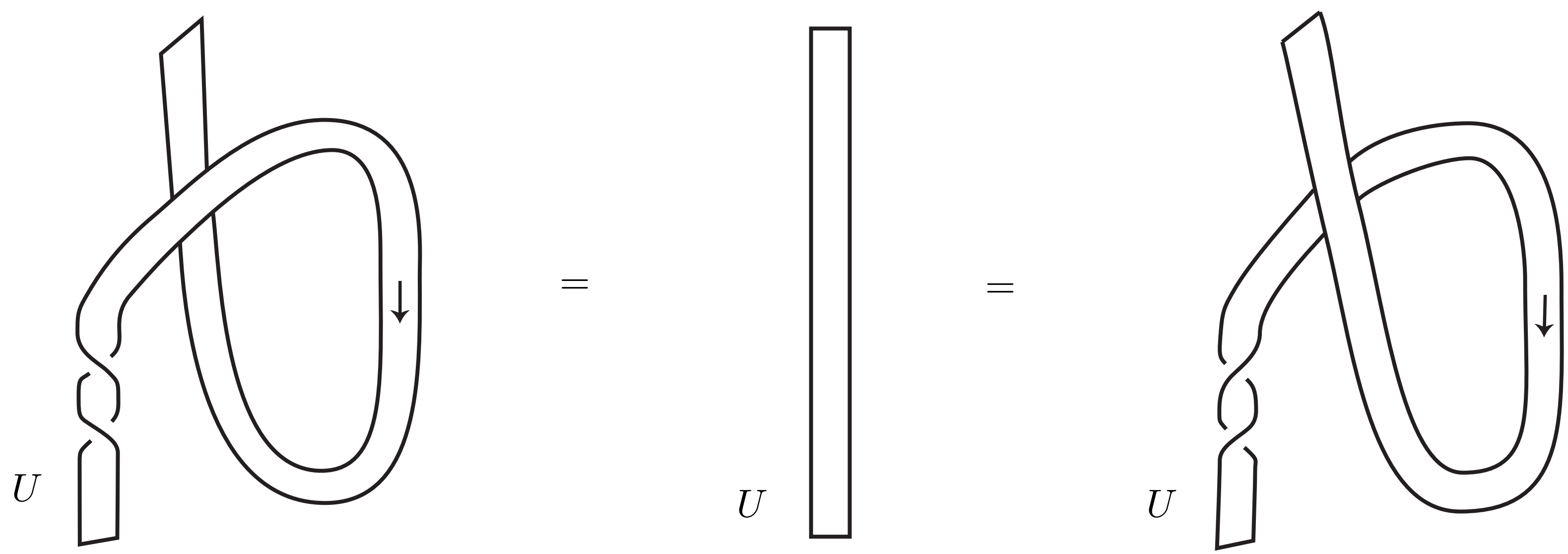




Superposing



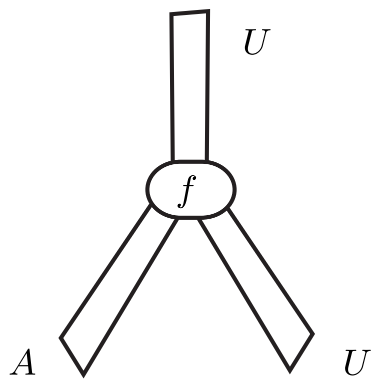
Yanking



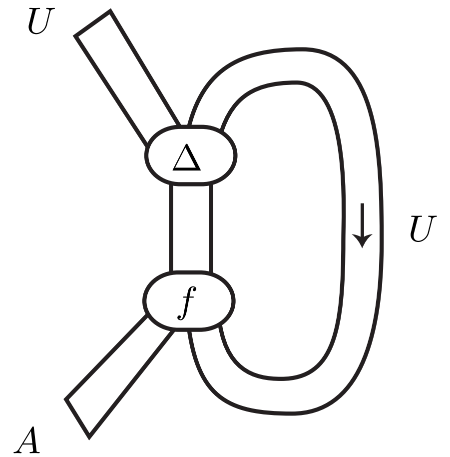
Traces = fixpoints
(Hasegawa - Hyland 1997)

In cartesian categories:

Fix :



$\mapsto$



Well-behaved parametric fixpoint operator.

Original question

When does a trace in the category $\mathbb{L}$ lifts to a trace in the category $\mathbb{M}$?

$$\begin{tikzcd} \mathbb{M} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{L} \arrow[ll, bend left, "M"] \end{tikzcd}$$

$\boxed{\textbf{Observation: } \text{the functor } L \text{ is usually } \textbf{faithful}.}$

Derived question

Characterize when a **faithful** balanced functor

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

between **balanced categories** transport a trace in $\mathbb{D}$ to a trace in $\mathbb{C}$.

Characterization

There exists a trace on $\mathbb{C}$ preserved by the functor F

$$\Longleftrightarrow$$

for all objects A, B, U and morphism

$$f \ : \ A \otimes U \longrightarrow B \otimes U$$

there exists a morphism

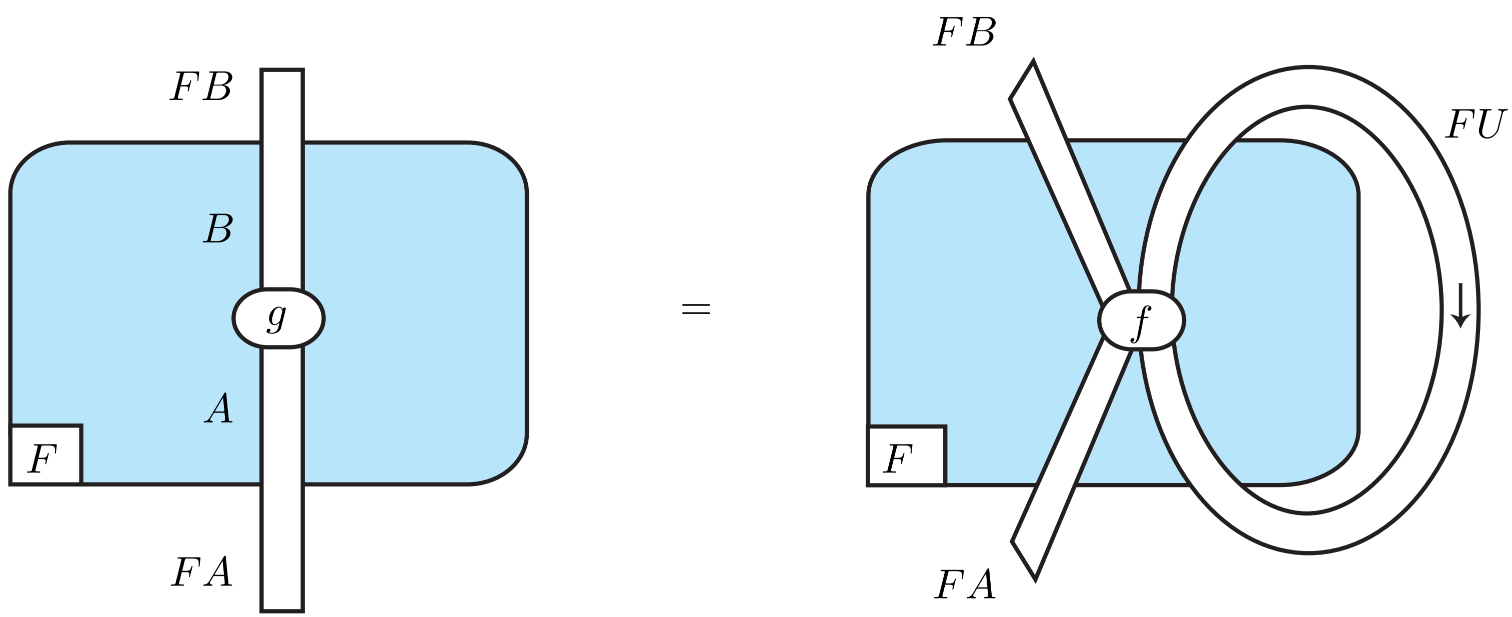
$$g : A \longrightarrow B$$

such that

$$F(g) \ = \ \mathsf{Tr}^{FU}_{FA,FB}(m^{-1}_{[A,B]} \circ F(f) \circ m_{[A,B]})$$

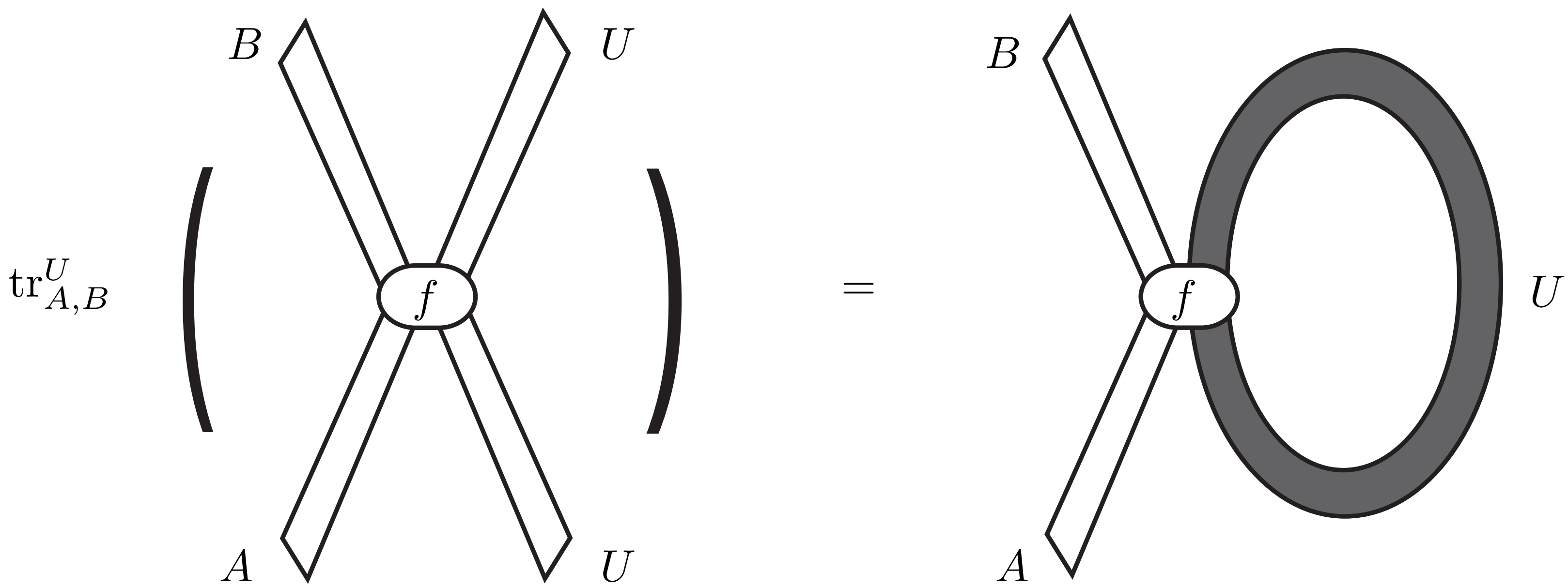
Pictorially...

The last equality is depicted as follows:

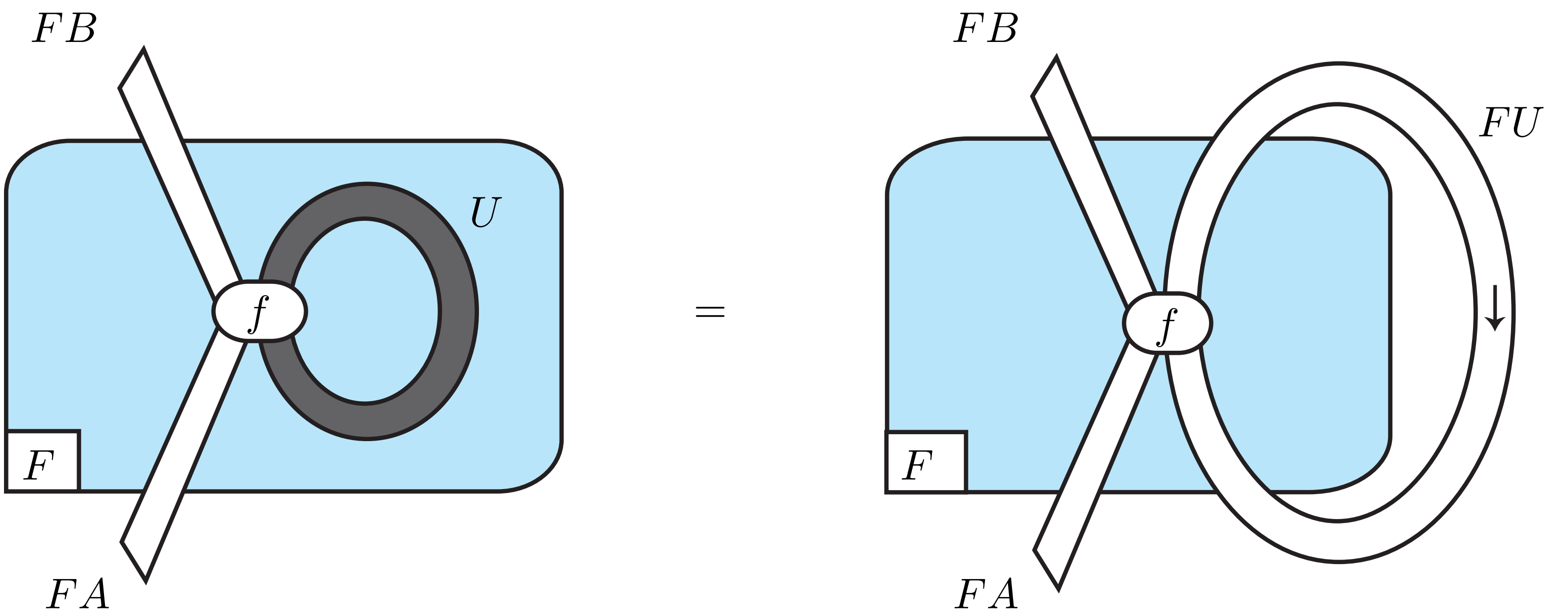


Proof sketch...

First step: define the operator



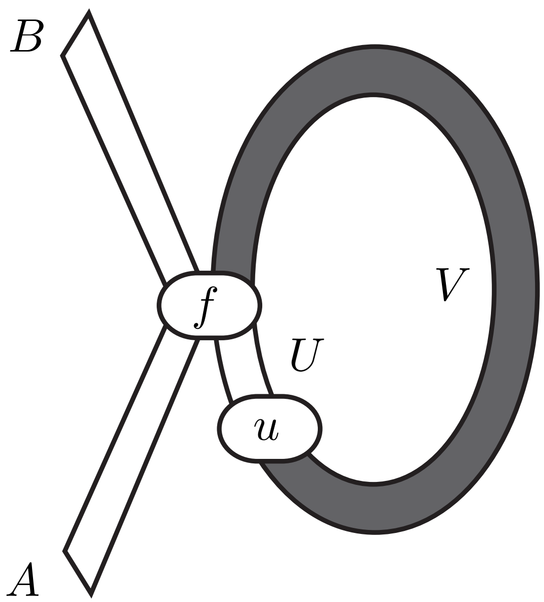
which transports every morphism f to the **unique** morphism such that



Second step: prove that tr satisfies the axioms of a trace operator.

Illustration: sliding (1)

We want to show that



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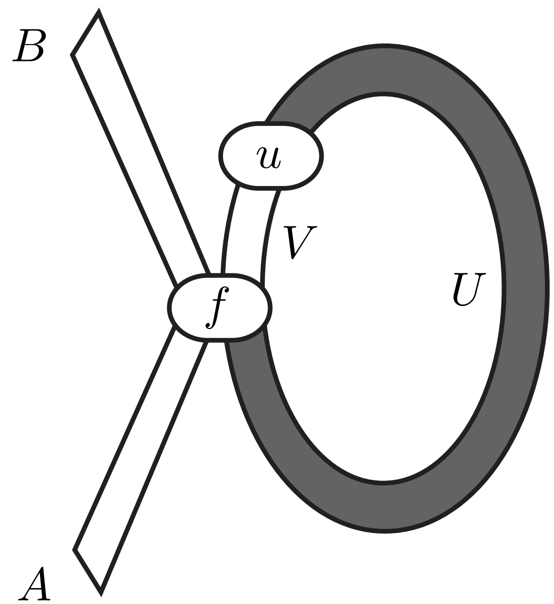
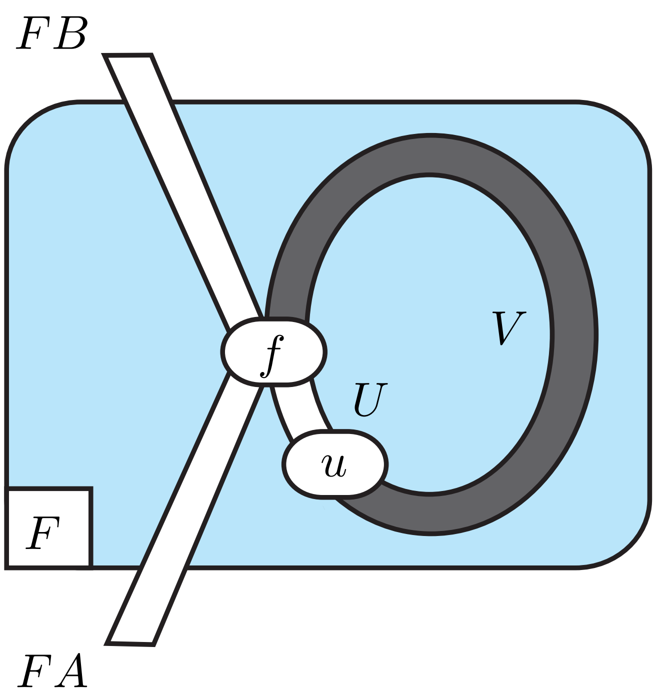


Illustration: sliding (2)

Because the functor F is **faithful**, this reduces to



$=$

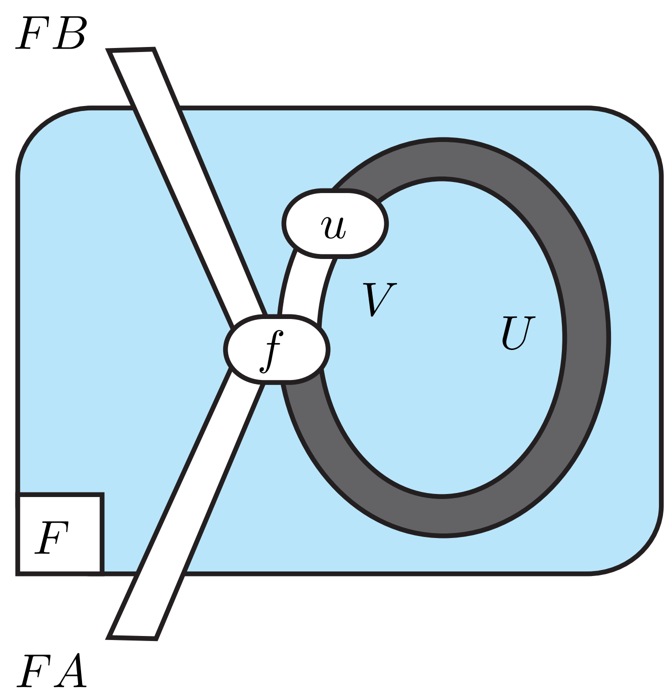


Illustration: sliding (3)

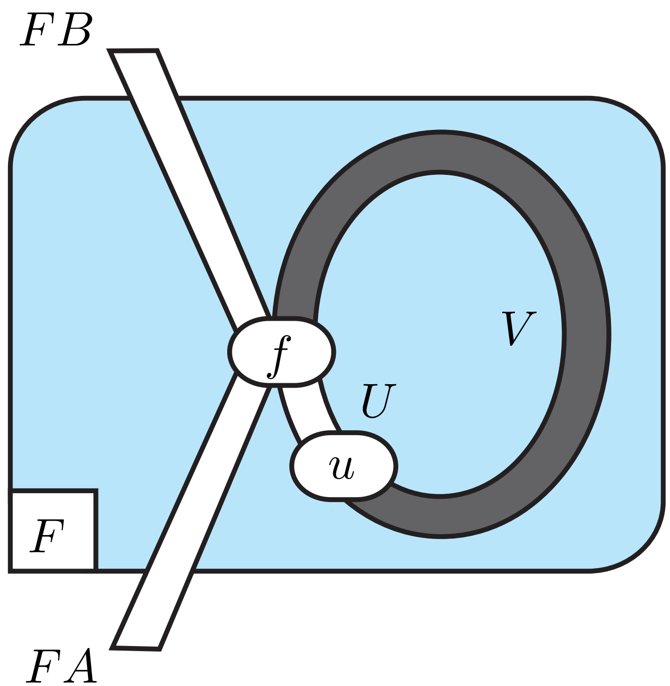


Illustration: sliding (4)

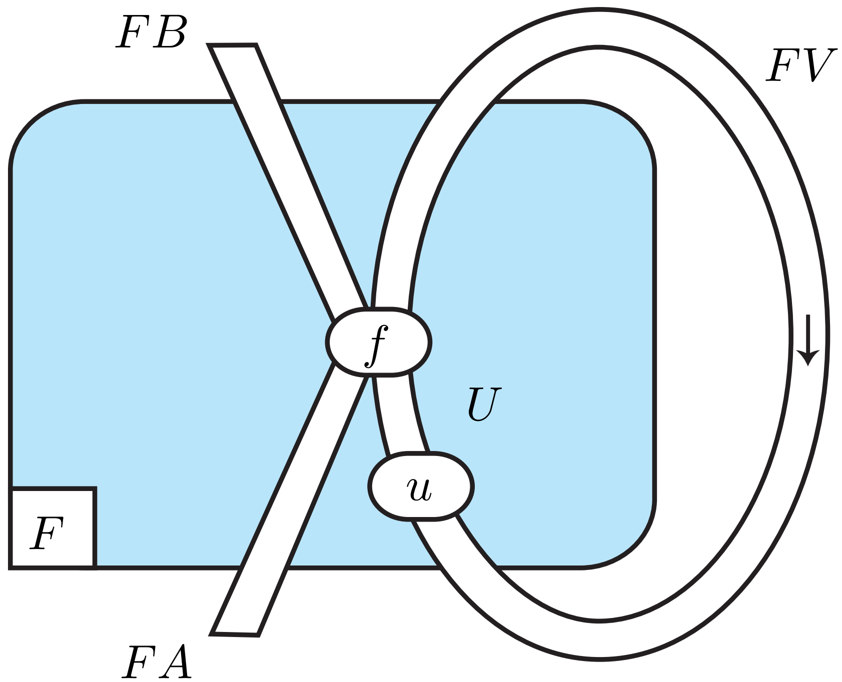


Illustration: sliding (5)

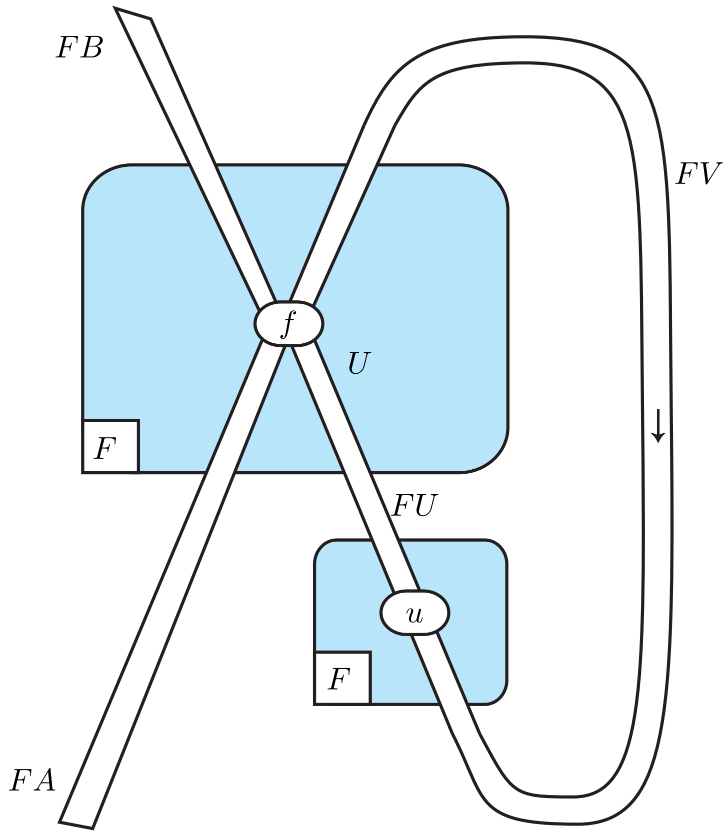


Illustration: sliding (6)

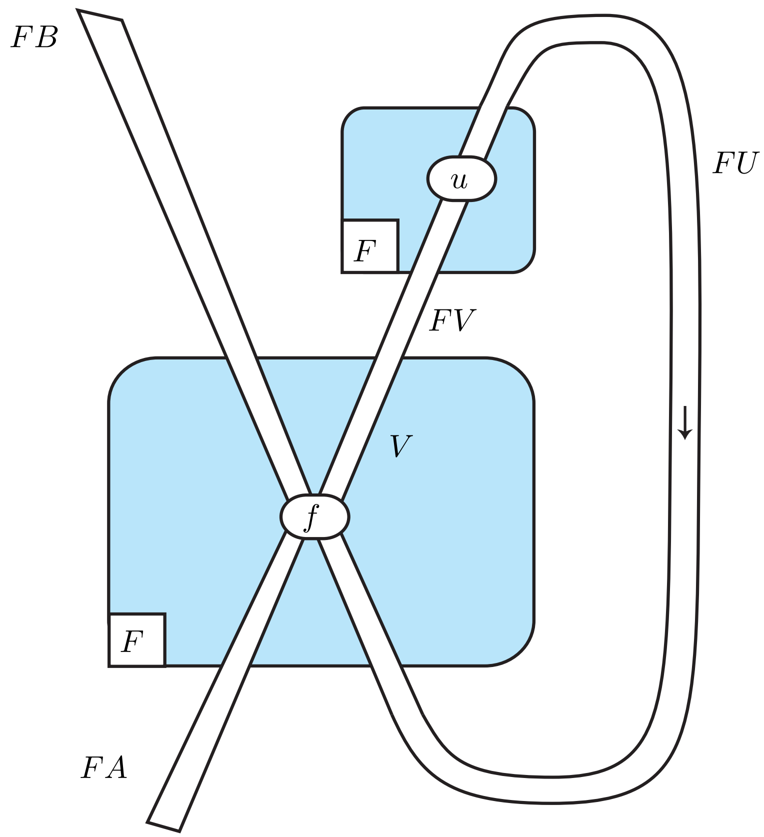


Illustration: sliding (7)

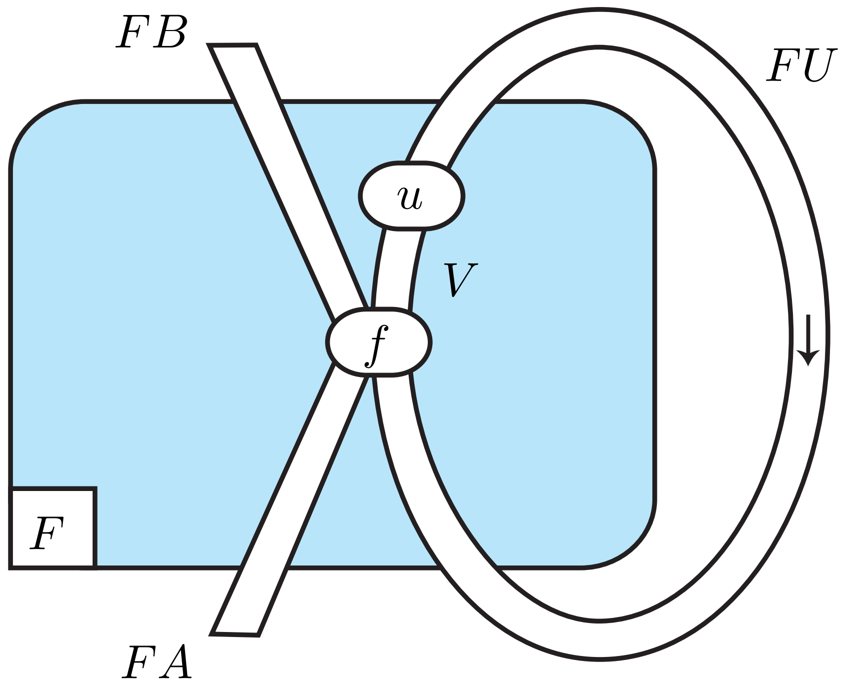
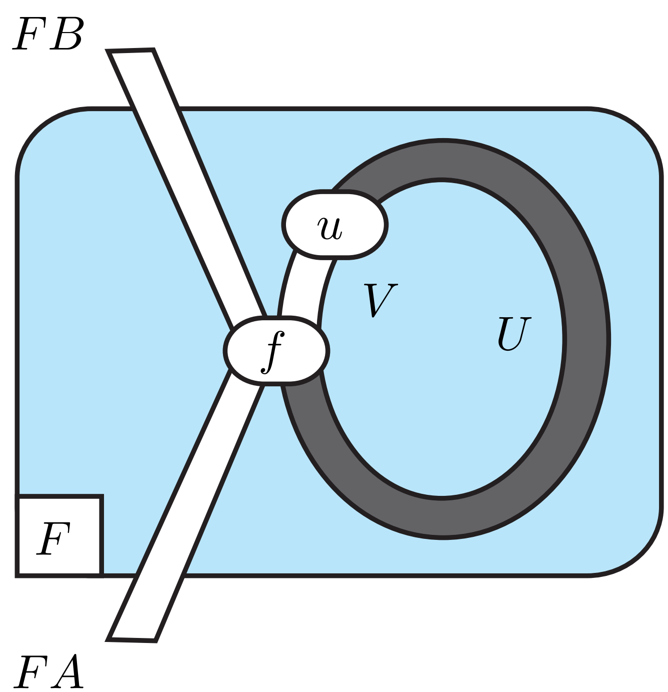


Illustration: sliding (8)



Examples

- The relational model of linear logic
- Game semantics (Conway games)

$\boxed{\text{Provides well-behaved parametric fixpoints in game semantics}}$

Non-Example (Masahito Hasegawa)

The adjunction generated by the **powerset monad**:

category of coalgebras

$$\begin{tikzcd} Set \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & Rel \arrow[ll, bend left, "M"] \end{tikzcd}$$

Kleisli category

$\boxed{\text{The trace of a function in } Rel \text{ is not a function anymore}}$

Currently investigating Ryu Hasegawa's model of linear logic.

Exercise 2

Decomposing

the exponential box of linear logic

The categorical semantics of linear logic

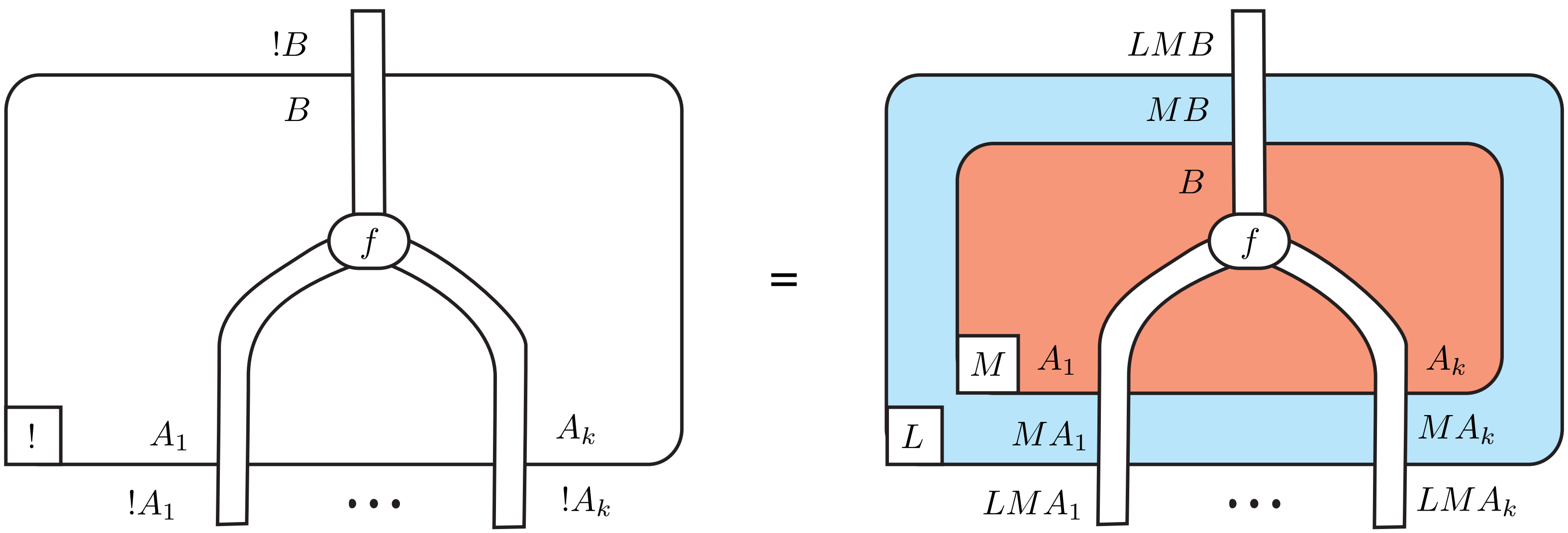
A **symmetric monoidal adjunction**

$$\begin{tikzcd} \mathbb{M} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{L} \arrow[ll, bend left, "M"] \end{tikzcd}$$

$\mathbb{M}$ cartesian $\qquad\qquad$ $\mathbb{L}$ symmetric monoidal closed

$$! \quad = \quad L \circ M$$

Decomposition of the exponential box



Decomposition of the contraction node

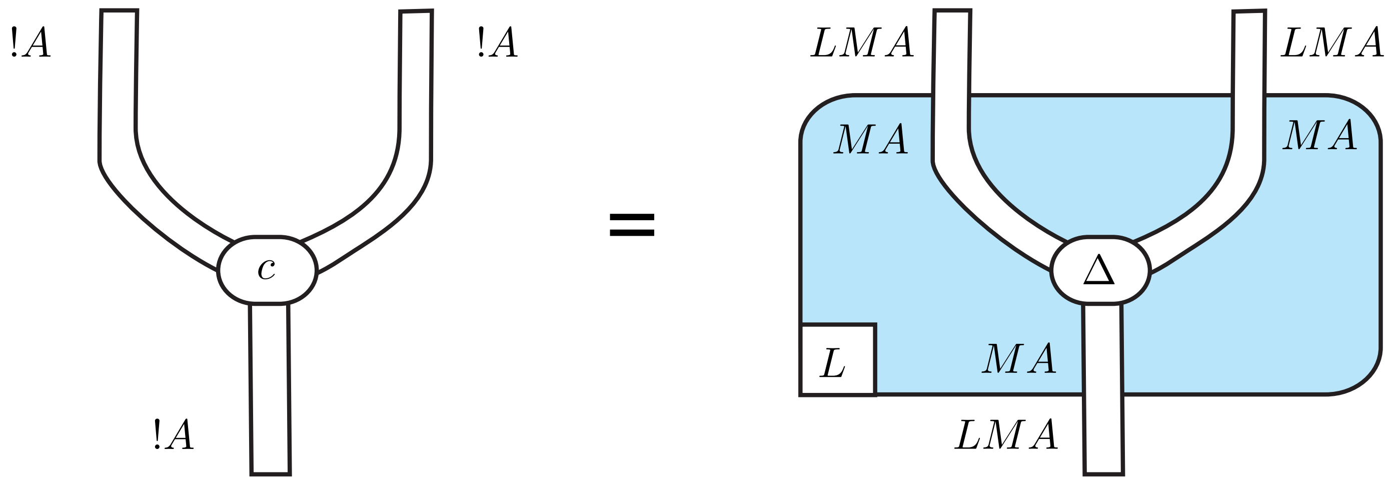
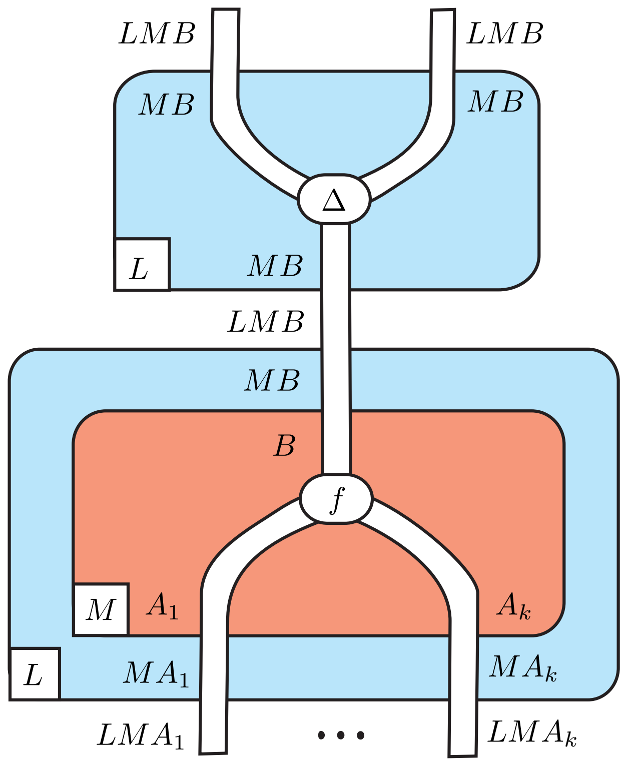
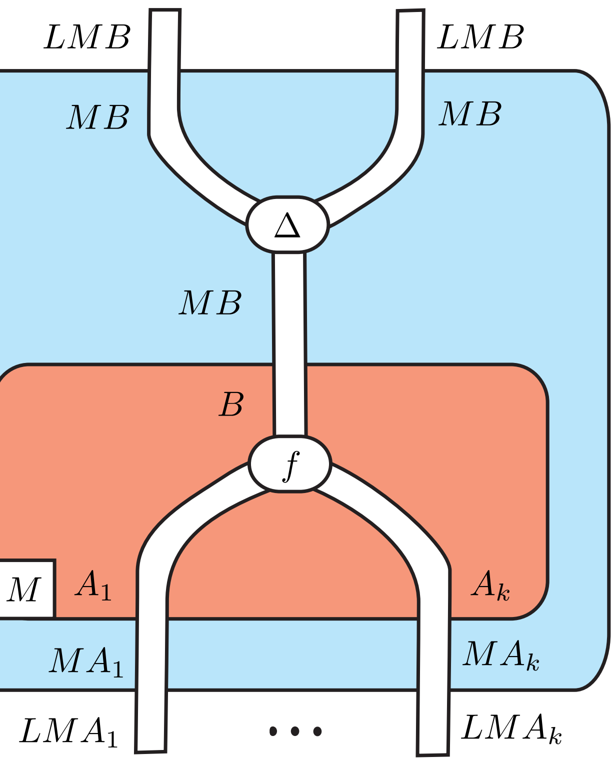


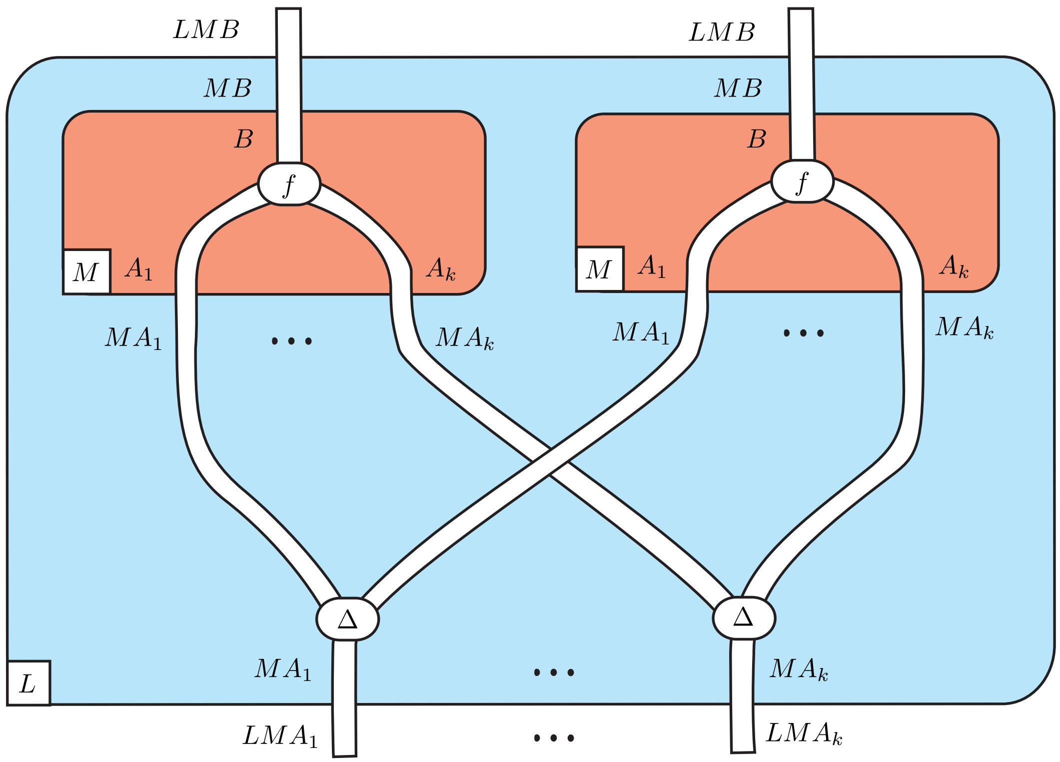
Illustration: duplication of the exponential box



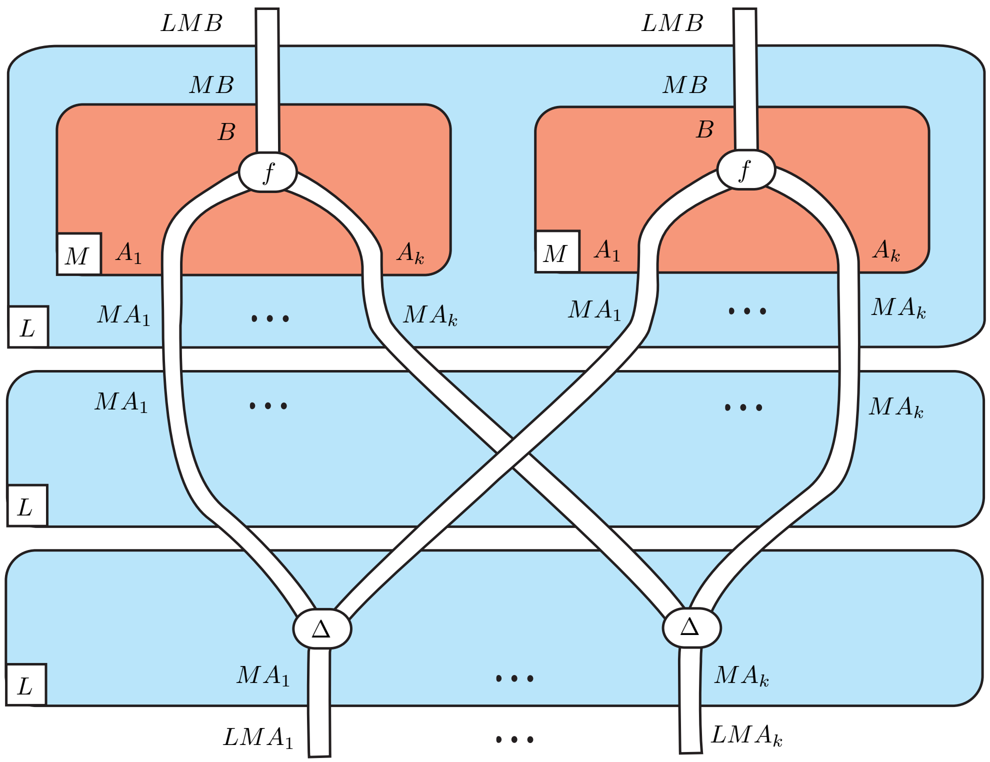
Duplication (step 1)



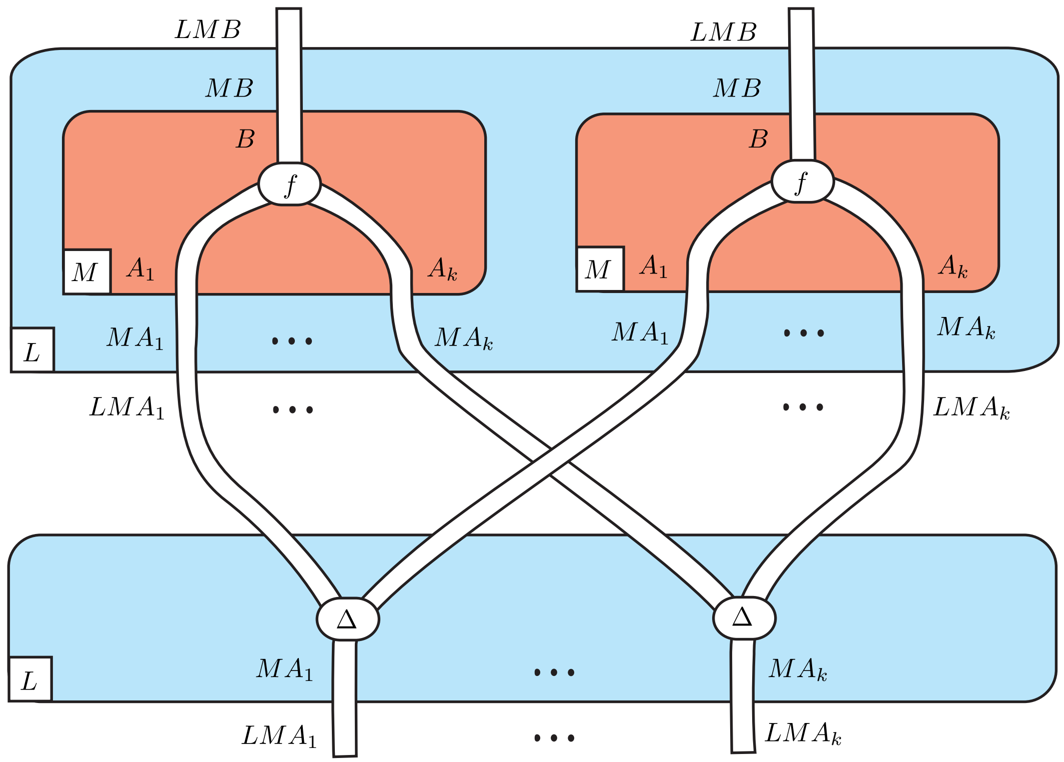
Duplication (step 2)



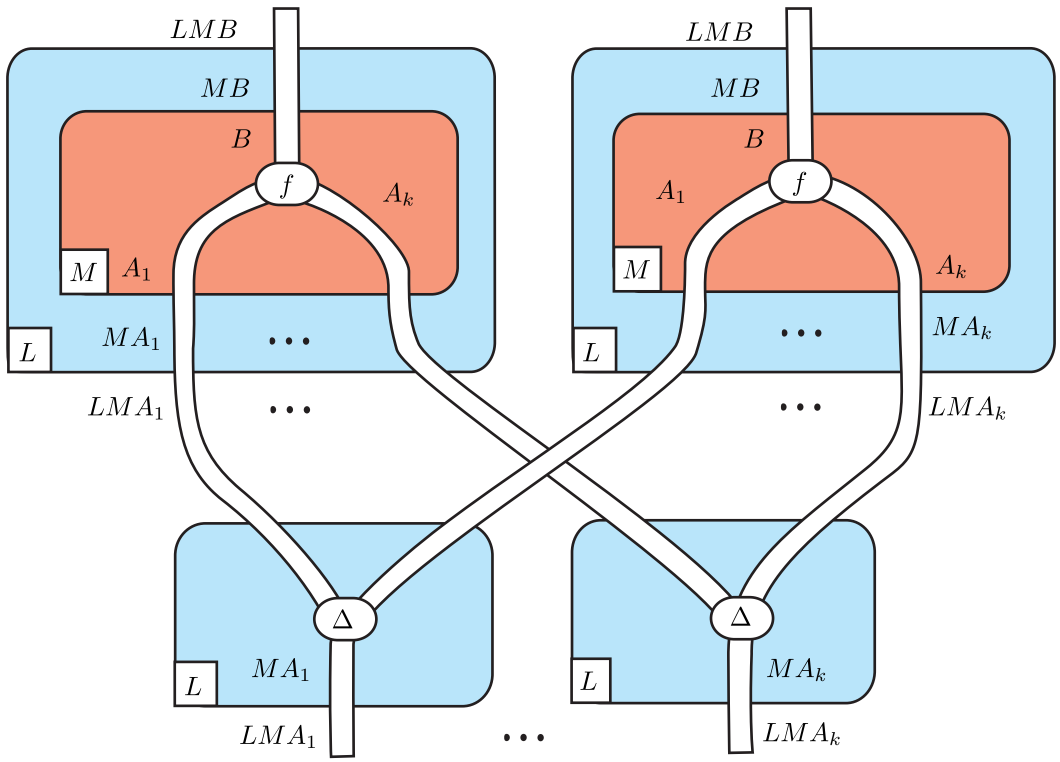
Duplication (step 3)



Duplication (step 4)



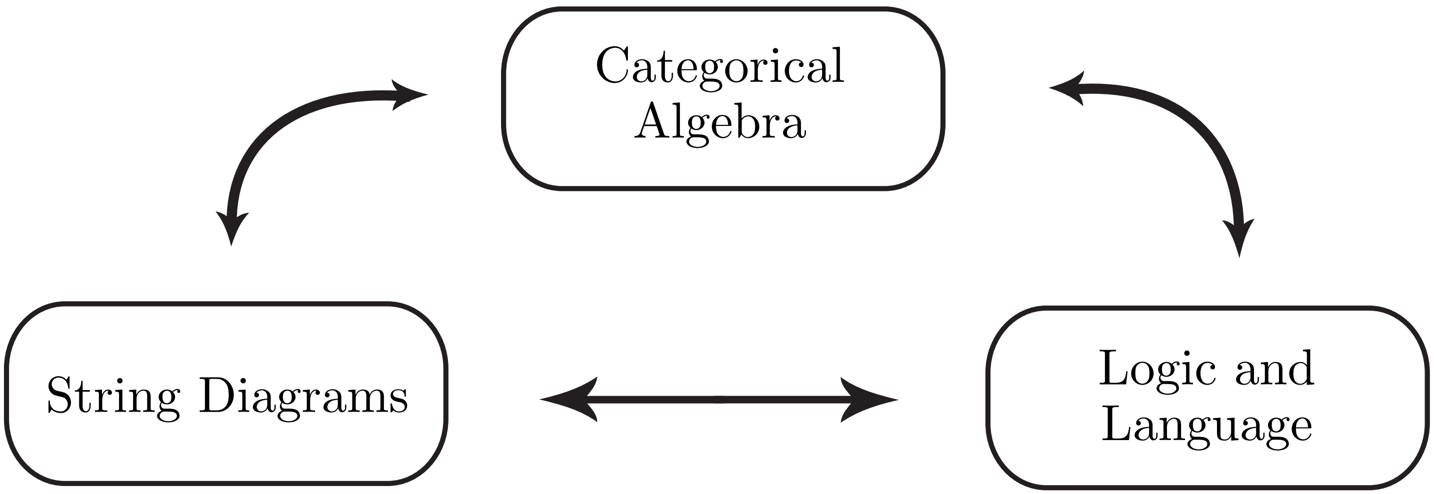
Duplication (step 5)



Five steps instead of one!

Follows **faithfully** the categorical proof of **soundness**.

Philosophy



Thank you!