CSI 2101 Discrete Structures — Winter 2011
Prof. Lucia Moura — University of Ottawa

Homework Assignment #1 (100 points, weight 6.25%)
Due: Tuesday Feb 8, at 2:30 p.m. (in lecture);
assignments with lateness between 1min-24hs will have a discount of 10%; after 24hs, not accepted; please drop off late assignments under my office door (STE5027).

All exercise numbers correspond to the textbook by Rosen, 6th edition.

Propositional Logic

1. (12 points) Use logical equivalences to show that $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$ is a tautology.

2. (12 points) Recall that a collection of logical operators is *functionally complete* if every compound proposition is logically equivalent to a compound proposition using only these logical operators. The logical operator NAND, denoted by $|$, is true when p or q, or both, are false, and is false otherwise. Show that $\{|\}$ is a functionally complete collection of operators, by proving the following steps.

   (a) Use truth tables to show that $p|p$ is logically equivalent to $\neg p$.

   (b) Use truth tables to show that $(p|q)|(p|q)$ is logically equivalent to $p \wedge q$.

   (c) Complete the argument by using parts (a), (b) and the fact that $\{\neg, \wedge\}$ is functionally complete.

Predicate Logic

3. (15 points) (Ex. 32, p 48)
   For each of the following statements, let the domain be all animals in the world.

   1 - Express each of the statements using quantifiers and propositional functions.
   2 - Form the negation of the statement so that no negation is to the left of the quantifier.
   3 - Express the negation in simple English. (Do not simply use the words "it is not the case that...").

   (a) All dogs have fleas.

   (b) There is a horse that can add.

   (c) Every koala can climb.

   (d) No monkey can speak French.

   (e) There exists a pig that can swim and catch fish.

4. (11 points) (Ex. 6, p.58) Let $C(x, y)$ mean that "student x is enrolled in class y", where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of the following statements in simple English sentence.

(a) $C(\text{John Smith}, \text{CSI2101})$

(b) $\exists x C(x, \text{MAT1348})$

(c) $\exists y C(\text{Bob Marley}, y)$

(d) $\exists x (C(x, \text{MAT1322}) \wedge C(x, \text{CSI2101}))$

(e) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$

(f) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

5. (10 points) (Ex. 30, p. 61). Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives). Show your steps.

   (a) $\neg \exists y \exists x P(x, y)$

   (b) $\neg \forall x \exists y P(x, y)$

   (c) $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$

   (d) $\neg \exists y (\exists x R(x, y) \vee \forall x S(x, y))$

   (e) $\neg \exists y (\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$

6. (10 points) (ex. 48, p. 49) Prove these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty. You will probably have to use a proof by cases.
   Hint: exercise 49, page 49, has a similar style and its solution is at the back of the textbook.

   (a) $\forall x (A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

   (b) $\exists x (A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$

7. (10 points) Ex 48, part of 50, p.62.

   (a) Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$ are logically equivalent, where all quantifiers have the same nonempty domain. Hint: Use case analysis, and check a similar exercise solution at the back of the textbook, Ex.49, page 60.

   (b) A statement is in prenex normal form (PNF) if and only if all quantifiers occur at the beginning of the statement (without negations), followed by a predicate involving no quantifiers. Put the following statement in prenex normal form:

   $$\neg(\forall x P(x) \vee \forall x Q(x))$$

Rules of Inference

8. (10 points) (Ex.10, p. 73) For each of the following set of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

(a) If I play hockey, then I am sore.
I use the whirlpool if I am sore.
I did not use the whirlpool.

(b) If I work is either sunny or partially sunny.
I worked last Monday or I worked last Friday.
It was not sunny on Tuesday.
It was not partially sunny on Friday.

(c) Every student has an Internet account.
Homer does not have an internet account.
Maggie has an internet account.

(d) I am either dreaming or hallucinating.
I am not dreaming.
If I am hallucinating, I see elephants running down the road.

9. (10 points) Consider the example from Lewis Carroll given in Example 27, Section 1.3. Give a formal proof, using known rules of inference, to establish the conclusion of the argument (4th statement) using the first 3 statements as premises.
Remember that a formal proof is a sequence of steps, each with a reason noted besides it; each step is either a premise, or is obtained from previous steps using inference rules.

EXAMPLE 27 Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\forall x(P(x) \to S(x)).$$
$$\neg\exists x(Q(x) \land R(x)).$$
$$\forall x(\neg R(x) \to \neg S(x)).$$
$$\forall x(P(x) \to \neg Q(x)).$$

(Note we have assumed that "small" is the same as "not large" and that "dull in color" is the same as "not richly colored." To show that the fourth statement is a valid conclusion of the first three, we need to use rules of inference that will be discussed in Section 1.5.)