Université d'Ottawa
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University of Ottawa
Faculty of Engineering

School of Information
Technology and Engineering

CSI 2101 Discrete Structures – Final Exam
Instructor: Lucia Moura

April 18, 2010
19:00-22:00
Duration: 3hs

Closed book, no calculators

Last name: ____________________

First name: ____________________

Student number: ____________________

There are 8 questions and 100 marks total.

This exam paper should have 16 pages,
including this cover page.
Theorems regarding recurrence relations are provided
in pages 15-16.

1 – Predicate Logic - short answers	/ 10
2 – Induction 1	/ 10
3 – Induction 2	/ 10
4 – Number theory 1	/ 10
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6 – Recurrence relations	/ 20
7 – Graphs	/ 16
8 – Program correctness	/ 10
Total	/ 100

1 Predicate Logic - short answers — 10 points

Part A — 6 points

Suppose $P(x, y)$ is a predicate and the universe for the variables x and y is $\{1, 2, 3\}$. Suppose $P(1, 3)$, $P(2, 1)$, $P(2, 2)$, $P(2, 3)$, $P(3, 1)$, $P(3, 2)$ are true, and $P(x, y)$ is false otherwise.

Determine whether the following statements are true or false:

- [true/false] $\forall x \exists y P(x, y)$
- [true/false] $\exists x \forall y P(x, y)$
- [true/false] $\neg \exists x \exists y (P(x, y) \wedge \neg P(y, x))$
- [true/false] $\forall y \exists x (P(x, y) \rightarrow P(y, x))$
- [true/false] $\forall x \forall y ((x \neq y) \rightarrow (P(x, y) \vee P(y, x))$
- [true/false] $\forall y \exists x ((x \leq y) \wedge P(x, y))$

Part B — 4 points Suppose the variable x represents people, and
$F(x)$: x is friendly $T(x)$: x is tall $A(x)$: x is angry

Write the statement using these predicates and any needed quantifiers:

- Some tall angry people are friendly.

- If a person is friendly, then that person is not angry.

2 Induction 1 — 10 points

Use the principle of mathematical induction to prove that $2|(n^2 + n)$ for all $n \geq 0$.
(recall that the symbol "$|$" means "divides")

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3 Induction 2 — 10 points

We are given a chocolate bar with $m \times n$ squares of chocolate, and our task is to divide it into mn individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is 2×2. The first split makes two pieces, both 2×1. Each of these pieces requires one more split to form single squares. This gives a total of three splits.

Use strong induction to conclude the following:
"To divide up a chocolate bar with $m \times n$ squares, we need at most $mn - 1$ splits."

Hint: Use strong induction on k, the number of squares in the chocolate bar ($k = mn$).

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4 Number theory 1 — 10 points

Part A — 5 points Find the inverse of 21 modulo 44 using the extended Euclidean Algorithm.

Part B — 5 points Using the solution above, find all integer solutions to the following linear congruence:

$$21x \equiv 3 \pmod{44}.$$

... continued

5 Proof methods/number theory — 14 points

Part A — 4 points Let m and n be integers. Use a proof by **contraposition** to show that if mn is even then m is even or n is even.

Part B — 4 points Use a proof by **contradiction** to prove that at least one of the numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers, $(a_1 + a_2 + \cdots, a_n)/n$.

Part C — 6 points Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod 8$.

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(more space to solve question 5)

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6 Recurrence relations — 20 points

For this question, you can refer to the theorems provided in pages 15-16.

Part A — 10 points

Consider the recurrence relation $a_n = 3a_{n-1} + 5^n$.

A. Write the associated homogeneous recurrence relation.

B. Find the general solution to the associated homogeneous recurrence relation.

C. Find a particular solution to the given recurrence relation.

D. Write the general solution to the given recurrence relation.

E. Find the particular solution to he given recurrence relation when $a_0 = 1$.

... continued

(space to continue solution...)

Part B — 10 points

Consider the following recursive algorithm:

procedure LongIntegerMultiply(X, Y, n: X and Y are n-bit integers, n is a power of 2)
  if $n = 1$ then return X*Y   /* line 1: single-bit multiplication */
  else
    split X into X_1, X_2 and Y into Y_1, Y_2 such that $X = 2^{n/2}X_1 + X_2$ and $Y = 2^{n/2}Y_1 + Y_2$
    $U \leftarrow$LongIntegerMultiply($X_1, Y_1, n/2$)
    $V \leftarrow$LongIntegerMultiply ($X_2, Y_2, n/2$)
    $W \leftarrow$LongIntegerMultiply($X_1 - X_2, Y_1 - Y_2, n/2$)
    $Z \leftarrow U + V - W$
    return $2^n U + 2^{n/2} Z + V$

A. Set up a divide-and-conquer recurrence relation for the number of single-bit multiplications (done in line 1) required to compute the product of two n-bit integers X and Y, where n is a power of 2 (i.e. $n = 2^k$ for some integer k), using the algorithm above.

B. Use the recurrence relation above and the Master theorem to derive a big-O estimate for the number of single-bit multiplications used in the algorithm above.

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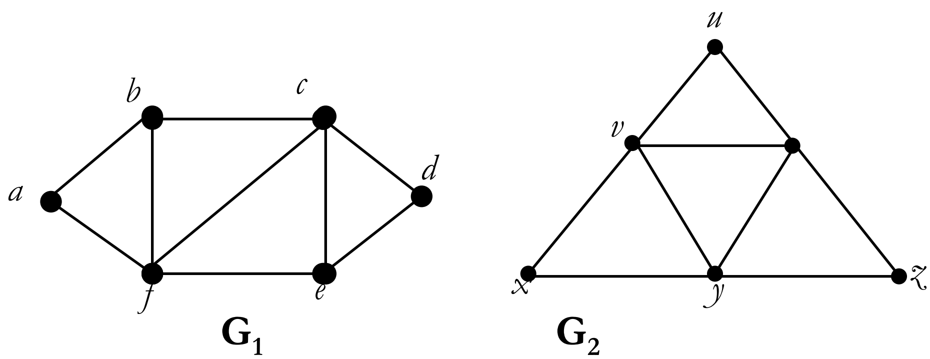
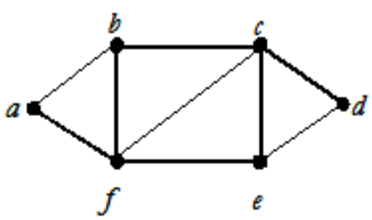
7 Graphs — 16 points

Part A — 4 points Are the following graphs isomorphic? Explain your answer.



Part B — 4 points Consider the following graph

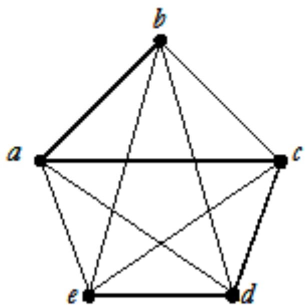


A. Does it have an Euler circuit? If yes, state it. If no, explain why.

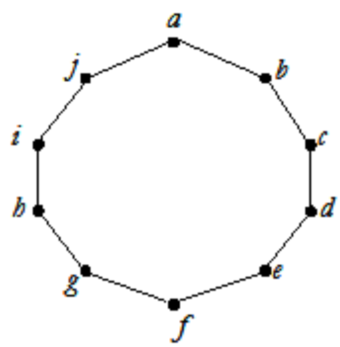
B. Does it have a Hamilton circuit? If yes, state it. If no, explain why.

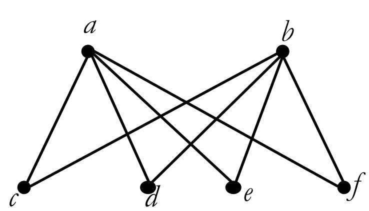
Part C — 6 points Graph colouring

- Is the following graph 4-colourable? [yes/no] Justify:

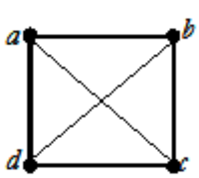


- What is the chromatic number of each of the following graphs? Justify.





Part D — 2 points Is the following graph planar? [yes/no] Justify:



8 Program correctness — 10 points

Consider the following program segment S:

$i \leftarrow 1$
$total \leftarrow 1$
while $i < n$ do
    $i \leftarrow i + 1$
    $total \leftarrow total + i$
endwhile

Part A — 5 points Let p be the proposition "$total = \frac{i(i+1)}{2}$ and $i \leq n$". Prove that p is a loop invariant for the while loop.

Part B — 5 points Use program verification techniques to show that S is correct with respect to the initial assertion (precondition) "$n \geq 1$" and the final assertion (postcondition) "$total = \frac{n(n+1)}{2}$". You may use the loop invariant in part A, even if you didn't prove it.

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(more space for question 8)

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Recurrence relation theorems:

MASTER THEOREM Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

THEOREM 1 Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

THEOREM 2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

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Recurrence relation theorems (cont'd):

THEOREM 5 If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$