CSI 2101 / Partial Orderings (§8.6)

Introduction

Lexicographic Order

Maximal/minimal, greatest/least elements

Lattices

Topological Sorting

Partial Orderings - Introduction

There is a total ordering for numbers: we can compare any two numbers and decide which is “larger or equal than”.

What about comparing strings? Or sets? Or formulas?

We can come up with a scheme to compare all strings

- lexicographic ordering, will talk about it shortly

However, quite often what we have is that some objects are genuinely incomparable, while for others we can say that **a** is less or equal then **b**.

- In an organization: A being the boss of B
- For sets: A being the subset of B

Captured in the notion of **partial ordering**

Partial Ordering – Definition & Examples

Definition: Let S be a set and R be a binary relation on S. If R is *reflexive*, *antisymmetric* and *transitive*, then we say that R is a partial ordering of S. The set S together with the partial ordering R is called a poset (partially ordered set) and denoted by (S,R). Members of S are called elements of the poset.

We will often use infix notation $a\ R\ b$ to denote $R(a,b) = T$

Example 1: S is a set of sets, and the partial order is a set inclusion $\subseteq$

Example 2: S is a set of integers, and R is 'divides': $aRb \equiv a|b$

Example 3: S is a set of formulae, aRb iff a is a sub-formula of b

Example 4: S is the set of tasks to be scheduled, aRb means *"a must finish before b can run"*

Partial Ordering - Comparability

We can have the same set S with different relation R:

Example 5: S is a set of strings, aRb iff a is a prefix of b

Example 6: S is a set of strings, aRb iff a is a substring of b

Example 7: S is a set of formulae, aRb iff $(b \rightarrow a)$

Example 8: S is a set of integers, R is the normal $\leq$

The last one is different then previous ones. How?

- every two integers are **comparable**

Definition: The elements a and b of the poset (S,R) are **comparable** iff aRb or bRa. Otherwise, they are called **incomparable**.

Definition: If (S,R) is a poset and every two elements of S are comparable, we say that S is **totally ordered** (or **linearly ordered**) set and R is called a **total** (or **linear**) **order**.

Partial Ordering – Lexicographic Order

Lexicographic order – a commonly used technique to construct total order on the Cartesian product of totally ordered sets

- let (A_1, R_1) and (A_2, R_2) be totally ordered sets
- $(A_1 \times A_2)$ is a Cartesian product of A_1 and A_2, essentially a set of the ordered pairs (a,b) such that $a \in A_1$ and $b \in A_2$

How can we define a total order R on $(A_1 \times A_2)$?

- $(a_1, a_2)\ R\ (b_1, b_2)$ iff $(a_1\ R_1\ b_1)$ or $(a_1 = b_1)$ and $(a_2\ R_2\ b_2)$

Straightforwardly extends from 2-tuples to n-tuples

OK, it works for tuples, can we use it for strings?

- works nicely for strings of equal lengths
- what to do if two strings a and b are of different lengths $n<m$?
  - compare a with the prefix of b of length n
  - if they are equal, then aRb

Partial Ordering – Hasse Diagrams

How to visualize posets?

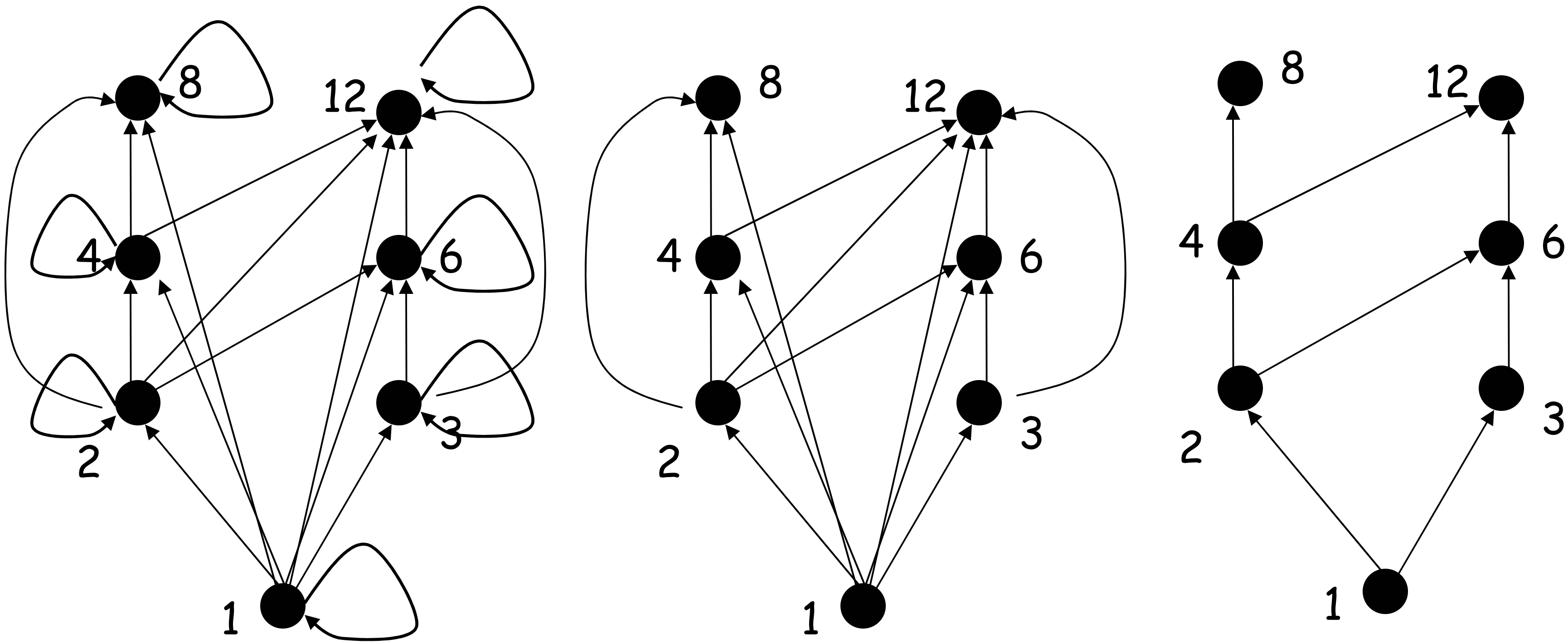
- represent elements by nodes
- draw a directed edge from **a** to b iff **aRb**

Natural, but messy, with lots of redundant information

Hasse Diagrams:

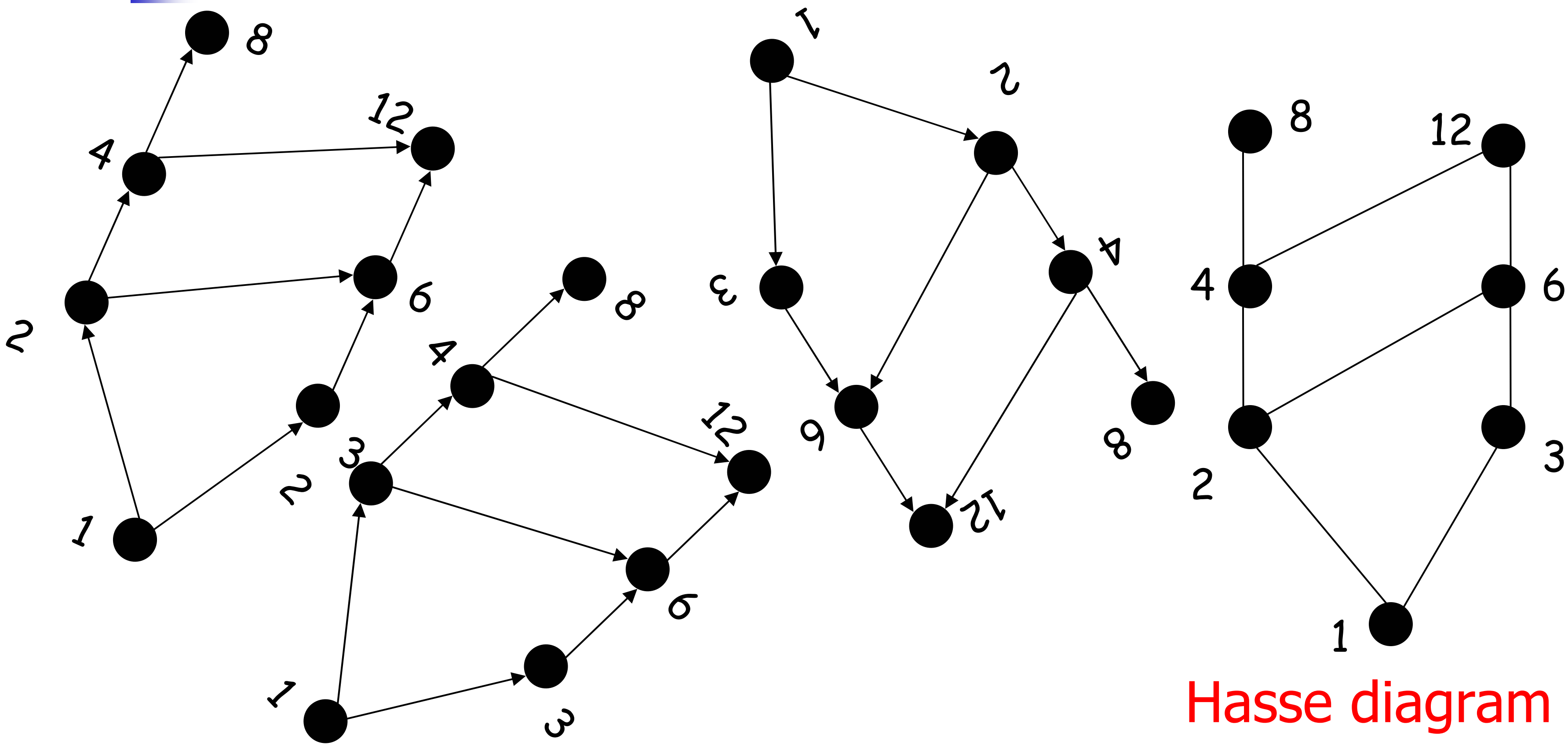
- use the above approach, but delete the edge that must obviously be there by the definition (i.e. the redundant information because of the transitivity and reflexivity of the relation)
- drop the self-cycles
- drop the bridging edges that follow from transitivity
- draw the 'larger' elements higher and drop the arrows
- what is left is the Hasse Diagram of the poset

Hasse diagram of a poset



the poset {1, 2, 3, 4, 6, 8, 12} with the divisibility relation

Hasse diagram of a poset



the poset {1, 2, 3, 4, 6, 8, 12} with the divisibility relation

Hasse diagram or

Upward drawing

Partial Ordering – Special Elements

Not all elements in the poset are equal

- $a \in S$ is **maximal** in (S,R), iff $\neg(\exists b: (aRb \wedge a \neq b))$
- $a \in S$ is **minimal** in (S,R), iff $\neg(\exists b: (bRa \wedge a \neq b))$
- $a \in S$ is a **greatest element** in (S,R), iff $\forall b: bRa$
- $a \in S$ is a **least element** in (S,R), iff $\forall b: aRb$

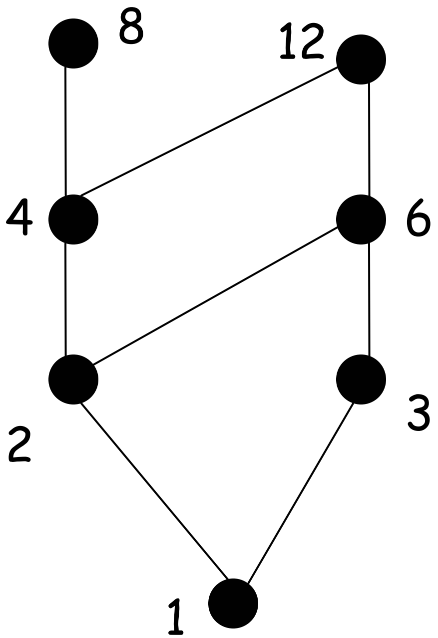
Is it true that every maximal element is also a greatest element?

Is it true that every greatest element is also a maximal element?

Is it true that a poset might have several maximal elements?

Is it true that a poset might have several least elements?

Partial Ordering – Special Elements



Maximal elements: 8 and 12

Minimal elements: 1

Greatest element: NONE

Least element: 1

Partial Ordering – Special Elements

Let **A** be a subset of **S**, where **(S,R)** is a poset

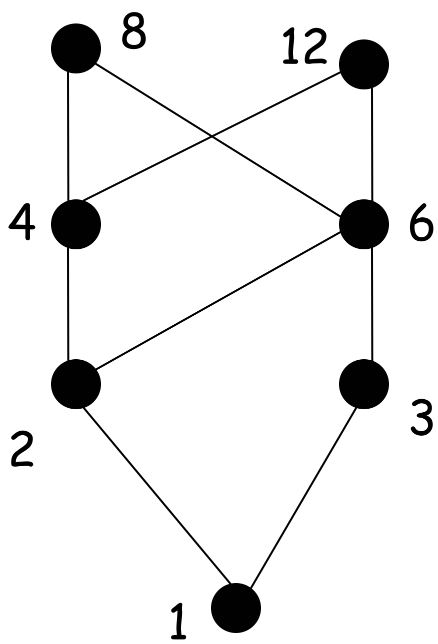
We say that **b** is an **upper bound** of the set **A** iff $\forall a \in A$, **aRb**

We say that **b** is a **lower bound** of the set **A** iff $\forall a \in A$, **bRa**

We say that **c** is the least upper bound (denoted by **lub(A)**) bound of the set **A** iff for every upper bound **b** of **A** holds **cRb**

We say that **c** is the greatest lower bound (denoted by **glb(A)** of the set **A** iff for every lower bound **b** of **A** holds **bRc**

Partial Ordering – Special Elements



8 is an upper bound for the set {4, 6}

12 is an upper bound for the set {4, 6}

But there are no least upper bound for the set {4, 6}

1 and 2 are both lower bounds for the set {4, 6}

2 is the greatest lower bounds for the set {4, 6}

Note that for the poset **(Z+, |)** , **glb(A)** coincides with **GCD(A)** and **lub(A)** coincides with **LCM(A)**.

Partial Ordering – Lattices

The posets for which every pair of elements have a **glb/lub** have many special properties and applications and are of special interest

- by induction, we can prove that every finite non-empty subset **A** of **S** has **glb/lub**, i.e. considering only pairs is not limiting

Such posets are called **lattices**

Is **$(Z^+, |)$** a lattice?

Is **$(\{1,2,4,8,12,24\}, |)$** a lattice?

Is **$(Z^+, \leq)$** a lattice?

Let S be a set with n elements. Is the sets of all subsets of S ordered by inclusion "power set of S" a lattice?

Partial Ordering – Topological Sorting

Given a set of tasks and dependences between them, find a schedule (for one computer) to execute them all.

What we want is a **compatible** total order **R'**:

- $\forall$**a,b**$\in$**S**: if **aRb** then also **aR'b**
- also called **topological sorting** of **(S,R)**

Actually, **R'** does not need to be total order to say that it is compatible with **R**

- but to say that **R'** is topological sorting, we require that **R'** is total

Partial Ordering – Topological Sorting

Constructing a Topological Sorting:

Input: A finite poset **(S,R)**

```
k = 1;
while (S ≠∅) {
    ak = a minimal element of S; // such element must exist
    S = S – {ak};
    k++;
}
```

Output: The sequence $\{a_1, a_2, \ldots, a_n\}$ defining the total order **R'** of **S**.

Partial Ordering: Well-Founded Sets

How do we prove a statement $P(x)$ for every element x of the poset?

Quite often, the following approach is very natural:

- **Basis step:** Directly prove $P(x)$ for every minimal element x
- **Induction step:** Prove that if $P(y)$ for all y such that yRx then also $P(x)$

Does this work for all posets?

- no, as there are posets that have infinite chains of smaller and smaller elements (e.g. $(Z, \leq)$)
- it works for **well-founded** posets
  - every non-empty set has a minimal element
- differs from the **well-ordered sets** by not requiring that the order relation be total