Quantum decoherence

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Outline

Quantum decoherence:

1. Basics of quantum mechanics
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1. Basics of quantum mechanics

The* very *basics

Quantum mechanics is a fundamental branch of physics that replaces classical mechanics and classical electromagnetism at atomic and subatomic level. For instance, Newtonian mechanics and classical electromagnetism cannot explain why an electron is staying in its orbital whereas quantum mechanics does.

Quantum mechanics is concerned with quanta which refers to discrete units that quantum theory assigns to physical quantity e.g.: Energy of an atom at rest, spin of a particle etc.

Quantum phenomena

Some phenomena that do not appear in classical physics that quantum mechanics describes:

- Quantisation or discretisation of certain physical quantities
- Wave-particle duality
- Uncertainty principle
- etc.

Probabilities

The ‘orthodox’ description of quantum mechanics – which I will present today – is a probabilistic one. Indeed, quantum mechanics is content to give a probabilistic description of the quantum world. The point is important since the ‘explanation’ of the measurement process I will give here via quantum decoherence do not explain how the quantum system finally evolves in a stable deterministic state but stops at the level of probabilities.

The postulates

Here is a summary of the mathematical framework in which quantum mechanics is expressed:

(i) To each physical system is associated a (separable) complex Hilbert space $\mathcal{H}$. One dimensional subspaces of $\mathcal{H}$ represents states of the system.

(ii) Evolution is described by means of unitary transformations on $\mathcal{H}$.

(iii) Observables are self-adjoint operators on $\mathcal{H}$.

(iv) A composite system is the Hilbert space obtained via the tensor product of the component state spaces.

Measurement in-depth

Note: From now on, I'll speak about finitary quantum mechanics i.e. Dim$(\mathcal{H}) < \infty$.

As I said in the previous slide, a measurable quantity (observable) is represented by a self-adjoint operator σ. The eigenvalues of σ $\{\lambda_i\} \subset \mathbb{R}$ represents the set of possible outcomes of the measurement. If I measure λ_j then, the state of the system after the measurement is the eigenstate corresponding to the eigenvalue λ_j.

Measurement in-depth

Recall that a hermitian matrix can be unitarily diagonalised. The eigenvectors obtained during the process spans the state space of the system in fact, this says that a state can be expressed as a superposition of the eigenstates of the observable.

An example

Now, let us consider the simplest quantum system: the qubit (quantum bit). A qubit is represented by a normalized vector $|\phi\rangle \in \mathcal{H}$ i.e.

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle; \quad \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1.$$

Now, consider the following observable on $\mathcal{H}$:

$$\sigma := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 1P_0 - 1P_1$$

Then expected value of – the probability to observe – 0 if I measure $|\phi\rangle$ is $\langle\phi|P_0|\phi\rangle = \overline{\alpha}\alpha \geq 0$.

Density operators

Now, note that:

$$\langle\phi|P_0|\phi\rangle = \text{Tr}(P_0|\phi\rangle\langle\phi|).$$

There, $\rho_\phi := |\phi\rangle\langle\phi|$ is called the *density operator* associated to $|\phi\rangle$. It is the projector that describes the one-dimensional subspace of $\mathcal{H}$ spanned by $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$; it is a positive matrix of trace one:

$$\begin{bmatrix} \alpha\overline{\alpha} & \alpha\overline{\beta} \\ \beta\overline{\alpha} & \beta\overline{\beta} \end{bmatrix}$$

The unitaries U then act on ρ as $U\rho U^\dagger$.

Density operators

Density operator formalism has many advantages. If two state vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ produce the same density matrix then, they are indistinguishable physically. Moreover, we can also consider probabilistic mixture of density operators.

Density operators

Concretely, Given a set of states and probabilities $\{|\phi_i\rangle, p_i\}$ with $\sum_i p_i = 1$, one can form the following density operator:

$$\sum_i p_i |\phi_i\rangle\langle\phi_i|$$

It corresponds to a probabilistic mixture of the states $|\phi_i\rangle$ hence, the points in the closure of the convex hull of pure states – states of the form $|\psi\rangle\langle\psi|$– are density operators.

Density operators

The set of physical transformation on density operators is given by superoperators i.e. completely positive trace preserving operators. This includes, isometries, traces, measurements, adjoining an ancilla etc.

2. The measurement problem

The measurement problem

So far, I have just given the mathematical formalism describing quantum mechanics and I have said that an observable is just some hermitian operator. Returning to our toy-model, before the measurement, the qubit is in the state:

$$\begin{bmatrix} \alpha\overline{\alpha} & \alpha\overline{\beta} \\ \beta\overline{\alpha} & \beta\overline{\beta} \end{bmatrix}$$

after the measurement, it will be in the state $|0\rangle\langle 0|$ with probability $\alpha\overline{\alpha}$ and in the state $|1\rangle\langle 1|$ with probability $\beta\overline{\beta}$. This is the wavefunction collapse.

The measurement problem

We can also think of the measurement as an operator which has the following effect on our qubit:

$$\begin{bmatrix} \alpha\overline{\alpha} & \alpha\overline{\beta} \\ \beta\overline{\alpha} & \beta\overline{\beta} \end{bmatrix} \xrightarrow{M} \begin{bmatrix} \alpha\overline{\alpha} & 0 \\ 0 & \beta\overline{\beta} \end{bmatrix}$$

Of course, this is nothing but $P_0\rho_\phi P_0 + P_1\rho_\phi P_1$. However, these probabilities are a mathematical artefact, we are left with either one of the two possible states in our hand.

The measurement problem

Vanishing of the off-diagonal elements in the density operator says that we no longer have a coherent superposition. Moreover, the outcome state is a classical mixture. Finally, note that the passage from coherent superposition to mixture is not unitary and fundamentally irreversible.

3. Quantum decoherence

To this date, there seems to be no consensus on how we should explain the full process of measurement. However, there is some consensus on how we pass from a coherent quantum state to a statistical mixture (classical). Here is the key intuition:

An observer is macroscopic (classical) while the objects exhibiting quantum phenomena are very small (a grain of dust is too big). However, for an observer to learn the outcome of a measurement on a quantum system, this system must become correlated with a measurement apparatus (big, classical) and this gives the illusion of a collapse.

Quantum decoherence

The remainder of this talk will be about explaining how this transition occurs that is, from the density operator point of view, the process of passing from a coherent superposition to a statistical mixture and that’s what *quantum decoherence* is about.

Quantum decoherence

There are a few concepts I need to introduce in order to speak of decoherence namely,

- Traces and
- Quantum entanglement.

Traces

Working with density matrix yields a notion of trace in an obvious sense: If I have a density operator ρ acting on $\mathcal{H} \bigotimes \mathcal{H}'$ (which is a joint system), I can trace out the part which act over $\mathcal{H}$ or the part acting over $\mathcal{H}'$. Now what does the trace mean? It is just the expression of the ignorance of what happens in some subsystem (which is traced out) to which some observer doesn't have access.

Quantum entanglement

A state described by ρ acting on $\mathcal{H} \otimes \mathcal{H}'$ is *unentangled* if it factors as $\rho_{\mathcal{H}} \otimes \rho_{\mathcal{H}'}$ and it is said to be *entangled* otherwise. For instance, the 2 qubit state

$$\rho_{AB} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is entangled; simply put, it is quantum correlation.

Quantum entanglement

Indeed, suppose that the 2 qubit state described in the previous slide is shared between two observers (Alice and Bob) and Alice measures her share and obtain ‘0’, then the state collapse to

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After, if Bob measure his share of the state then, he will obtain ‘0’ with certainty.

All this together

Now, suppose that Alice and Bob know that they share an entangled qubit however, they can't communicate then, from their respective point of view, their state looks like

$$\mathsf{Tr}_A(\rho_{AB}) = \mathsf{Tr}_B(\rho_{AB}) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is the completely mixed state. However, if they can communicate, Alice measures her share and communicates the result to Bob then, Bob knows the state of his qubit without having to measure it.

Back to decoherence. What actually happens during a measurement? There is a quantum object that interacts with some macroscopic apparatus in such a way that the apparatus produces a signal which states the outcome of the measurement. In other words, there is an interaction between a quantum system, a macroscopic apparatus and the environment where the wave function becomes correlated (entangled) with both the environment, the apparatus and, moreover, the environment is inaccessible (traced) to observers.

Quantum decoherence

Formally, we have a state that represents a system $\mathcal{S}$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

and a measurement apparatus $\mathcal{A}$ s.t.

$$|0\rangle \otimes |A_0\rangle \mapsto |0\rangle \otimes |A_0\rangle \quad \text{and} \quad |1\rangle \otimes |A_0\rangle \mapsto |1\rangle \otimes |A_1\rangle$$

and an environment $\mathcal{E}$ in the state

$$|\epsilon\rangle.$$

The joint system is, before the measurement in the state

$$|\psi\rangle \otimes |A_0\rangle \otimes |\epsilon\rangle.$$

The measurement process is given by

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |A_0\rangle \otimes |\epsilon\rangle \mapsto \alpha|0\rangle \otimes |A_0\rangle \otimes |\epsilon_0\rangle + \beta|1\rangle \otimes |A_1\rangle \otimes |\epsilon_1\rangle$$

s.t. $\langle\epsilon_0|\epsilon_1\rangle = 0$. As I said before, the environment is inaccessible to the observer thus, the reduced density operator is

$$\rho_{\mathcal{S}\mathcal{A}} = \mathrm{Tr}_{\mathcal{E}}(\rho_{\mathcal{S}\mathcal{A}\mathcal{E}}) = |\alpha|^2|0\rangle|A_0\rangle\langle 0|\langle A_0| + |\beta|^2|1\rangle|A_1\rangle\langle 1|\langle A_1|$$

Which contains only classical correlations.