Induction and program correctness

1. (20 points) **Mathematical Induction**
   Use induction to prove that for very positive integer \( n \),
   \[
   \sum_{k=1}^{n} k2^k = (n - 1)2^{n+1} + 2.
   \]

2. (25 points) **Strong induction**
   Use strong induction to show that every positive integer \( n \) can be written as a sum of distinct powers of two, that is, as a sum of the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8 \), and so on.
   
   Hint: for the inductive step, separately consider the case where \( k+1 \) is even and where it is odd. When it is even, note that \( (k+1)/2 \) is an integer.

3. (25 points) **Structural Induction**
   
   (a) Give a recursive definition of the function \( \text{ones}(s) \), which counts the number of ones in a bit string \( s \) (a bitstring is a string over the alphabet \( \Sigma = \{0, 1\} \)).
   
   (b) Use structural induction to prove that \( \text{ones}(s \cdot t) = \text{ones}(s) + \text{ones}(t) \); where the symbol \( \cdot \) denotes concatenation of strings.
   
   Hint: in some of your steps you need to rely on the recursive definition of strings and of concatenation given in the textbook, as well as on the definition of \( \text{ones}(s) \) given by you.

4. (30 points) **Correctness of recursive algorithms**
   
   Prove that Algorithm 6 (recursive binary search algorithm) in page 314 (Section 4.4) is correct, as follows. Consider the following statement:

   \[ P(k) : \text{“ If } n \text{ is an integer and } a_1, a_2, \ldots, a_n \text{ are integers in increasing order, and } i, j, x \text{ are integers such that } 1 \leq i \leq n, 1 \leq j \leq n \text{ and } j - i = k, \text{ then procedure } \text{binarysearch}(i, j, x) \text{ calculates } \text{location}, \text{ where } \text{location} = 0 \text{ if there exists no } l, i \leq l \leq j, \text{ with } a_l = x, \text{ or } \text{location} = m \text{ and } a_m = x \text{ with } i \leq m \leq j, \text{ otherwise.”} \]

   Use strong induction to prove that \( P(k) \) is true for all \( k \geq 0 \).