Formal Homotopy Quantum Field Theories and Traces.

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- Also for an algebra, L, $(U(L), Aut(L), \delta)$ is a crossed module, where $U(L)$ is the group of units of L and δ maps a unit e to the automorphism given by conjugation by e.

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- Extra special case: X, CW-complex, $A = X_1$, 1-skeleton of X,

$$\pi_2(X, X_1) \stackrel{\partial}{?} \pi_1(X_1)$$

determines 2-type of X. NB. $\ker \partial = \pi_2(X), \text{coker} \partial = \pi_1(X)$.

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  ; is the unit for the monoidal structure on d-cobord

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Usual picture with usual provisos on associativity etc.

- Get that a TQFT is simply a monoidal functor:

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- $T(M)$ is a linear map from $\mathbb{C}$ to itself, i.e., is speci ed by a single complex number, a numerical invariant of M

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- Partial solution by Turaev (1999): Replace X by characteristic structure map, $g : X \; ! \; B$, where B is a `background' space, e.g. BG.
- For the cobordisms, want $F : M \; ! \; B$ agreeing with the structure maps on the ends, but F will only be given *up to homotopy relative to that boundary*.

- Get a monoidal category d $Hocobord(B)$: (Rodrigues, 2000)

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- De nition: A HQFT is a monoidal functor
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General result (Rodrigues) : d-dimensional HQFTs over B depend on the $(d + 1)$-type of B only, so can assume $\pi_n(B)$ is trivial for $n > d + 1$.
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- d-manifold = disjoint union of oriented circles,
- cobordism = oriented surface possibly with boundary,
- can restrict to B being a 2-type.

$B = K(\pi, 1)$

Here

- $\pi_1(B) = \pi$, $\pi_i(B) = 1$ for $i > 1$, and `extra structure' = `isomorphism class of principal π-bundles'.
- (Turaev, 1999) HQFTs over $K(\pi, 1)$!

  π-graded algebras with inner product and `extra structure'

  = *crossed π-algebras.*

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(iii) $\rho(ab, c) = \rho(a, bc)$ for any a, b, c 2 L.

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  - (iii) for any $g, h \ 2 \ \pi$, $a \ 2 \ L_g$, $b \ 2 \ L_h$, $\phi_h(a)b = ba$;
  - (iv) (**Trace formula**) for any $g, h \ 2 \ \pi$ and $c \ 2 \ L_{ghg^{-1}h^{-1}}$,

$$Tr(c\phi_h : L_g \ ! \ L_g) = Tr(\phi_{g^{-1}} c : L_h \ ! \ L_h),$$

where Tr denotes the K-valued trace of the endomorphism.

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  Progress so far: Notion of Formal HQFT based on combinatorial models of structural maps. `Crossed C-algebras' giving classi cation for this formal model.

Formal C- map on a manifold M

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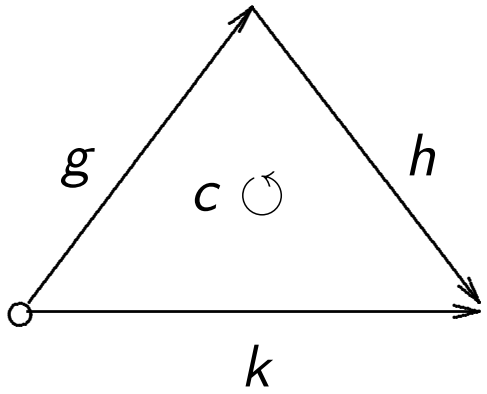
- labelling of edges with elements of P;
- labelling of faces with elements of C;
- boundary condition:



with $g; h; k \ 2 \ P$ and $c \ 2 \ C$, and where $@c = kh^{-1}g^{-1}$:

- cocycle condition from any 3-simplices.

Formal HQFTs - definition

A *formal HQFT* with background C assigns

- to each formal C-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, a K-vector space $\tau(\mathbf{g})$, and by extension, to each formal C-map on a 1-manifold S, given by a list, $\mathbf{g} = f\mathbf{g}_i \, j \, i = 1, 2, \ldots, mg$ of formal C-circuits, a vector space $\tau(\mathbf{g})$ and an isomorphism,

$$\tau(\mathbf{g}) = \bigotimes_{i=1,\ldots,m} \tau(\mathbf{g}_i),$$

identifying $\tau(\mathbf{g})$ as a tensor product;

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(ii) For formal C-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \ ! \ \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \ ! \ \mathbf{g}_2$$

with composite $\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}$, we have

$$\tau(\mathbf{F} \#_{\mathbf{g}_1} \mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \ ! \ \tau(\mathbf{g}_2).$$

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(iii) $\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}$.

Crossed C-algebras - the definition

Let $C = (C, P, \partial)$ be a crossed module. A *crossed C-algebra* consists of a crossed P-algebra, $L = \bigoplus_{g \in P} L_g$, together with elements $\epsilon\ 2\ L_{\partial c}$, for $c\ 2\ C$, such that

(a) $\tilde{1} = 1\ 2\ L_1$;

(b) for $c, c'\ 2\ C$, $\widetilde{(c'c)} = \tilde{c}'\ \ \epsilon$;

(c) for any $h\ 2\ P$, $\phi_h(\epsilon) = {}^{\widetilde{h}}c$.

We note that the rst two conditions make `tilderisation' into a group homomorphism $(\sim) : C\ \ !\ \ U(L)$, the group of units of the algebra, L.
In fact:

Suppose that L is a crossed C-algebra. The diagram

$$\begin{array}{ccc} C & \xrightarrow{(\tilde{})} & U(L) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\delta} \\ P & \xrightarrow[\phi]{} & Aut(L) \end{array} \qquad (1)$$

is a morphism of crossed modules from C to $\mathfrak{Aut}(L)$.

Classification Theorem for 2-D FHQFTS

Main theorem

Formal HQFTs over C correspond to crossed C-algebras up to isomorphism.

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2. Formal C-maps specify C-bundles, and the whole theory `shea es' well, allowing coe cients in a sheaf on the gros topos, e.g. de Rham structures...do it!
3. Open-closed HQFTs (Moore-Segal) correspond to K-theoretic invariants of the crossed algebras for $B = K(G, 1)$. Generalise.

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3. Look for higher dimensional trace formulae which give more information on the objects.
4. ..... and a lot more!

The End

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