Introduction to Combinatorial Algorithms

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Introduction to the course

What are :

- Combinatorial Structures?
- Combinatorial Algorithms?
- Combinatorial Problems?

Combinatorial Structures

Combinatorial structures are *collections* of k-subsets/k-tuple/permutations from a parent set (finite).

- **Undirected Graphs:**
  Collections of 2-subsets (edges) of a parent set (vertices).

$$V = \{1,2,3,4\} \quad E = \{\{1,2\},\{1,3\},\{1,4\},\{3,4\}\}$$

- **Directed Graphs:**
  Collections of 2-tuples (directed edges) of a parent set (vertices).

$$V = \{1,2,3,4\} \quad E = \{(2,1),(3,1),(1,4),(3,4)\}$$

- **Hypergraphs or Set Systems:**
  Similar to graphs, but hyper-edges are sets with possibly more than two elements.

$$V = \{1,2,3,4\} \quad E = \{\{1,3\},\{1,2,4\},\{3,4\}\}$$

Building blocks: finite sets, finite lists (tuples)

- **Finite Set**
  $X = \{1, 2, 3, 5\}$
  - undordered structure, no repeats
    $\{1, 2, 3, 5\} = \{2, 1, 5, 3\} = \{2, 1, 1, 5, 3\}$
  - cardinality (size) = number of elements, $|X| = 4$.

  A k-subset of a finite set X is a set $S \subseteq X$, $|S| = k$.
  For example: $\{1, 3\}$ is a 2-subset of X.

- **Finite List (or Tuple)**
  $L = [1, 5, 2, 1, 3]$
  - ordered structure, repeats allowed
    $[1, 5, 2, 1, 3] \neq [1, 1, 2, 3, 5] \neq [1, 2, 3, 5]$
  - length = number of items, length of L is 5.

  An n-tuple is a list of length n.
  A permutation of an n-set X is a list of length n such that every element of X occurs exactly once.

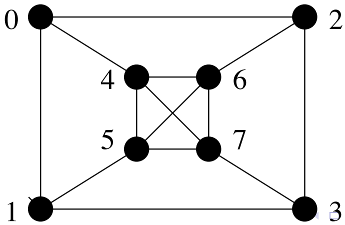
Graphs

Definition

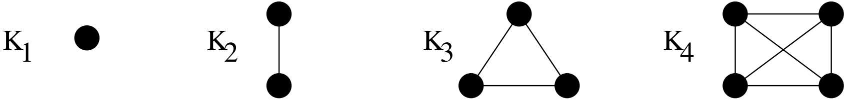
A graph is a pair (V, E) where:
V is a finite set (of vertices).
E is a finite set of 2-subsets (called edges) of V.

$G_1 = (V, E)$

$V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ $\quad E = \{\{0,4\},\{0,1\},\{0,2\},\{2,3\},\{2,6\},\{1,3\},\{1,5\},\{3,7\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\}$



Complete graphs are graphs with all possible edges.

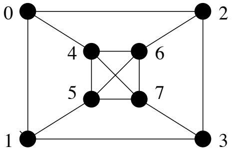


Substructures of a graph: hamiltonian cycle

Definition

A hamiltonian cycle is a closed path that passes through each vertex once.

The list $[0, 1, 5, 4, 6, 7, 3, 2]$ describes a hamiltonian cycle in the graph:
(Note that different lists may describe the same cycle.)



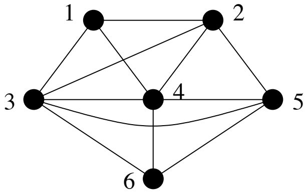
Problem (Traveling Salesman Problem)

Given a weight/cost function $w : E \to R$ on the edges of G, find a smallest weight hamiltonian cycle in G.

Substructures of a graph: cliques

Definition

A clique in a graph $G = (V, E)$ is a subset $C \subseteq V$ such that $\{x, y\} \in E$, for all $x, y \in C$ with $x \neq y$.
(Or equivalently: the subgraph induced by C is complete).



- Some cliques: $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{4, 6\}$, $\{1\}$, $\emptyset$
- Maximum cliques (largest): $\{1, 2, 3, 4\}$, $\{3, 4, 5, 6\}$, $\{2, 3, 4, 5\}$

Famous problems involving cliques

Problem (Maximum clique problem)

Find a clique of maximum cardinality in a graph.

Problem (All cliques problem)

Find all cliques in a graph without repetition.

Set systems/Hypergraphs

Definition

A set system (or hypergraph) is a pair $(X, \mathcal{B})$ *where:*
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  A partition is a set system $(X, \mathcal{B})$ such that
  $B_1 \cap B_2 = \emptyset$ for all $B_1, B_2 \in \mathcal{B}, B_1 \neq B_2$, and $\cup_{B \in \mathcal{B}} B = X$.

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- **Steiner triple system (a type of combinatorial designs):**
  $\mathcal{B}$ is a set of 3-subsets of X such that for each $x, y \in X, x \neq y$, there exists eactly one block $B \in \mathcal{B}$ with $\{x, y\} \subseteq B$.
  $X = \{0, 1, 2, 3, 4, 5, 6\}$
  $\mathcal{B} = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$

Combinatorial algorithms

Combinatorial algorithms are algorithms for investigating combinatorial structures.

- **Generation**
  Construct all combinatorial structures of a particular type.
- **Enumeration**
  Compute the number of all different structures of a particular type.
- **Search**
  Find at least one example of a combinatorial structures of a particular type (if one exists).
  Optimization problems can be seen as a type of search problem.

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  - Generate all maximum cliques of a graph.
  - Generate all Steiner triple systems of a finite set.

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  - Compute the number of cliques of a graph.
  - Compute the number of maximum cliques of a graph.
  - Compute the number of Steiner triple systems of a finite set.

- **Search**

  Find at least one example of a combinatorial structures of a particular type (if one exists).

  Optimization problems can be seen as a type of search problem.

  - Find a Steiner triple system on a finite set. (feasibility)
  - Find a maximum clique of a graph. (optimization)
  - Find a hamiltonian cycle in a graph. (feasibility)
  - Find a smallest weight hamiltonian cycle in a graph. (optimization)

Hardness of Search and Optimization

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- **P** = class of decision problems that can be **solved** in polynomial time. (e.g. Shortest path in a graph is in **P**)
  NP = class of decision problems that can be **verified** in polynomial time. (e.g. Hamiltonian path in a graph is in **NP**)
  Therefore, **P** $\subseteq$ **NP**.
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- There are several approaches to deal with **NP-hard** problems.

Approaches for dealing with NP-hard problems

- **Exhaustive Search**
  - exponential-time algorithms.
  - solves the problem exactly

  (Backtracking and Branch-and-Bound)
- **Heuristic Search**
  - algorithms that explore a search space to find a feasible solution that is hopefully "close to" optimal, within a time limit
  - approximates a solution to the problem

  (Hill-climbing, Simulated annealing, Tabu-Search, Genetic Algor's)
- **Approximation Algorithms**
  - polynomial time algorithm
  - we have a provable guarantee that the solution found is "close to" optimal.

  (not covered in this course)

Types of Search Problems

1) **Decision Problem**:
A yes/no problem
Problem 1: Clique (decision)
Instance: graph $G = (V, E)$,
target size k
Question:
Does there exist a clique C
of G with $|C| = k$?

2) **Search Problem**
Find the guy.
Problem 2: Clique (search)
Instance: graph $G = (V, E)$,
target size k
Find:
a clique C of G
with $|C| = k$, if one exists.

3) **Optimal Value**:
Find the largest target size.
Problem 3: Clique (optimal value)
Instance: graph $G = (V, E)$,
Find:
the maximum value of $|C|$,
where C is a clique

4) **Optimization**:
Find an optimal guy.
Problem 4: Clique (optimization)
Instance: graph $G = (V, E)$,
Find:
a clique C such that
$|C|$ is maximize (max. clique)

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   Sequential generation (successor), rank, unrank.
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   Backtracking algorithms (exhaustive generation, exhaustive search, optimization)
   Branch-and-bound (exhaustive search, optimization)

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   Backtracking algorithms (exhaustive generation, exhaustive search, optimization)
   Branch-and-bound (exhaustive search, optimization)
3. **Heuristic Search** [text Chap 5]
   Hill-climbing, Simulated annealing, Tabu-Search, Genetic Algs.

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3. **Heuristic Search** [text Chap 5]
   Hill-climbing, Simulated annealing, Tabu-Search, Genetic Algs.
4. **Computing Isomorphism and Isomorph-free Exhaustive Generation** [text Chap 7 + Kaski&Ostergard's book Chap 3,4]
   Graph isomorphism, isomorphism of other structures.
   Computing invariants. Computing certificates.
   Isomorph-free exhaustive generation.

Course evaluation

- 45% Assignments
  3 assignments, 15% each
  covering: theory, algorithms, implementation
- 55% Project: individual, chosen by student
  5% Project proposal (up to 1 page)
  40% Project paper (10-15 page)
  10% Project presentation (15-20 minute talk)
  research (reading papers related to course topics),
  original work (involving one or more of: modelling, application, algorithm design, implementation, experimentation, analysis)