Recurrence Relations

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Winter 2017

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$$a_n = 2a_{n-1} - a_{n-2}, \text{ for } n \geq 2$$

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- $a_n = 5$, for all $n \geq 0$.

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  The following sequences are solutions of this recurrence relation:
  - $a_n = 3n$, for all $n \geq 0$,
  - $a_n = 5$, for all $n \geq 0$.
- The **initial conditions** for a sequence specify the terms before n_0 (before the recurrence relation takes effect).
  The recurrence relations together with the initial conditions uniquely determines the sequence. For the example above, the initial conditions are: $a_0 = 0$, $a_1 = 3$; and $a_0 = 5$, $a_1 = 5$; respectively.

Modeling with Recurrence Relations (used for advanced counting)

- Compound interest: A person deposits \$10,000 into savings that yields 11% per year with interest compound annually. How much is in the account in 30 years?
- Growth of rabbit population on an island:
  A young pair of rabbits of opposite sex are placed on an island. A pair of rabbits do not breed until they are 2 months old, but then they produce another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months.
- The Hanoi Tower:
  Setup a recurrence relation for the sequence representing the number of moves needed to solve the Hanoi tower puzzle.
- Find a recurrence relation for the number of bit strings of length n that do not have two consecutive 0s, and also give initial conditions.

Linear Homogeneous Recurrence Relations

We will study more closely **linear homogeneous recurrence relations of degree k with constant coefficients**:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

linear $=$ previous terms appear with exponent 1 (not squares, cubes, etc),
homogeneous $=$ no term other than the multiples of a_i's,
degree $k=$ expressed in terms of previous k terms
constant coefficients $=$ coefficients in front of the terms are constants, instead of general functions.

This recurrence relation plus k initial conditions uniquely determines the sequence.

Which of the following are linear homogeneous recurrence relations of degree k with constant coefficients? If yes, determine k; if no, explain why not.

- $P_n = (1.11)P_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$
- $H_n = 2H_{n-1} + 1$
- $a_n = a_{n-5}$
- $a_n = a_{n-1} + a_{n-2}^2$
- $B_n = nB_{n-1}$

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Theorem (1)

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then, the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: ($\Leftarrow$) If $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then $\{a_n\}$ is a solution for the recurrence relation.
($\Rightarrow$) If $\{a_n\}$ is a solution for the recurrence relation, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for some constants α_1 and α_2.

Exercises:

1. Solve: $a_n = a_{n-1} + 2a_{n-2}$ with $a_o = 2$ and $a_1 = 7$
2. Find explicit formula for the Fibonacci Numbers.

Root with multiplicity 2...

Theorem (2)

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2 \ldots$, where α_1 and α_2 are constants.

Exercise:

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, with initial conditions $a_0 = 1$, $a_1 = 6$.

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Solution:
$r^2 - 6r + 9 = 0$ has only 3 as a root.
So the format of the solution is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$. Need to determine α_1 and α_2 from initial conditions:

$$\begin{aligned} a_0 &= 1 = \alpha_1 \\ a_1 &= 6 = \alpha_1 \cdot 3 + \alpha_2 3 \end{aligned}$$

Solving these equations we get $\alpha_1 = 1$ and $\alpha_2 = 1$.
Therefore, $a_n = 3^n + n3^n$.

Question: how can you double check this answer is right?

Theorem (3)

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then, a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots,$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Exercise:

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3},$$

with the initial conditions $a_0 = 2$, $a_1 = 5$,and $a_2 = 15$.

Theorem (4)

Let $c_1, c_2, \ldots, c_k$ *be real numbers. Suppose that the characteristic equation* $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ *has* t *distinct roots* $r_1, r_2, \ldots, r_t$ *with multiplicities* $m_1, m_2, \ldots, m_t$*, respectively, so that* $m_i \geq 1$ *and* $m_1 + m_2 + \cdots + m_t = k$*. Then, a sequence* $\{a_n\}$ *is a solution of the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + \cdots \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots,$ *where* $\alpha_{i,j}$ *are constants for* $1 \leq i \leq t$*,* $0 \leq j \leq m_i - 1$.

Exercise:

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3},$$

with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Non-homogeneous Recurrence Relations

We look not at **linear non-homogeneous recurrence relation with constant coefficients**, that is, one of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.
The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

is the **associated homogeneous recurrence relation**.

Solving Non-homogeneous Linear Recurrence Relations

Theorem (5)

If $\{a_n^{(p)}\}$ is a particular solution for the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Key: find a particular solution to the non-homogeneous case and we are done, since we know how to solve the homogeneous one.

Finding a particular solution

Theorem (6)

Suppose that $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$, where $b_0, b_1, \ldots, b_t$ and s are real numbers.
When s is NOT a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Exercises: (roots of characteristic polynomial are given to simplify your work)
Find all solutions of

- $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$? (root: $r_1 = 3$)
- $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ (root: $r_1 = 3, r_2 = 2$)

What is the form of a particular solution to

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n),$$

when:

- $F(n) = 3^n$,
- $F(n) = n3^n$,
- $F(n) = n^2 2^n$,
- $F(n) = (n^2 + 1)3^n$.

(root: $r_1 = 3$, multiplicity 2)

Divide-and-Conquer Recurrence Relations

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  - use some extra operations to combine the individual solutions into the final solution for the problem of size n, say $g(n)$ steps.
- Examples: binary search, merge sort, fast multiplication of integers, fast matrix multiplication.
- A divide-and-conquer recurrence relation, expresses the number of steps $f(n)$ needed to solve the problem:

$$f(n) = af(n/b) + cn^d.$$

(for simplicity assume this is defined for n that are multiples of b; otherwise there are roundings up or down to closest integers)

Examples

Give the recurrence relations for:

- Mergesort
- Binary search
- Finding both maximum and minimum over a array of length n by dividing it into 2 pieces and the comparing their individual maxima and minima.

Master Theorem for Divide-and-Conquer Recurrence Relations

Theorem (Master Theorem)

Let f be an increasing function that satisfies the recurrence relation:

$$f(n) = af(n/b) + cn^d,$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d non-negative. Then,

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Proof of the master theorem

We can prove the theorem by showing the following steps:

1. Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.
   Once this is shown, it is clear that if $a = b^d$ then $f(n) \in O(n^d \log n)$.
2. Show that if $a \neq b^d$ and n is a power of b, then $f(n) = c_1 n^d + c_2 n^{log_b a}$, where $c_1 = b^d c/(b^d - a)$ and $c_2 = f(1) + b^d c/(a - b^d)$.
3. Once the previous is shown, we get:
   if $a < b^d$, then $\log_b a < d$, so
   $f(n) = c_1 n^d + c_2 n^{\log_b a} \leq (c_1 + c_2)n^d \in O(n^d)$.
   if $a > b^d$, then $\log_b a > d$, so
   $f(n) = c_1 n^d + c_2 n^{\log_b a} \leq (c_1 + c_2)n^{\log_b a} \in O(n^{\log_b a})$.

Proving item 1:

Lemma

If $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.

Proof:

Let $k = \log_b n$; i. e. $n = b^k$. We will prove the lemma by induction on k.
Basis: $k = 0$ $(n = 1)$. In this case,
$f(1)n^d + cn^d \log_b n = f(1)1^d + c1^d 0 = f(1) = f(n)$.
Inductive step: $k \geq 1$ and we assume the equality is true for $k-1$, i.e. we assume $f(n/b) = f(b^{k-1}) = f(1)(b^{k-1})^d + c(b^{k-1})^d(k-1)$.

$$\begin{aligned}
f(n) &= af(n/b) + cn^d = af(b^{k-1}) + c(b^k)^d \\
&= a(f(1)(b^{(k-1)d}) + c(b^{(k-1)d})(k-1)) + c(b^k)^d \\
&= b^d(f(1)(b^{(k-1)d}) + c(b^{(k-1)d})(k-1)) + c(b^k)^d \\
&= f(1)(b^k)^d + c(b^k)^d(k) \\
&= f(1)n^d + cn^d \log_b n.
\end{aligned}$$

Proving item 2:

Lemma

If $a \neq b^d$ and n is a power of b, then $f(n) = c_1 n^d + c_2 n^{log_b a}$, where $c_1 = b^d c/(b^d - a)$ and $c_2 = f(1) + b^d c/(a - b^d)$.

Proof:

Let $k = \log_b n$; i. e. $n = b^k$. We will prove the lemma by induction on k.
Basis: If $n = 1$ and $k = 0$, then
$c_1 n^d + c_2 n^{\log_b a} = c_1 + c_2 = b^d c/(b^d - a) + f(1) + b^d c/(a - b^d) = f(1)$.
Inductive step: Assume lemma is true for k, where $n = b^k$. Then, for
$n = b^{k+1}$, $f(n) = af(n/b) + cn^d =$
$a((b^d c/(b^d - a))(n/b)^d + (f(1) + b^d c/(a - b^d))(n/b)^{\log_b a})) + cn^d =$
$(b^d c/(b^d - a))n^d a/b^d + (f(1) + b^d c/(a - b^d))n^{\log_b a} + cn^d =$
$n^d[ac/(b^d - a) + c(b^d - a)/(b^d - a)] + [f(1) + b^d c/(a - b^d)]n^{\log_b a} =$
$(b^d c/(b^d - a))n^d + (f(1) + b^d c/(a - b^d))n^{\log_b a} = c_1 n^d + c_2 n^{log_b a}$. □

Use the master theorem to determine the asymptotic growth of the following recurrence relations:

- binary search: $b(n) = b(n/2) + 2$;
- mergesort: $M(n) = 2M(n/2) + n$;
- maximum/minima: $m(n) = 2m(n/2) + 2$.

You have divided and conquered; have you saved in all cases?