Homework Assignment #1 (100 points, weight 10%)
Due: Thursday, October 10, at 10:00 a.m. (in lecture)

1. (8 points) Asymptotic notation properties.
Let \( f \) and \( g \) be asymptotically positive functions. Prove or disprove each of the following conjectures. (In order to disprove a statement, it is sufficient to show a counter-example, as long as you prove it to violate the statement).

a. (2 points) If \( g_1 \in O(f_1) \) and \( g_2 \in O(f_2) \) then \( g_1 \cdot g_2 \in O(f_1 \cdot f_2) \).

b. (2 points) \( g \in O(f) \) if and only if \( f \in \Omega(g) \).

c. (2 points) \( f \in \Theta(f(n/2)) \).

d. (2 points) If \( g \in o(f) \) then \( f + g \in \Theta(f) \).

2. (4 points) Encodings
For each of the following types of encodings, give the encoding for the integer number 7 and the length of the encoding:

a. (2 points) unary encoding (1-symbol alphabet); binary encoding (2-symbol alphabet); ternary encoding (3-symbol alphabet); hexadecimal encoding (16-symbol alphabet).

b. (2 points) For which of the above encodings the following algorithm runs in polynomial time? Justify.

Algorithm A(<n>): { for i=1 to n do print i; }

3. (13 points) Turing machines, RAM programs and analysis of algorithms.
Consider the language:
\[ \text{Odd} = \{ x \in \{1\}^* : x \text{ is the unary representation of an odd number } \} \]

a. (5 points) Draw the state diagram for a single tape Turing machine \( M_1 \) that decides \( \text{Odd} \). Analyse the complexity of \( M_1 \) by giving an asymptotic upper bound (big-Oh notation) for the number of steps taken by the machine on strings of length \( n \).

b. (8 points) Convert the Turing machine \( M_1 \) found on part a. into a RAM program, following the recipe given in the proof of Theorem 2 in pages 47-48 of the lecture notes. Similarly to part a., analyse the complexity of the RAM program.

4. (10 points) Multitape Turing machines
A string is said to be palindrome if it reads the same forward and backward, such as: 0110, 10101, 100001. Consider the language \( \text{Palindrome} = \{ x \in \{0, 1\}^* : x \text{ is palindrome} \} \).

a. (5 pts) Draw the state diagram for a 2-tape Turing machine that decides \( \text{Palindrome} \).

b. (5 pts) Draw the state diagram for a 1-tape Turing machine that decides \( \text{Palindrome} \).
5. (15 points) **P, NP and co-NP.**

We define the complexity class **co-NP** as: **co-NP** = \{L \subseteq \{0, 1\}^* : such that L \in **NP** \}.

a. (5 points) Prove that the class **P**, viewed as a set of languages, is closed under union, intersection, concatenation and complement. That is, prove that:

- if \(L_1, L_2 \in \textbf{P}\), then \(L_1 \cup L_2 \in \textbf{P}\);
- if \(L_1, L_2 \in \textbf{P}\), then \(L_1 \cap L_2 \in \textbf{P}\);
- if \(L_1, L_2 \in \textbf{P}\), then \(L_1L_2 \in \textbf{P}\);
- if \(L \in \textbf{P}\), then \(\overline{L} \in \textbf{P}\);

b. (5 points) Prove that \(\textbf{P} \subseteq \textbf{co-NP}\).

c. (5 points) Prove that if \(\textbf{NP} \neq \textbf{co-NP}\) then \(\textbf{P} \neq \textbf{NP}\).

6. (20 points) **Decision problems vs optimization problems.**

Define the optimization problem \textsc{LongestPathLength} as the relation that associates each instance of an undirected graph and two vertices with the number of edges in the longest simple path between the two vertices. Define the following decision problem:

\[
\textsc{LongestPath} = \{ <G, u, v, k > : G = (V,E) \text{ is an undirected graph,} \\
\quad u, v \in V, k \geq 0 \text{ is an integer, and there exists a} \\
\quad \text{simple path from } u \text{ to } v \text{ having at least } k \text{ edges} \}
\]

Show that the optimization problem \textsc{LongestPathLength} can be solved in polynomial time if and only if \(\textsc{LongestPath} \in \textbf{P}\).

7. (30 points) **Proving NP-completeness.**

Consider the languages: \textsc{LongestPath} as defined in the previous exercise, and

\[
\textsc{ShortestPath} = \{ <G, u, v, k > : G = (V,E) \text{ is an undirected graph,} \\
\quad u, v \in V, k \geq 0 \text{ is an integer, and there exists a} \\
\quad \text{simple path from } u \text{ to } v \text{ having at most } k \text{ edges} \}
\]

a. (5 points) Show that \(\textsc{ShortestPath} \in \textbf{P}\).

b. (25 points) Show that \(\textsc{LongestPath}\) is NP-complete.

You may assume that \(\textsc{Hampath}\) is NP-complete, where:

\[
\textsc{Hampath} = \{ <G, u, v > : G = (V,E) \text{ is an undirected graph,} \\
\quad u, v \in V, \text{ and there exists a hamiltonian path from } u \text{ to } v \}
\]

**IMPORTANT:**

Part of the marks will be for conciseness and clarity.

You must read and comply with the policy on plagiarism stated in the course web page.