CSI 4105 Design and Analysis of Algorithms II — Fall 2002
Computer Science — University of Ottawa

Homework Assignment #1 (100 points, weight 10%)
Due: Thursday, October 10, at 10:00 a.m. (in lecture)

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1. (8 points) **Asymptotic notation properties.**
Let f and g be asymptotically positive functions. Prove or disprove each of the following conjectures. (In order to disprove a statement, it is sufficient to show a counter-example, as long as you prove it to violate the statement).

- **a.** (2 points) If $g_1 \in O(f_1)$ and $g_2 \in O(f_2)$ then $g_1 \cdot g_2 \in O(f_1 \cdot f_2)$.
- **b.** (2 points) $g \in O(f)$ if and only if $f \in \Omega(g)$.
- **c.** (2 points) $f \in \Theta(f(n/2))$
- **d.** (2 points) If $g \in o(f)$ then $f + g \in \Theta(f)$.

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2. (4 points) **Encodings**
For each of the following types of encodings, give the encoding for the integer number 7 and the length of the encoding:

- **a.** (2 points) unary encoding (1-symbol alphabet); binary encoding (2-symbol alphabet); ternary encoding (3-symbol alphabet); hexadecimal encoding (16-symbol alphabet).
- **b.** (2 points) For which of the above encodings the following algorithm runs in polynomial time? Justify. `algorithm A(<n>):` { `for i=1 to n do print i;` }

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3. (13 points) **Turing machines, RAM programs and analysis of algorithms.**
Consider the language:
ODD $= \{x \in \{1\}^* : x$ is the unary representation of an odd number $\}$.

- **a.** (5 points) Draw the state diagram for a single tape Turing machine M_1 that decides ODD. Analyse the complexity of M_1 by giving an asymptotic upper bound (big-Oh notation) for the number of steps taken by the machine on strings of length n.
- **b.** (8 points) Convert the Turing machine M_1 found on part **a.** into a RAM program, following the recipe given in the proof of Theorem 2 in pages 47-48 of the lecture notes. Similarly to part **a.**, analyse the complexity of the RAM program.

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4. (10 points) **Multitape Turing machines**
A string is said to be palindrome if it reads the same forward and backward, such as: 0110, 10101, 100001. Consider the language PALINDROME $= \{x \in \{0,1\}^* : x$ is palindrome$\}$.

- **a.** (5 pts) Draw the state diagram for a 2-tape Turing machine that decides PALINDROME.
- **b.** (5 pts) Draw the state diagram for a 1-tape Turing machine that decides PALINDROME.

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5. (15 points) **P, NP and co-NP.**
We define the **complexity class co-NP** as: **co-NP** $= \{L \subseteq \{0,1\}^* :$ such that $\overline{L} \in$ **NP** $\}$.

a. (5 points) Prove that the class **P**, viewed as a set of languages, is closed under union, intersection, concatenation and complement. That is, prove that:

- if $L_1, L_2 \in \mathbf{P}$, then $L_1 \cup L_2 \in \mathbf{P}$;
- if $L_1, L_2 \in \mathbf{P}$, then $L_1 \cap L_2 \in \mathbf{P}$;
- if $L_1, L_2 \in \mathbf{P}$, then $L_1 L_2 \in \mathbf{P}$;
- if $L \in \mathbf{P}$, then $\overline{L} \in \mathbf{P}$;

b. (5 points) Prove that $\mathbf{P} \subseteq$ **co-NP**.

c. (5 points) Prove that if **NP** $\neq$ **co-NP** then $\mathbf{P} \neq \mathbf{NP}$.

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6. (20 points) **Decision problems vs optimization problems.**
Define the optimization problem LongestPathLength as the relation that associates each instance of an undirected graph and two vertices with the number of edges in the longest simple path between the two vertices. Define the following decision problem:

$$\text{LongestPath} = \{< G, u, v, k > \ : \ G = (V, E) \text{ is an undirected graph, } u, v \in V,\ k \geq 0 \text{ is an integer, and there exists a simple path from } u \text{ to } v \text{ having at } \textbf{least } k \text{ edges}\}$$

Show that the optimization problem LongestPathLength can be solved in polynomial time if and only if LongestPath $\in P$.

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7. (30 points) **Proving NP-completeness.**
Consider the languages: LongestPath as defined in the previous exercise, and

$$\text{ShortestPath} = \{< G, u, v, k > \ : \ G = (V, E) \text{ is an undirected graph, } u, v \in V,\ k \geq 0 \text{ is an integer, and there exists a simple path from } u \text{ to } v \text{ having at } \textbf{most } k \text{ edges}\}.$$

a. (5 points) Show that ShortestPath $\in P$.

b. (25 points) Show that LongestPath is NP-complete.
You may assume that Hampath is NP-complete, where:

$$\text{Hampath} = \{< G, u, v > \ : \ G = (V, E) \text{ is an undirected graph, u, v} \in \text{V, and there exists a hamiltonian path from u to v}\}.$$

IMPORTANT:
Part of the marks will be for conciseness and clarity.
You **must read and comply** with the policy on plagiarism stated in the course web page.