Traces of intruders

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— Slides from the Fields Institute Workshop on Traces —
Ottawa, April 28, 2007

Outline

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Loop categories

Definition

Given a small strict symmetric monoidal category

$$\mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C} \xleftarrow{I} 1$$

define

$$\begin{aligned} |\mathbb{C}^{\circlearrowright}| &= |\mathbb{C}| \\ \mathbb{C}^{\circlearrowright}(A, B) &= \oint_{U \in |\mathbb{C}|} \mathbb{C}(A \otimes U, B \otimes U) \end{aligned}$$

Loop categories

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define

$$\begin{aligned}
|\mathbb{C}^{\circlearrowleft}| &= |\mathbb{C}| \\
\mathbb{C}^{\circlearrowleft}(A, B) &= \left(\sum_{U \in |\mathbb{C}|} \mathbb{C}(A \otimes U, B \otimes U) \right) \Big/ \sim
\end{aligned}$$

. . . where ∼ is the coend equivalence. . .

$$\begin{tikzcd}
 & \mathbb{C}(A\otimes U, B\otimes U) \arrow[dr, dashed] & \\
\mathbb{C}(A\otimes V, B\otimes U) \arrow[ur, "(-)\circ(A\otimes u)"] \arrow[dr, "(B\otimes u)\circ(-)"'] \arrow[rr, dashed] & & \mathbb{C}^{\circlearrowleft}(A,B) \\
 & \mathbb{C}(A\otimes V, B\otimes V) \arrow[ur, dashed] &
\end{tikzcd}$$

. . . where ∼ is the coend equivalence. . .

$$
\begin{tikzcd}[column sep=small]
 & A \otimes U \arrow[dl, "f"'] \\
B \otimes V \arrow[dr, "B\otimes u"'] & \\
 & B \otimes U
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}[column sep=small]
A \otimes V \arrow[dr, "A\otimes u"] & \\
 & A \otimes U \arrow[dl, "f"] \\
B \otimes V &
\end{tikzcd}
$$

...extended to factor out c

$$
\begin{tikzcd}
A \otimes A \arrow[d, "c"'] \\
A \otimes A
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
A \arrow[d, "id"] \\
A
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes U \otimes V \arrow[d, "h"'] \\
B \otimes U \otimes V
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
A \otimes V \otimes U \arrow[r, "A \otimes c"] & A \otimes U \otimes V \arrow[d, "h"] \\
B \otimes V \otimes U & B \otimes U \otimes V \arrow[l, "B \otimes c"]
\end{tikzcd}
$$

Composition

Given

- $f \in \mathbb{C}^{\circlearrowleft}(A, B)$ as $A \otimes U \xrightarrow{f_0} B \otimes U$, and
- $g \in \mathbb{C}^{\circlearrowleft}(B, C)$ as $B \otimes V \xrightarrow{g_0} C \otimes V$,

the composite

- $f \circ g \in \mathbb{C}^{\circlearrowleft}(A, C)$ can be viewed as

$$\begin{tikzcd}
A \otimes U \otimes V \arrow[r, "f_0 \otimes V"] & B \otimes U \otimes V \arrow[d, "B \otimes c"] & C \otimes U \otimes V \\
& B \otimes V \otimes U \arrow[r, "g_0 \otimes U"] & C \otimes V \otimes U \arrow[u, "C \otimes c"]
\end{tikzcd}$$

Composition

Given

- $f \in \mathbb{C}^{\circlearrowleft}(A,B)$ as $A \otimes U \xrightarrow{f_0} B \otimes U$, and
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- $f \circ g \in \mathbb{C}^{\circlearrowleft}(A,C)$ can be viewed as

$$\begin{tikzcd} A \otimes U \otimes V \arrow[r, "f_0 \otimes V"] & B \otimes U \otimes V \arrow[d, "B \otimes c"] & \\ A \otimes V \otimes U \arrow[u, "A \otimes c"] & B \otimes V \otimes U \arrow[r, "g_0 \otimes U"] & C \otimes V \otimes U \end{tikzcd}$$

Tensor

Given

- $f \in \mathbb{C}^{\circlearrowleft}(A, B)$ as $A \otimes U \xrightarrow{f_0} B \otimes U$, and
- $h \in \mathbb{C}^{\circlearrowleft}(C, D)$ as $C \otimes V \xrightarrow{h_0} D \otimes V$,

the tensor product

- $f \otimes h \in \mathbb{C}^{\circlearrowleft}(A \otimes C, B \otimes D)$ can be viewed as

$$\begin{tikzcd}
A \otimes C \otimes U \otimes V \arrow[d, "A \otimes c \otimes V"] & B \otimes D \otimes U \otimes V \\
A \otimes U \otimes C \otimes V \arrow[r, "f_0 \otimes h_0"'] & B \otimes U \otimes D \otimes V \arrow[u, "B \otimes c \otimes V"']
\end{tikzcd}$$

Trace?

Given

- $f \in \mathbb{C}^{\circlearrowleft}(A \otimes U, B \otimes U)$ as

$$(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V$$

its trace

- $Tr^{U}_{AB} f \in \mathbb{C}^{\circlearrowleft}(A, B)$ can be viewed as

$$A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)$$

i.e. as itself, modulo associativity.

Yes, trace

Proposition

The operators $Tr^U_{AB} : \mathbb{C}^{\circlearrowleft}(A \otimes U, B \otimes U) \longrightarrow \mathbb{C}^{\circlearrowleft}(A, B)$ satisfy the trace axioms.

Sketch of a proof

- dinaturality (sliding), yanking $\Leftarrow$ imposed by $\sim$
- naturality (tightening) $\Leftarrow$ def'n of composition in $\mathbb{C}^{\circlearrowleft}$
- vanishing, superposition $\Leftarrow$ inspection

Loop monad

Monad data

- 2-category $\mathcal{SM}$ of small symmetric monoidal cats
- 2-functor $\circlearrowleft : \mathcal{SM} \longrightarrow \mathcal{SM}$
- unit functors

$$\eta_{\mathbb{C}} \; : \; \mathbb{C} \longrightarrow \mathbb{C}^{\circlearrowleft}$$

$$(A \xrightarrow{f} B) \longmapsto [A \otimes I \xrightarrow{f \otimes I} B \otimes I]_{\sim}$$

- evaluation functors

$$\mu_{\mathbb{C}} \; : \; \mathbb{C}^{\circlearrowleft\circlearrowleft} \longrightarrow \mathbb{C}^{\circlearrowleft}$$

$$[[(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V]_{\sim}]_{\sim} \longmapsto [A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)]_{\sim}$$

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$$\begin{aligned} \mu_{\mathbb{C}} \ &: \ \mathbb{C}^{\circlearrowleft\circlearrowleft} \longrightarrow \mathbb{C}^{\circlearrowleft} \\ [[(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V]_{\sim}]_{\sim} &\longmapsto [A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)]_{\sim} \end{aligned}$$

Traces of intruders

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Trace as algebra
Loop categories
Loop monad
Traced cats are loop algs
Uniform trace
Applications
Intruders
Summary

Loop monad

Monad data

- 2-category $\mathcal{SM}$ of small symmetric monoidal cats
- 2-functor $\circlearrowleft : \mathcal{SM} \longrightarrow \mathcal{SM}$
- unit functors

$$\begin{aligned} \eta_{\mathbb{C}} &: \mathbb{C} \longrightarrow \mathbb{C}^{\circlearrowleft} \\ (A \xrightarrow{f} B) &\longmapsto [A \otimes I \xrightarrow{f \otimes I} B \otimes I]_{\sim} \end{aligned}$$

- evaluation functors

$$\begin{aligned} \mu_{\mathbb{C}} &: \mathbb{C}^{\circlearrowleft\circlearrowleft} \longrightarrow \mathbb{C}^{\circlearrowleft} \\ [[(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V]_{\sim}]_{\sim} &\longmapsto [A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)]_{\sim} \end{aligned}$$

Traced categories are loop algebras

Theorem

The loop category $\mathbb{C}^{\circlearrowleft}$ is the free traced category generated by the symmetric monoidal category $\mathbb{C}$.

The loop algebra structures $T : \mathbb{C}^{\circlearrowleft} \longrightarrow \mathbb{C}$ are just the trace operators, expressed in a functorial form.

[Syntactic construction: Abramsky, Kelly-Laplaza]

Proof

For a loop algebra $T : \mathbb{C}^{\circlearrowleft} \longrightarrow \mathbb{C}$, and every pair $A, B \in \mathbb{C}$,

- the families $\left\{ Tr^U_{AB} : \mathbb{C}(A \otimes U, B \otimes U) \longrightarrow \mathbb{C}(A, B) \right\}_{U \in \mathbb{C}}$ are in one-to-one correspondence with
- the arrow part $T_{AB} : \mathbb{C}^{\circlearrowleft}(A, B) \longrightarrow \mathbb{C}(A, B)$

along

$$\sum_U \mathbb{C}(A \otimes U, B \otimes U) \twoheadrightarrow \oint_U \mathbb{C}(A \otimes U, B \otimes U) \xrightarrow{Tr_{AB}} \mathbb{C}(A, B)$$

Proof

The operators $Tr^U_{AB} : \mathbb{C}(A \otimes U, B \otimes U) \longrightarrow \mathbb{C}(A, B)$ satisfy the trace axioms because:

- naturalities, yanking $\iff$ factor by $\oint_U \mathbb{C}(A \otimes U, B \otimes U)$,
- superposition $\iff$ $T \circ \boldsymbol{\eta}_{\mathbb{C}} = id_{\mathbb{C}}$
- vanishing $\iff$ $T \circ \boldsymbol{\mu}_{\mathbb{C}} = T \circ T^{\circlearrowleft}$

Uniform trace

Definition

A trace operator is *uniform* if

$$Tr^U_{AB}(f) \quad = \quad Tr^V_{AB}(g)$$

holds whenever there is some h which makes the diagram

$$\begin{tikzcd} A\otimes U \arrow[r, "A\otimes h"] \arrow[d, "f"'] & A\otimes V \arrow[d, "g"] \\ B\otimes U \arrow[r, "B\otimes h"'] & B\otimes V \end{tikzcd}$$

commute

Strict loop categories

Definition

Given a small symmetric monoidal category

$$\mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C} \xleftarrow{I} 1$$

define

$$\begin{aligned} |\mathbb{C}^{\hookleftarrow}| &= |\mathbb{C}| \\ \mathbb{C}^{\hookleftarrow}(A,B) &= \left(\sum_{U \in |\mathbb{C}|} \mathbb{C}(A \otimes U, B \otimes U) \right) \Big/ \approx \end{aligned}$$

. . . where $\approx$ strengthens the coend equivalence

$$\begin{tikzcd}
& \mathbb{C}(A\otimes U, B\otimes U) \arrow[dr, "(B\otimes h)\circ(-)"] \arrow[drr, dashed] & & \\
\mathbb{C}(A\otimes V, B\otimes U) \arrow[ur, "(-)\circ(A\otimes u)"] \arrow[dr, "(B\otimes u)\circ(-)"'] & & \mathbb{C}(A\otimes U, B\otimes V) \arrow[r, dashed] & \mathbb{C}^{\hookleftarrow}(A,B) \\
& \mathbb{C}(A\otimes V, B\otimes V) \arrow[ur, "(-)\circ(A\otimes h)"'] \arrow[urr, dashed] & &
\end{tikzcd}$$

... where $\approx$ strengthens the coend equivalence

by

$$\begin{tikzcd} A\otimes U \arrow[r, dashed, "A\otimes h"] \arrow[d, "f"'] \arrow[dr, phantom, "\approx"] & A\otimes V \arrow[d, "g"] \\ B\otimes U \arrow[r, dashed, "B\otimes h"] & B\otimes V \end{tikzcd}$$

Uniform traced categories are strict loop algebras

Theorem

The strict loop category $\mathbb{C}^{\looparrowleft}$ is the free uniform traced category generated by the symmetric monoidal category $\mathbb{C}$.

The strict loop algebra structures $T : \mathbb{C}^{\looparrowleft} \longrightarrow \mathbb{C}$ are just the uniform trace operators, expressed in a functorial form.

Applications

The upshot of the monadic view is that the structure of

- loop categories,
- trace algebras,
- trace homomorphisms

can often be **effectively calculated**.

Examples

$$\mathbb{N} \quad \subseteq \quad \mathbb{N}[\mathcal{T}] \quad \subseteq \quad \mathbb{N}[\mathcal{T},\mathcal{A}]$$

(sets) (clone) (action category)

The loop category of finite sets

Consider the monoid of natural numbers

$$\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N} \xleftarrow{0} 1$$

as the category of sets $n = \{0, 1, \ldots, n-1\}$ and functions. Then

$$\mathbb{N}^{\circlearrowleft}(a, b) \quad = \quad \sum_{u \in \mathbb{N}} \left\{ a + u \xrightarrow{f} b + u \mid \forall y \in u \exists x.\ f(x) = y \right\}$$

The loop category of finite sets

Notation

$$\text{Write } a + u \xrightarrow{f} b + u \text{ where } \forall y \in u \exists x.\ f(x) = y$$

$$\text{as} \qquad \hat{a} \xrightarrow{f} \hat{b} \qquad \text{i.e. } \begin{aligned} \hat{a} &= a + u \\ \hat{b} &= b + u \end{aligned}$$

The loop category of finite sets

$$
\begin{array}{rcl}
Tr^{v}_{ab} \;:\; \mathbb{N}^{\circlearrowleft}(a+v, b+v) & \longrightarrow & \mathbb{N}^{\circlearrowleft}(a,b) \\
\left(\hat{a}+v \xrightarrow{f} \hat{b}+v\right) & \longmapsto & \left(\hat{a}+\hat{v} \xrightarrow{\hat{f}} \hat{b}+\hat{v}\right)
\end{array}
$$

where

$$
\begin{tikzcd}
\hat{a}+\hat{v} \arrow[rr, "\hat{f}"] \arrow[d, dotted, hook] & & \hat{b}+\hat{v} \arrow[d, dotted, hook] \\
\hat{a}+v_{i+1} \arrow[rr, "f_{i+1}"] \arrow[dd, hook] & & \hat{b}+v_{i+1} \arrow[dd, hook] \\
& \hat{b}_{i+1}+v_{i+1} \arrow[ur, hook] \arrow[dr, hook] & \\
\hat{a}+v_i \arrow[ur, two heads] \arrow[rr, "f_i"'] \arrow[d, dotted, hook] & & \hat{b}+v_i \arrow[d, dotted, hook] \\
\hat{a}+v_0 \arrow[rr, "f_0=f"'] & & \hat{b}+v_0
\end{tikzcd}
$$

The loop category of finite sets

Then

$$f = f_0 \sim f_1 \sim \cdots \sim f_i \sim f_{i+1} \sim \cdots \overset{*}{\sim} \hat{f}$$

because

$$\begin{tikzcd}[column sep=small] & \hat{a}+v_i \arrow[dl, "\hat{f}_i"'] \\ \hat{b}+v_{i+1} \arrow[dr, hook] & \\ & \hat{b}+v_i \end{tikzcd} \quad \sim \quad \begin{tikzcd}[column sep=small] \hat{a}+v_{i+1} \arrow[dr, hook] & \\ & \hat{a}+v_i \arrow[dl, "\hat{f}_i"] \\ \hat{b}+v_{i+1} & \end{tikzcd}$$

*NB The chain must be finite, because the sets are finite.

The strict loop category of finite sets

$$\mathbb{N}^{\leftrightarrow}(a,b) = \sum_{u\in\mathbb{N}} \Big\{ a+u \xrightarrow{f} b+u \mid \forall y \in u.\ f(y) = y \\ \vee \left(\exists i.\ f^i(y) \in b \\ \wedge \exists x.\ f(x) = y\right) \Big\}$$

The strict loop category of finite sets

Notation

$$\text{Write } a + w \xrightarrow{f} b + w \text{ where } \forall y \in w.\ f(y) = y$$
$$\vee \quad (\exists i.\ f^i(y) \in b$$
$$\wedge\ \exists x.\ f(x) = y$$

$$\text{as} \qquad \tilde{a} \xrightarrow{f} \tilde{b} \qquad \text{i.e. } \tilde{a} = a + w$$
$$\tilde{b} = b + w$$

The strict loop category of finite sets

$$
\begin{array}{rcl}
Tr^v_{ab} \; : \; \mathbb{N}^{\looparrowleft}(a+v, b+v) & \longrightarrow & \mathbb{N}^{\looparrowleft}(a,b) \\
(\tilde a + v \xrightarrow{f} \tilde b + v) & \longmapsto & \left(\tilde a + \tilde v \xrightarrow{\tilde f} \tilde b + \tilde v\right)
\end{array}
$$

where

$$
\begin{tikzcd}
\tilde a+\tilde v \arrow[rr, "\tilde f"] \arrow[d, equal] & & \tilde b+\tilde v \arrow[d, equal] \\
\tilde a+v^b+v^\bullet & & \tilde b+v^b+v^\bullet \\
\tilde a+v^b+v^\circlearrowleft \arrow[u, two heads] \arrow[d, equal] & & \tilde b+v^b+v^\circlearrowleft \arrow[u, two heads] \arrow[d, equal] \\
\tilde a+\hat v \arrow[rr, "\hat f"'] & & \tilde b+\hat v
\end{tikzcd}
$$

$$
\begin{array}{rcl}
v^b & = & \{y \in v \mid \exists i.\ f^i(y) \in b\} \\
v^\circlearrowleft & = & \{y \in v \mid \forall i.\ f^i(y) \in v\} \\
v^\bullet & = & v^\circlearrowleft / \approx \\
 & & x \approx y \iff \exists k\ell \geq 0.\ g^k(x) = g^\ell(y)
\end{array}
$$

Clones (Lawvere theories)

Given an algebraic theory $\mathcal{T} = \langle \Sigma_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ where

- $\Sigma = \Sigma_{\mathcal{T}}$ is a signature, and
- $E = E_{\mathcal{T}}$ is a set of equations

adjoin to $\mathbb{N}$

- an arrow $m \xrightarrow{\varphi} n$ for every m-tuple $\langle \varphi_i(x_1, \ldots x_n) \rangle_{i \le m}$ of well-formed Σ-operations, and
- identify them modulo E

to form (the dual of) the *clone* (or *Lawvere theory*)

- $\mathbb{N}[\mathcal{T}] = \mathbb{N}[\Sigma; E]$.

NB since the well-formed operations include projections, the arrows of $\mathbb{N}[\mathcal{T}]$ include the variables. A clone is thus a form of *polynomial category* (cf. Lambek-Scott).

Clones (Lawvere theories)

(à la Milner)

$$
\begin{aligned}
\left|\mathbb{N}[\mathcal{T}]\right| &= |\mathbb{N}| \\
\mathbb{N}[\mathcal{T}](m,n) &= \left\{ (x_1,\ldots,x_n)\langle \varphi_1,\ldots,\varphi_m \rangle \right\} \Big/ \alpha
\end{aligned}
$$

Iterative algebras

Definition.

An algebraic theory $\mathcal{T}$ is *iterative* if every system

$$\begin{aligned}
y_1 &= f_1(y_1, y_2, \ldots, y_k, \ldots, y_\ell) \\
y_2 &= f_2(y_1, y_2, \ldots, y_k, \ldots, y_\ell) \\
&\cdots \\
y_k &= f_k(y_1, y_2, \ldots, y_k, \ldots, y_\ell)
\end{aligned}$$

has a unique solution

$$\begin{aligned}
f_1^\dagger(y_{k+1}, \ldots, y_\ell) &= f_1(f_1^\dagger, f_2^\dagger, \ldots, f_k^\dagger, \ldots, y_\ell) \\
f_2^\dagger(y_{k+1}, \ldots, y_\ell) &= f_2(f_1^\dagger, f_2^\dagger, \ldots, f_k^\dagger, \ldots, y_\ell) \\
&\cdots \\
f_k^\dagger(y_{k+1}, \ldots, y_\ell) &= f_k(f_1^\dagger, f_2^\dagger, \ldots, f_k^\dagger, \ldots, y_\ell)
\end{aligned}$$

provided that all equations are *guarded*, i.e. that none of the operations f_j is a projection.

Traced clones

Theorem

A clone $\mathbb{N}[\mathcal{T}]$ has a uniform trace if and only if the corresponding algebraic theory $\mathcal{T}$ is iterative.

Proof (1)

An arrow $f \in \mathbb{N}[\mathcal{T}](a+v, b+v)$ is a tuple

$$
\begin{aligned}
f_1^b &= f_1^b(y_1^a, \ldots, y_a^a, y_1^v, \ldots, y_v^v) \\
&\cdots \\
f_b^b &= f_b^b(y_1^a, \ldots, y_a^a, y_1^v, \ldots, y_v^v) \\
f_1^v &= f_1^v(y_1^a, \ldots, y_a^a, y_1^v, \ldots, y_v^v) \\
&\cdots \\
f_v^v &= f_v^v(y_1^a, \ldots, y_a^a, y_1^v, \ldots, y_v^v)
\end{aligned}
$$

Proof (2)

Rearranging the equations, we can achieve that

$$\begin{aligned} f_1^v &= f_1^v(\ldots, y_1^v, \ldots, y_v^v) \\ &\cdots \\ f_k^v &= f_k^v(\ldots, y_1^v, \ldots, y_v^v) \end{aligned}$$

are guarded operations, whereas

$$\begin{aligned} f_{k+1}^v &= f_{k+1}^v(\ldots, y_1^v, \ldots, y_v^v) \\ &\cdots \\ f_v^v &= f_v^v(\ldots, y_1^v, \ldots, y_v^v) \end{aligned}$$

are projections.

Proof (3)

The second set just induces some identifications of variables.

This gives a $(v - k)$-tuple $y^*_{k+1}, y^*_{k+2}, \ldots y^*_v$, possibly with repetitions.

Proof (4)

Now solve

$$
\begin{aligned}
y_1^v &= f_1^v(y_1^v, \ldots, y_k^v, y_{k+1}^*, \ldots y_v^*) \\
&\cdots \\
y_k^v &= f_k^v(y_1^v, \ldots, y_k^v, y_{k+1}^*, \ldots y_v^*)
\end{aligned}
$$

to get

$$
\begin{aligned}
f_1^\dagger &= f_1^v(f_1^\dagger, \ldots, f_k^\dagger, y_{k+1}^*, \ldots, y_v^*) \\
&\cdots \\
f_k^\dagger &= f_k^v(f_1^\dagger, \ldots, f_k^\dagger, y_{k+1}^*, \ldots, y_v^*)
\end{aligned}
$$

Proof (5)

$Tr^{v}_{ab}(f) \in \mathbb{N}[\mathcal{T}](a,b)$ is now the tuple

$$
\begin{aligned}
f^{\bullet}_{1} &= f^{b}_{1}(y^{a}_{1},\ldots,y^{a}_{a},f^{\dagger}_{1},\ldots,f^{\dagger}_{k},y^{*}_{k+1},\ldots,y^{*}_{v})\\
f^{\bullet}_{2} &= f^{b}_{2}(y^{a}_{1},\ldots,y^{a}_{a},f^{\dagger}_{1},\ldots,f^{\dagger}_{k},y^{*}_{k+1},\ldots,y^{*}_{v})\\
&\cdots\\
f^{\bullet}_{b} &= f^{b}_{b}(y^{a}_{1},\ldots,y^{a}_{a},f^{\dagger}_{1},\ldots,f^{\dagger}_{k},y^{*}_{k+1},\ldots,y^{*}_{v})
\end{aligned}
$$

Action categories

(Milner 95, DP 97)

$$
\begin{aligned}
\left|\mathbb{N}[\mathcal{T};\mathcal{A}]\right| &= |\mathbb{N}| \\
\mathbb{N}[\mathcal{T};\mathcal{A}](m,n) &= \left\{ (x_1,\ldots,x_n)[P]\langle\varphi_1,\ldots,\varphi_m\rangle \right\} \Big/ \alpha
\end{aligned}
$$

Traced action categories

Theorem

An action category $\mathbb{N}[\mathcal{T};\mathcal{A}]$ *has a uniform trace if and only if*
the algebraic theory $\mathcal{T}$ *is iterative, and*
the pomsets in $\mathcal{A}$ *are consistent.*

Intruder in the Middle

Solving the Turing Test

[[this part was not presented]]

Tracing out the Middle

[[this part was not presented]]

Summary

- the trace operators can be viewed in a functorial form
  - as algebras for the loop monad
- the trace structure can be freely adjoined to process models
  - hiding = tracing out
- intrusion can be modeled in terms of the Int-composition
  - security analysis becomes unwinding the trace