University of Ottawa
CSI 4105 – Midterm Test
Instructor: Lucia Moura

February 6, 2010
10:00 am
Duration: 1:50 hs

Closed book

Last name: ______________________

First name: ______________________

Student number: ______________________

There are 4 questions and 100 marks total.

This exam paper should have 10 pages,
including this cover page.

1 – Short answers	/ 25
2 – Search versus Decision problem	/ 25
3 – NP-completeness reductions	/ 25
4 – CLIQUE is NP-complete	/ 25
Total	/ 100

1 Short answers — 25 points

The questions below are of the type "true or false"; briefly justify your answer.
Note: You may use any result shown in class or in homework without proving it. You may use the fact that you know particular problems are polynomial-time solvable or NP-complete.

- For all problems $X \in \mathcal{NP}$, $X \leq_P$ 3D-Matching. [TRUE] [FALSE]
  Justify:

- If Vertex-Cover $\in \mathcal{P}$ then Sat $\in \mathcal{P}$. [TRUE] [FALSE]
  Justify:

- If $\mathcal{P} = \mathcal{NP}$ then Shortest-Path is NP-complete. [TRUE] [FALSE]

- It is possible that Independent-Set $\in \mathcal{P}$ and Ham-cycle $\notin \mathcal{P}$. [TRUE] [FALSE]
  Justify:

- Let X_1 and X_2 be decision problems in $\mathcal{NP}$, and assume $\mathcal{P} \neq \mathcal{NP}$.

  If $X_1 \leq_P X_2$ and $X_2 \leq_P X_1$, then both X_1 and X_2 are NP-complete. [TRUE] [FALSE]

  Justify:

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2 Search versus Decision problem — 25 points

Recall that 3-SAT is the following decision problem:
"Given a formula ϕ which is a conjunction of k clauses over a set of variables $\{x_1, x_2, \ldots, x_n\}$, does there exist a satisfying truth assignment for ϕ?"
Consider the analogous problem 3-SATSEARCH that **computes a satisfying assignment** for ϕ, if one exists, or outputs "ϕ is unsatisfiable", otherwise.

Show that if 3-SAT can be solved in polynomial time, then 3-SATSEARCH can be solved in polynomial time.
Hint 1: Do this by providing an algorithm that solves 3-SATSEARCH by doing calls to the polynomial-time algorithm that decides 3-SAT.
Justify that your algorithm runs in polynomial time. (Note, your formulas may grow in size, and you must be careful and justify that the calls for 3-SAT have time that remain polynomial on the size of ϕ, even when using for your transformed larger formulas).

Hint 2 : One can force the value of a variable x_i in formula ϕ, by transforming ϕ into $\phi \wedge \phi'_{i,v}$, where $\phi'_{i,v}$ is given below, and uses two extra variables y and z:

- Forcing $x_i = 1$: $\phi'_{i,1} = (x_i \vee y \vee z) \wedge (x_i \vee y \vee \overline{z}) \wedge (x_i \vee \overline{y} \vee z) \wedge (x_i \vee \overline{y} \vee \overline{z})$
- Forcing $x_i = 0$: $\phi'_{i,0} = (\overline{x_i} \vee y \vee z) \wedge (\overline{x_i} \vee y \vee \overline{z}) \wedge (\overline{x_i} \vee \overline{y} \vee z) \wedge (\overline{x_i} \vee \overline{y} \vee \overline{z})$

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Algorithm 3-SatSearchALG(φ)
    Input:    a fomula φ in conjunctive normal form with tree literals per clause,
              having n-variables and k-clauses.
    Output:   a truth assignment to x1, x2 ..., xn that satisfies φ,
              or ''φ is not satisfiable''
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3 NP-completeness reductions — 25 points

In this question, you are asked to apply some reductions discussed in class to the examples given. For problems X_2 and X_2 shown in each part below, we have studied a reduction algorithm that shows that $X_1 \leq_P X_2$; you should apply the algorithms discussed in class.

Part A — 10 points 3-SAT $\leq_P$ INDEPENDENTSET

Consider the following instance for 3-SAT: $\phi = (\overline{x_1} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_4)$.

- Give the corresponding instance for the independent set problem INDEPENDENT SET, according to the reduction algorithm for 3-SAT $\leq_P$ INDEPENDENTSET. You need to provide (G, k), where G is a graph and k is a target size for the independent set.

- Give a satisfying assignment for ϕ and show how it translate to an independent set of the right size, by marking the independent set in the picture above.

Part B — 15 points 3-SAT $\leq_P$ DIRECTEDHAMCYCLE

Use the same instance of 3-SAT as shown in the previous part.

- Give the corresponding instance for the directed Hamiltonian cycle problem, DIRECTED-HAMCYCLE, according to the reduction algorithm for 3-SAT $\leq_P$ DIRECTEDHAMCYCLE. You need to provide a graph.

- Show a hamiltonian cycle in the graph above, corresponding to a satisfying assignment for ϕ. You may use the same satisfying assignment as in the previous question.

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4 Clique is NP-complete — 25 points

Recall that an **independent set** in a graph $G = (V, E)$ is a subset $I \subseteq V$ such that for all $x, y \in I$, $\{x, y\} \notin E$.

A **clique** in a graph $G' = (V', E')$ is a subset $C \subseteq V'$ such that for all $x, y \in C$ with $x \neq y$, $\{x, y\} \in E'$.

Consider the following decision problems:

IndSet: Given (G, k), does G have an independent set of size at least k ?

Clique: Given (G, k), does G have a clique of size at least k ?

Recall the definition of the complement of a graph below, which in plain words say that you keep the same vertex set, but switch edges with non-edges.
The **complement** of a graph $G = (V, E)$ is a graph $\overline{G} = (V, E')$ such that for every $x, y \in V$, $x \neq y$, we have $\{x, y\} \in E'$ if and only if $\{x, y\} \notin E$.

Part A — 10 points Let $G = (V, E)$ be a graph, let $\overline{G}$ be its complement graph and let $S \subseteq V$. Prove that S is an independent set of G if and only if S is a clique of $\overline{G}$.

Example:

$\{1, 2, 3\}$ is an independent set of G	$\{1, 2, 3\}$ is a clique of $\overline{G}$

Part B — 15 points Prove that CLIQUE is NP-complete, by proving that:

- (5 points) CLIQUE $\in \mathcal{NP}$.
- (10 points) CLIQUE is NP-hard.
  Hint: Reduce from INDEPENDENTSET. You may use the result in Part A, even if you did not prove it.

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