Chapter 2 Linear Time-Invariant Systems

2.0 Introduction

- Many physical systems can be modeled as linear time-invariant (LTI) systems
- Very general signals can be represented as linear combinations of delayed impulses.
- By the principle of superposition, the response $y[n]$ of a discrete-time LTI system is the sum of the responses to the individual shifted impulses making up the input signal $x[n]$.

2.1 Discrete-Time LTI Systems: The Convolution Sum

2.1.1 Representation of Discrete-Time Signals in Terms of Impulses

A discrete-time signal can be decomposed into a sequence of individual impulses.

Example:

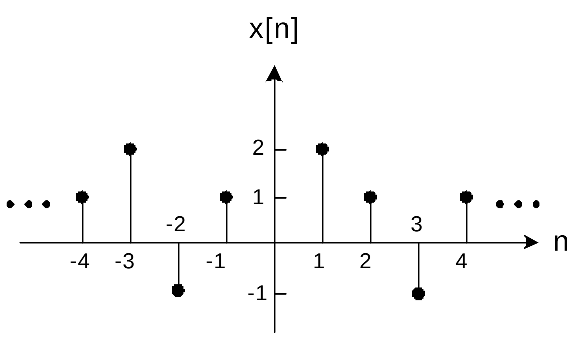


Fig. 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

The signal in Fig. 2.1 can be expressed as a sum of the shifted impulses:

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots \quad (2.1)$$

or in a more compact form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \quad (2.2)$$

This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulse $\delta[n-k]$, where the weights in the linear combination are $x[k]$. Eq. (2.2) is called the ***sifting property*** of the discrete-time unit impulse.

2.1.2 Discrete-Time Unit Impulse Response and the Convolution – Sum Representation of LTI Systems

Let $h_k[n]$ be the response of the LTI system to the shifted unit impulse $d[n-k]$, then from the superposition property for a linear system, the response of the linear system to the input $x[n]$ in Eq. (2.2) is simply the weighted linear combination of these basic responses:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]. \qquad (2.3)$$

If the linear system is time invariant, then the responses to time-shifted unit impulses are all time-shifted versions of the same impulse responses:

$$h_k[n] = h_0[n-k]. \qquad (2.4)$$

Therefore the impulse response $h[n] = h_0[n]$ of an LTI system ***characterizes*** the system completely. This is not the case for a *linear time-varying system*: one has to specify all the impulse responses $h_k[n]$ (an infinite number) to characterize the system.

For the LTI system, Eq. (2.3) becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \qquad (2.5)$$

This result is referred to as the convolution sum or superposition sum and the operation on the right-hand side of the equation is known as the convolution of the sequences of $x[n]$ and $h[n]$.

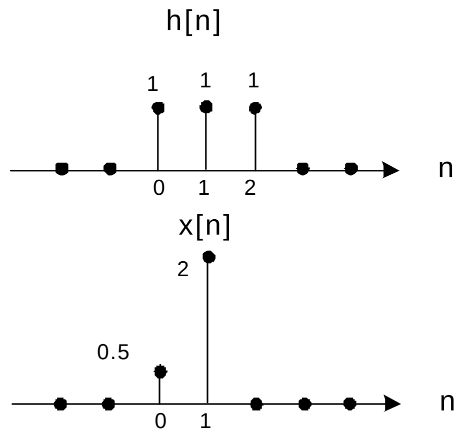
The convolution operation is usually represented symbolically as

$$y[n] = x[k] * h[n]. \qquad (2.6)$$

2.1.3 Calculation of Convolution Sum

- ***One way to visualize the convolution sum of Eq. (2.5) is to draw the weighted and shifted impulse responses one above the other and to add them up.***

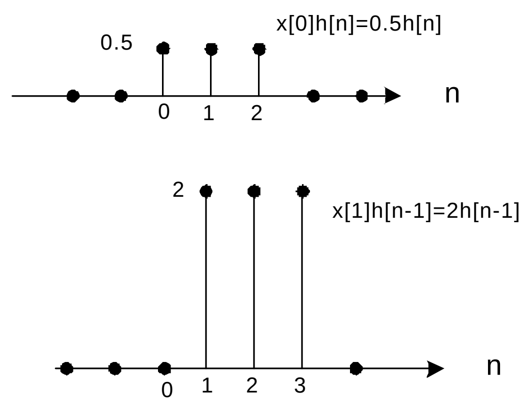
Example: Consider the LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in Fig. 2. 2.



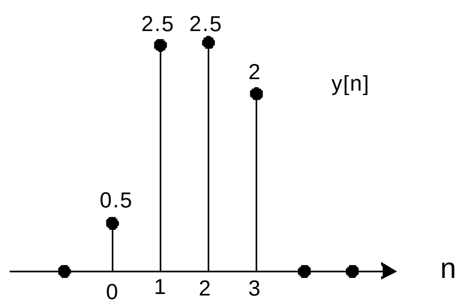
(a)

The output response based on Eq. (2.5) can be expressed

$$y[n] = \sum_{k=0}^{1} x[k]h[n-k] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1].$$



(b)



(c)

Fig. 2.2 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses to the nonzero values of the input; (c) the overall responses.

- ***Another way to visualize the convolution sum is to draw the signals $x[k]$ and $h[n-k]$ as functions of k (for a fixed n), multiply them to form the signal $g[k]$, and then sum all values of $g[k]$.***

Example: Calculate the convolution of $x[k]$ and $h[n]$ shown in Fig. 2.2 (a).

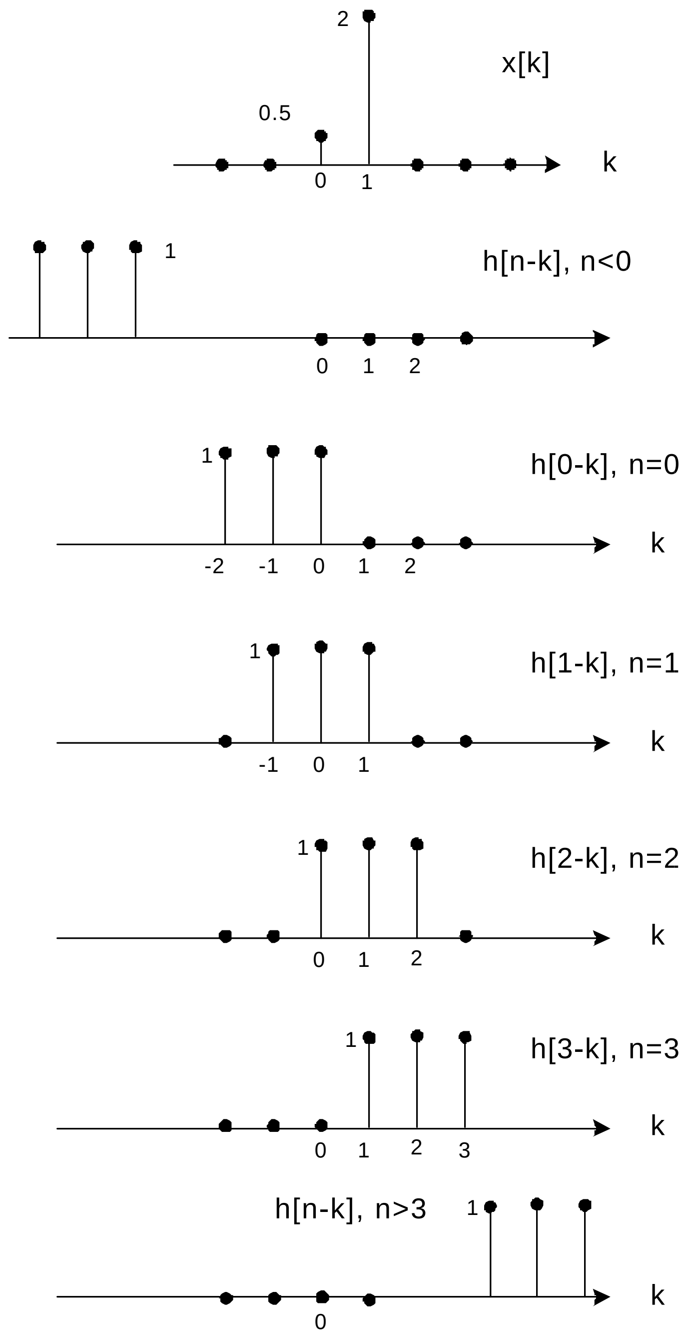


Fig. 2.3 Interpretation of Eq. (2.5) for the signals $x[k]$ and $h[n]$.

For $n < 0$, $y[n] = 0$

For $n = 0$, $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5$

For $n = 1$, $y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2 = 2.5$

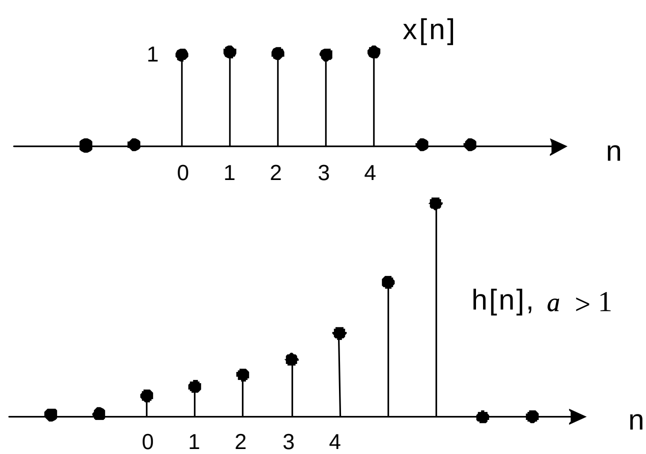
For $n = 2$, $y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2 = 2.5$

For $n = 3$, $y[3] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 2$

For $n > 3$, $y[n] = 0$

The resulting output values agree with those obtained in the preceding example.

Example: Compute the response of an LTI system described by its impulse response $h[n] = \begin{cases} a^n, & 0 \le n \le 6 \\ 0, & otherwise \end{cases}$ to the input signal $x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & otherwise \end{cases}$.



To do the analysis, it is convenient to consider five separate intervals:

For $n < 0$, there is no overlap between the nonzero portions of $x[n]$ and $h[n-k]$, and consequently, $y[n] = 0$.

For $0 \le n \le 4$, $x[k]h[n-k] = \begin{cases} a^{n-k}, & 0 \le k \le n \\ 0, & otherwise \end{cases}$, Thus, in this interval $y[n]=\sum_{k=0}^{n}a^{n-k}=a^{n}\sum_{k=0}^{n}a^{-k}=a^{n}\left(\frac{1-a^{-n-1}}{1-a^{-1}}\right)=\frac{1-a^{n+1}}{1-a}$

For $4<n\le 6$, $x[k]h[n-k]=\begin{cases}a^{n-k}, & 0\le k\le 4\\ 0, & otherwise\end{cases}$

$$y[n]=\sum_{k=0}^{4}a^{n-k}=a^{n}\sum_{k=0}^{4}\left(a^{-1}\right)^{k}=a^{n}\frac{1-\left(a^{-1}\right)^{5}}{1-a^{-1}}=\frac{a^{n-4}-a^{n+1}}{1-a}.$$

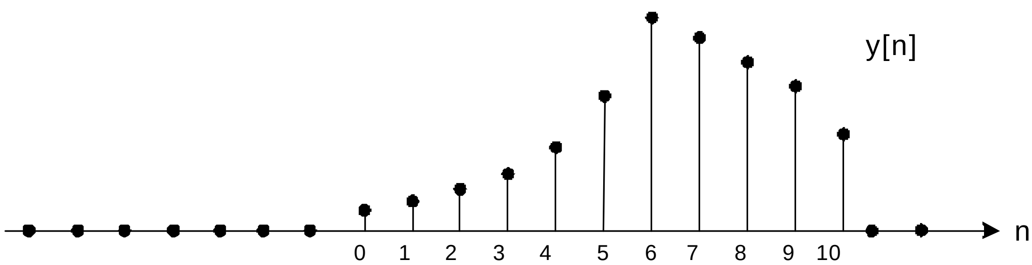
For $6<n\le 10$, $x[k]h[n-k]=\begin{cases}a^{n-k}, & (n-6)\le k\le 4\\ 0, & otherwise\end{cases}$

$$y[n]=\sum_{k=n-6}^{4}a^{n-k}.$$

Let $r=k-n+6$, $y[n]=\sum_{r=0}^{10-n}a^{6-r}=a^{6}\sum_{r=0}^{10-n}\left(a^{-1}\right)^{r}=a^{6}\frac{1-a^{n-11}}{1-a^{-1}}=\frac{a^{n-4}-a^{7}}{1-a}$.

For $n-6>4$, or $n>10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence, $y[n]=0$.

The output is illustrated in the figure below.



2.2 Continuous-Time LTI systems: the Convolution Integral

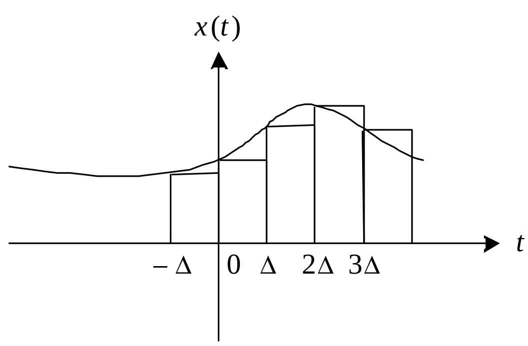
The response of a continuous-time LTI system can be computed by convolution of the impulse response of the system with the input signal, using a convolution integral, rather than a sum.

2.2.1 Representation of Continuous-Time Signals in Terms of Impulses

A continuous-time signal can be viewed as a linear combination of continuous impulses:

$$x(t) = \int_{-\infty}^{\infty} x(t)d(t-t)dt \ . \qquad (2.7)$$

The result is obtained by chopping up the signal $x(t)$ in sections of width Δ, and taking sum



Recall the definition of the unit pulse $d_{\Delta}(t)$; we can define a signal $\hat{x}(t)$ as a linear combination of delayed pulses of height $x(k\Delta)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)d_{\Delta}(t-k\Delta)\Delta \qquad (2.8)$$

Taking the limit as $\Delta \to 0$, we obtain the integral of Eq. (2.7), in which when $\Delta \to 0$

(1) The summation approaches to an integral
(2) $k\Delta \to t$ and $x(k\Delta) \to x(t)$
(3) $\Delta \to dt$
(4) $d_{\Delta}(t-k\Delta) \to d(t-t)$

Eq. (2.7) can also be obtained by using the sampling property of the impulse function. If we consider t is fixed and t is time variable, then we have $x(t)d(t-t)$ $= x(t)d(-(t-t)) = x(t)d(t-t)$. Hence

$$\int_{-\infty}^{\infty} x(t)d(t-t)dt = \int_{-\infty}^{\infty} x(t)d(t-t)dt = x(t)\int_{-\infty}^{\infty} d(t-t)dt = x(t). \quad (2.9)$$

As in discrete time, this is the ***sifting property*** of continuous-time impulse.

2.2.2 Continuous-Time Unit Impulse Response and the Convolution Integral Representation of an LTI system

The ***linearity property*** of an LTI system allows us to calculate the system response to an input signal $\hat{x}(t)$ using ***Superposition Principle***. Let $\hat{h}_{k\Delta}(t)$ be the pulse response of the linear-varying system to the unit pulses $d_{\Delta}(t-k\Delta)$ for $-\infty < k < +\infty$. The response of the system to $\hat{x}(t)$ is

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_{k\Delta}(t-k\Delta)\Delta . \quad (2.10)$$

Note that the response $\hat{h}_{k\Delta}(t)$ tends to the impulse response $h_t(t)$ as $\Delta \to 0$. Then at the limit, we obtain the response of the system to the input signal $x(t) = \lim_{\Delta \to 0} \hat{x}(t)$:

$$y(t) = \lim_{\Delta \to 0} \hat{y}(t) = \int_{-\infty}^{+\infty} x(t)h_t(t)dt . \quad (2.11)$$

For an LTI system, the impulse responses $h_t(t)$ are the same as $h_0(t)$, except they are shifted by t, that is, $h_t(t) = h_0(t-k)$. Then we may define the unit impulse response of the LTI system

$$h(t) = h_0(t), \quad (2.12)$$

and an LTI system is completely determined by its impulse response.

So the response to the input signal $x(t)$ can be written as a convolution integral:

$$y(t) = \int_{-\infty}^{+\infty} x(t)h(t-t)dt , \quad (2.13)$$

or it can be expressed symbolically

$$y(t) = x(t) * h(t). \quad (2.14)$$

2.2.3 Calculation of convolution integral

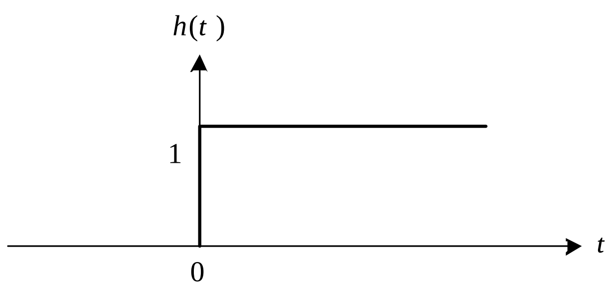
The output $y(t)$ is a weighted integral of the input, where the weight on $x(t)$ is $h(t-t)$. To evaluate this integral for a specific value of t,

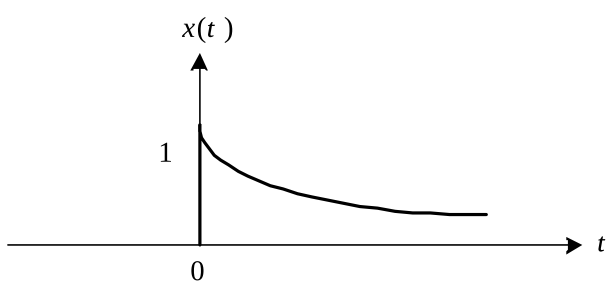
- First obtain the signal $h(t-t)$ (regarded as a function of t with t fixed) from $h(t)$ by a reflection about the origin and a shift to the right by t if t >0 or a shift to the left by $|t|$ is t <0.
- Then multiply together the signals $x(t)$ and $h(t-t)$.
- $y(t)$ is obtained by integrating the resulting product from $t=-\infty$ to $t=+\infty$.

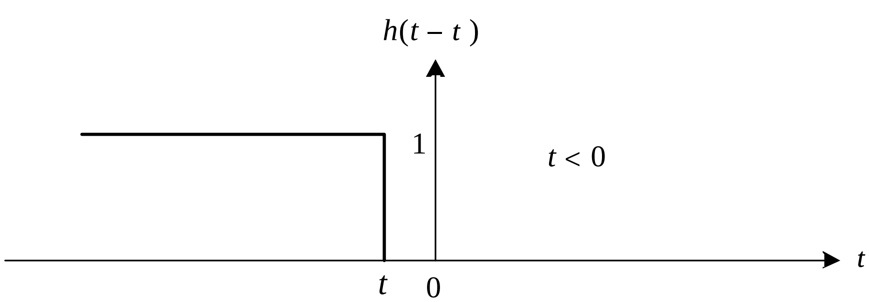
Example: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

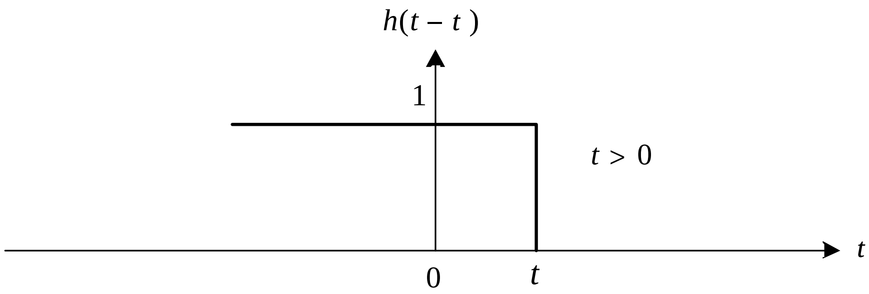
$$x(t)=e^{-at}u(t),\ a>0 \text{ and } h(t)=u(t).$$

Step1: The functions $h(t)$, $x(t)$ and $h(t-t)$ are depicted









Step 2: From the figure we can see that for $t < 0$, the product of the product $x(t)$ and $h(t-t)$ is zero, and consequently, $y(t)$ is zero. For $t > 0$

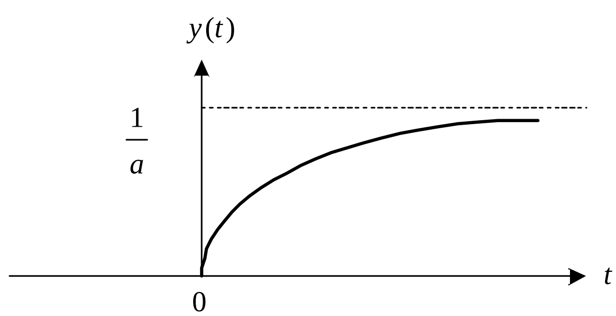
$$x(t)h(t-t) = \begin{cases} e^{-at}, & \text{t > 0} \\ 0, & otherwise \end{cases}$$

Step 3: Compute $y(t)$ by integrating the product for $t > 0$

$$y(t) = \int_0^t e^{-at}\,dt = -\frac{1}{a}e^{-at}\Big|_0^t = \frac{1}{a}(1-e^{-at}) .$$

The output of $y(t)$ for all t is

$y(t) = \frac{1}{a}(1-e^{-at})u(t)$, and is shown in figure below.



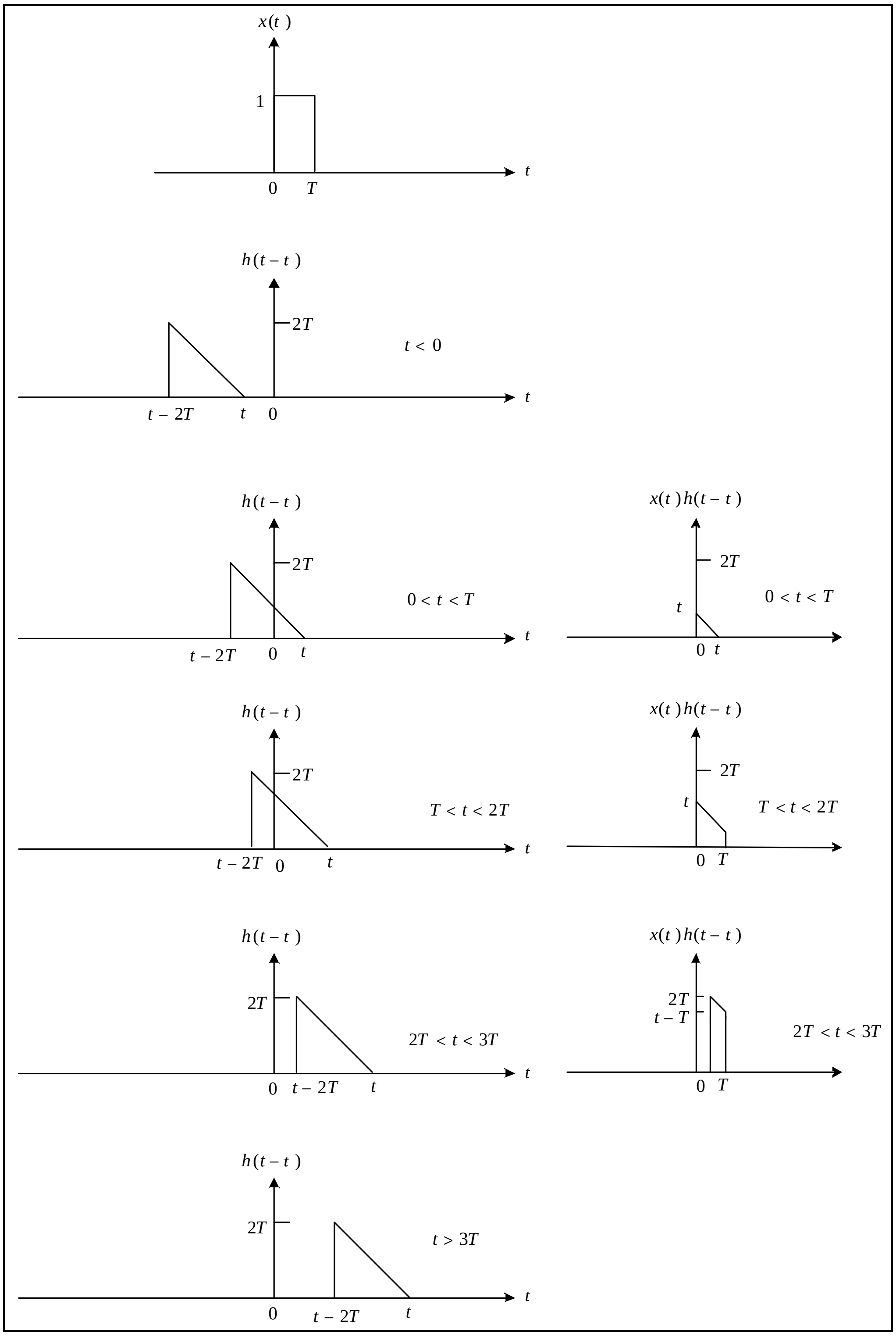
Example: Compute the convolution of the two signals below:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases} \text{ and } h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & otherwise \end{cases}$$

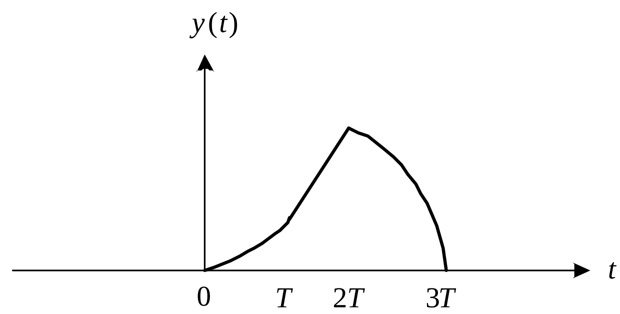
For this example, it is convenient to calculate the convolution in separate intervals. $x(t)$ is sketched and $h(t-t)$ is sketched in each of the intervals:

For $t < 0$, and $t > 3T$, $x(t)h(t-t) = 0$ for all the values of t, and consequently $y(t)$ =0.

For other intervals, the product $x(t)h(t-t)$ can be found in the figure on the next page. Thus for these three intervals, the integration can be calculated with the result shown below:


x(t)
1
0
T
t
h(t − t)
2T
t < 0
t − 2T
t
0
0 < t < T
x(t)h(t − t)
T < t < 2T
2T < t < 3T
t − T
t > 3T

$$y(t)=\begin{cases} 0, & t<0 \\ \frac{1}{2}t^2, & 0<t<T \\ Tt-\frac{1}{2}T^2, & T<t<2T \\ -\frac{1}{2}t^2+Tt+\frac{3}{2}T^2, & 2T<t<3T \\ 0, & t>3T \end{cases}$$



2.3 Properties of Linear Time-Invariant Systems

LTI systems can be characterized completely by their impulse response. The properties can also be characterized by their impulse response.

2.3.1 The Commutative Property of LTI Systems

A property of convolution in both continuous and discrete time is a ***Commutative Operation***. That is

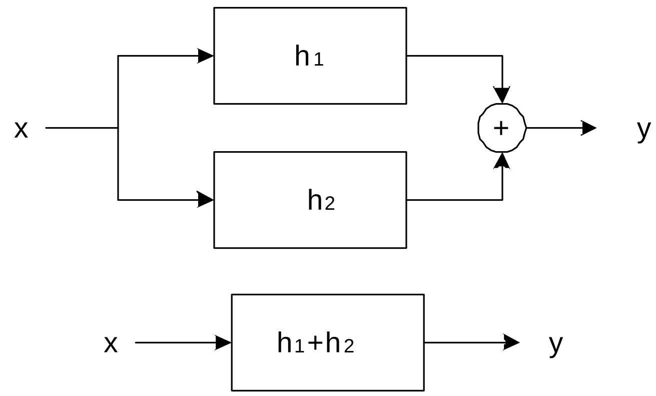
$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]k[n-k], \quad (2.15)$$

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau \ . \quad (2.16)$$

2.3.2 The Distributive Property of LTI Systems

$$x*(h_1+h_2)=x*h_1+x*h_2 \quad (2.17)$$

for both discrete-time and continuous-time systems. The property means that summing the outputs of two systems is equivalent to a system with an impulse response equal to the sum of the impulse response of the two individual systems, as shown in the figure below.



The distributive property of convolution can be exploited to break a complicated convolution into several simpler ones.

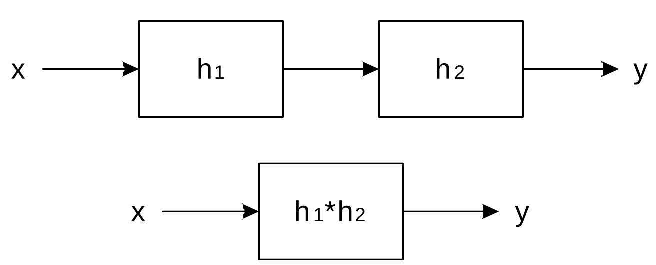
For example, an LTI system has an impulse response $h[n]=u[n]$, with an input $x[n]=\left(\frac{1}{2}\right)^n u[n]+2^n u[-n]$. Since the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. That is, $x_1[n]=\left(\frac{1}{2}\right)^n u[n]$, $x_2[n]=2^n u[-n]$, using the distributive property we have

$$y[n]=\left(x_1(t)+x_2(t)\right)*h(t)=x_1(t)*h(t)+x_2(t)*h(t)=y_1[n]+y_2[n]$$

2.3.3 The Associative Property

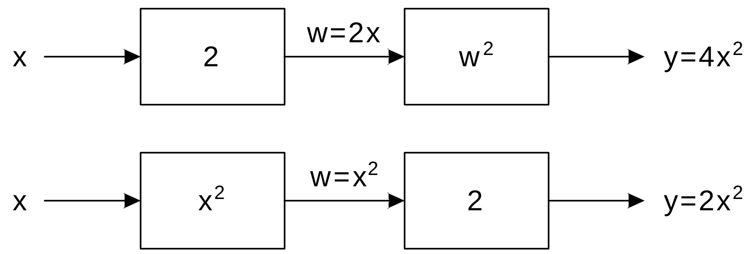
$$x*\left(h_1*h_2\right)=\left(x*h_1\right)*h_2 . \tag{2.18}$$

for both discrete-time and continuous-time systems.



- For LTI systems, the change of order of the cascaded systems will not affect the response.

- For nonlinear systems, the order of cascaded systems in general cannot be changed. For example, a two memoryless systems, one being multiplication by 2 and the other squaring the input, the outputs are different if the order is changed, as shown in the figure below.



2.3.4 LTI system with and without memory

A system is memoryless if its output at any time depends only on the value of its input at the same time. This is true for a discrete-time system, if $h[n]=0$ for $n \neq 0$. In this case, the impulse response has the form

$$h[n]=Kd[n], \tag{2.19}$$

where $K=h[0]$ is a constant and the convolution sum reduces to the relation

$$y[n]=Kx[n]. \tag{2.20}$$

Otherwise the LTI system has memory.

For continuous-time systems, we have the similar results if it is memoryless:

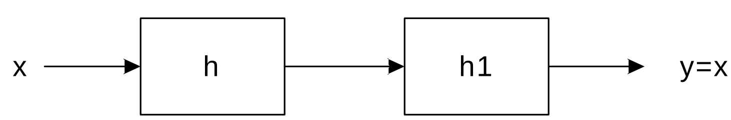
$$h(t)=Kd(t), \tag{2.21}$$

$$y(t)=Kx(t). \tag{2.22}$$

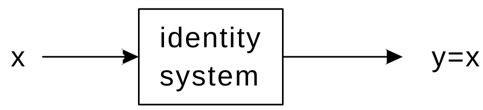
Note that if $K=1$ in Eqs. (2.19) and (2.21), the systems become ***identity systems***, with output equal to the input.

2.3.5 Invertibility of LTI systems

We have seen that a system S is invertible if and only if there exists an inverse system S^{-1} such that $S^{-1}S$ is an identity system.



Since the overall impulse response in the figure above is $h * h_1$, h_1 must satisfy for it to be the impulse response of the inverse system, namely $h * h_1 = d$.



Applications - channel equalization: for transmission of a signal over a communication channel such as telephone line, radio link and fiber, the signal at the receiving end is often processed through a filter whose impulse response is designed to be the inverse of the impulse response of the communication channel.

Example: Consider a system with a pure time shifted output $y(t) = x(t - t_0)$.

The impulse response of this system is $h(t) = d(t - t_0)$, since $x(t - t_0) = x(t) * d(t - t_0)$, that is, convolution of a signal with a shifted impulse simply shifts the signal

To recover the signal from the output, that is, to invert the system, all that is required is to shift the output back. So the inverse system should have a impulse response of $d(t + t_0)$, then

$$d(t - t_0) * d(t + t_0) = d(t)$$

Example: Consider the LTI system with impulse response $h[n] = u[n]$.

The response of this system to an arbitrary input is

$$y[n] = \sum_{-\infty}^{+\infty} x[k]u[n-k].$$

Considering that $u[n-k]$ is 0 for $n - k < 0$ and 1 for $n - k \geq 0$, so we have

$$y[n] = \sum_{-\infty}^{n} x[k].$$

This is a system that calculates the running sum of all the values of the input up to the present time, and is called a summer or accumulator. This system is invertible, and its inverse is given as

$$y[n] = x[n] - x[n-1],$$

It is a first difference operation. The impulse response of this inverse system is

$$h_1[n] = d[n] - d[n-1],$$

We may check that the two systems are really inverses to each other:

$$h[n]*h_1[n]=u[n]*\{d[n]-d[n-1]\}=u[n]-u[n-1]=d[n]$$

2.3.6 Causality for LTI systems

A system is causal if its output depends only on the past and present values of the input signal. Specifically, for a discrete-time LTI system, this requirement is $y[n]$ should not depend on $x[k]$ for $k>n$. Based on the convolution sum equation, all the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k>n$ must be zero, which means that the impulse response of a causal discrete-time LTI system should satisfy the condition

$h[n]=0$, for $n<0$ (2.23)

A causal system is causal if its impulse response is zero for negative time; this makes sense as the system should not have a response before impulse is applied.

A similar conclusion can be arrived for continuous-time LTI systems, namely

$h(t)=0$, for $t<0$ (2.24)

Examples: The accumulator $h[n]=u[n]$, and its inverse $h[n]=d[n]-d[n-1]$ are causal. The pure time shift with impulse response $y(t)=x(t-t_0)$ for $t_0>0$ is causal, but is not causal for $t_0<0$.

2.3.7 Stability for LTI Systems

Recall that a system is *stable* if every bounded input produces a bounded output.

For LTI system, if the input $x[n]$ is bounded in magnitude

$|x[n]|\le B$, for all n

If this input signal is applied to an LTI system with unit impulse response $h[n]$, the magnitude of the output

$$|y[n]|=\left|\sum_{k=-\infty}^{+\infty}h[k]x[n-k]\right|\le\sum_{k=-\infty}^{+\infty}|h[k]||x[n-k]|\le B\sum_{k=-\infty}^{+\infty}|h[k]| \quad (2.25)$$

$y[n]$ is bounded in magnitude, and hence is stable if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty . \tag{2.26}$$

So discrete-time LTI system is stable is Eq. (2.26) is satisfied.

The similar analysis applies to continuous-time LTI systems, for which the stability is equivalent to

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty . \tag{2.27}$$

Example: consider a system that is pure time shift in either continuous time or discrete time.

In discrete time, $\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=-\infty}^{+\infty} |d[n-n_0]| = 1$,

while in continuous time, $\int_{-\infty}^{+\infty} |h(t)| dt = \int_{-\infty}^{+\infty} |d(t-t_0)| dt = 1$,

and we conclude that both of these systems are stable.

Example: The accumulator $h[n] = u[n]$ is unstable because $\sum_{k=-\infty}^{+\infty} |h[k]| = \sum_{k=0}^{+\infty} |u[n]| = \infty$.

2.3.8 The Unit Step Response of an LTI System

The step response of an LTI system is simply the response of the system to a unit step. It conveys a lot of information about the system. For a discrete-time system with impulse response $h[n]$, the step response is $s[n] = u[n] * h[n]$. However, based on the commutative property of convolution, $s[n] = h[n] * u[n]$, and therefore, $s[n]$ can be viewed as the response to input $h[n]$ of a discrete-time LTI system with unit impulse response. We know that $u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k] . \tag{2.28}$$

From this equation, $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1] . \tag{2.29}$$

It can be seen the step response of a discrete-time LTI system is the ***running sum of its impulse*** response. Conversely, the impulse response of a discrete-time LTI system is the ***first difference*** of its step response.

Similarly, in continuous time, the step response of an LTI system is the running integral of its impulse response,

$$s(t) = \int_{-\infty}^{t} h(t\,)dt \;, \tag{2.30}$$

and the unit impulse response is the first derivative of the unit step response,

$$h(t) = \frac{ds(t)}{dt} = s'(t)\,. \tag{2.31}$$

Therefore, in both continuous and discrete time, the unit step response can also be used to characterize an LTI system.

2.4 Causal LTI Systems Described by Differential and Difference Equations

This is a class of systems for which the input and output are related through

- *A linear constant-coefficient differential equation in continuous time, or*
- *A linear constant-coefficient difference equation in discrete-time.*

2.4.1 Linear Constant-Coefficient Differential Equations

In a causal LTI difference system, the discrete-time input and output signals are related implicitly through a linear constant-coefficient differential equation.

Let us consider a first-order differential equation,

$$\frac{dy(t)}{dt} + 2y(t) = x(t)\,, \tag{2.32}$$

where $y(t)$ denotes the output of the system and $x(t)$ is the input.

This equation can be explained as the velocity of a car $y(t)$ subjected to friction force proportional to its speed, in which $x(t)$ would be the force applied to the car.

In general, an N^{th}-order linear constant coefficient differential equation has the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\,, \tag{2.33}$$

The solution of the differential equation can be obtained if we have the N initial conditions (or auxiliary conditions) on the output variable and its derivatives.

Recall that the solution to the differential equation is the sum of the ***homogeneous*** solution of the differential equation $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$ (a solution with input set to zero) and of a ***particular*** solution (a function that satisfy the differential equation).

Forced response of the system = particular solution (usually has the form of the input signal)
Natural response of the system = homogeneous solution (depends on the initial conditions and forced response).

Example: Solve the system described by $\frac{dy(t)}{dt} + 2y(t) = x(t)$. Given the input is $x(t) = Ke^{3t}u(t)$, where K is a real number.

As mentioned above, the solution consists of the homogeneous response and the particular solution:

$$y(t) = y_h(t) + y_p(t), \tag{2.34}$$

where the particular solution $y_p(t)$ satisfies $\frac{dy(t)}{dt} + 2y(t) = x(t)$ and homogenous solution $y_h(t)$ satisfies

$$\frac{dy(t)}{dt} + 2y(t) = 0. \tag{2.35}$$

For the particular solution for $t > 0$, $y_p(t)$ is a signal that has the same form as $x(t)$ for $t > 0$, that is

$$y_p(t) = Ye^{3t}. \tag{2.36}$$

Substituting $x(t) = Ke^{3t}u(t)$ and $y_p(t) = Ye^{3t}$ into $\frac{dy(t)}{dt} + 2y(t) = x(t)$, we get

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}, \tag{2.37}$$

Canceling the factor e^{3t} on both sides, we obtain $Y = K/5$, so that

$$y_p(t) = \frac{K}{5}e^{3t},\ t > 0 \tag{2.38}$$

To determine the natural response $y_h(t)$ of the system, we hypothesize a solution of the form of an exponential,

$$y_h(t) = Ae^{st}. \tag{2.39}$$

Substituting Eq. (3.38) into Eq. (3.35), we get

$$Ase^{st} + 2Ase^{st} = 0, \tag{2.40}$$

which holds for $s = -2$. With this value of s, Ae^{-2t} is a solution to the homogeneous equation $\frac{dy(t)}{dt} + 2y(t) = 0$ for any choice of A.

Combining the natural response and the forced response, we get the solution to the differential equation $\frac{dy(t)}{dt} + 2y(t) = x(t)$:

$$y(t) = y_h(t) + y_p(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \; t > 0 \tag{2.41}$$

Because the initial condition on $y(t)$ is not specified, so the response is not completely determined, as the value of A is not known.

For causal LTI systems defined by linear constant coefficient differential equations, the initial conditions are always $y(0) = \frac{dy(0)}{dt} = ... = \frac{dy^{N-1}(0)}{dt^{N-1}} = 0$, which is called ***initial rest***.

For this example, the initial rest implies that $y(0) = 0$, so that $y(0) = A + \frac{K}{5} = 0 \Rightarrow A = -\frac{K}{5}$, the solution is

$$y(t) = \frac{K}{5}(e^{3t} - e^{-2t}), \; t > 0 \tag{2.42}$$

For $t < 0$, the condition of initial rest and causality of the system implies that $y(t) = 0$, $t < 0$, since $x(t) = 0$, $t < 0$.

2.4.2 Linear Constant-Coefficient Difference Equations

In a causal LTI difference system, the discrete-time input and output signals are related implicitly through a linear constant-coefficient difference equation.

In general, an N^{th}-order linear constant coefficient difference equation has the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{2.43}$$

The solution of the differential equation can be obtained when we have the N initial conditions (or auxiliary conditions) on the output variable.

The solution to the difference equation is the sum of the ***homogeneous*** solution $\sum_{k=0}^{N} a_k y[n-k] = 0$ (a solution with input set to zero, or natural response) and of a ***particular*** solution (a function that satisfy the difference equation).

$$y[n] = y_h[n] + y_p[n], \tag{2.44}$$

The concept of ***initial rest*** of the LTI causal system described by difference equation means that $x[n] = 0$, $n < n_0$ implies $y[n] = 0$, $n < n_0$.

Example: consider the difference equation

$$y[n] - \frac{1}{2} y[n-1] = x[n], \tag{2.45}$$

The equation can be rewritten as

$$y[n] = \frac{1}{2} y[n-1] + x[n], \tag{2.46}$$

It can be seen from Eq. (2.46) that we need the previous value of the output, $y[n-1]$, to calculate the current value.

Suppose that we impose the condition of initial rest and consider the input

$$x[n] = Kd[n]. \tag{2.47}$$

Since $x[n] = 0$ for $n \le -1$, the condition of initial rest implies that $y[n] = 0$, for $n \le -1$, so that we have as an initial condition: $y[-1] = 0$. Starting from this initial condition, we can solve for successive values of $y[n]$ for $n \ge 0$:

$$y[0] = \frac{1}{2} y[-1] + x[0] = K,$$

$$y[1] = \frac{1}{2}y[0] + x[1] = \frac{1}{2}K ,$$

$$y[2] = \frac{1}{2}y[1] + x[2] = \left(\frac{1}{2}\right)^2 K ,$$

$$y[3] = \frac{1}{2}y[2] + x[3] = \left(\frac{1}{2}\right)^3 K ,$$

$$\cdots$$

$$y[n] = \frac{1}{2}y[n-1] + x[n] = \left(\frac{1}{2}\right)^n K .$$

Since for an LTI system, the input-output behavior is completely characterized by its impulse response. Setting $K = 1$, , $x[n] = d[n]$ we see that the impulse response for the system is

$$h[n] = \left(\frac{1}{2}\right)^n u[n] . \qquad (2.48)$$

Note that the causal system in the above example has an impulse response of infinite duration. In fact, if $N \geq 1$ in Eq. (2.43), the difference equation is recursive, it is usually the case that the LTI system corresponding to this equation together with the condition of initial rest will have an impulse response of infinite duration. Such systems are referred to as ***infinite impulse response* (*IIR*) *systems***.

2.4.3 Block Diagram Representations of 1st-order Systems Described by Differential and Difference Equations

Block diagram interconnection is very simple and nature way to represent the systems described by linear constant-coefficient difference and differential equations.

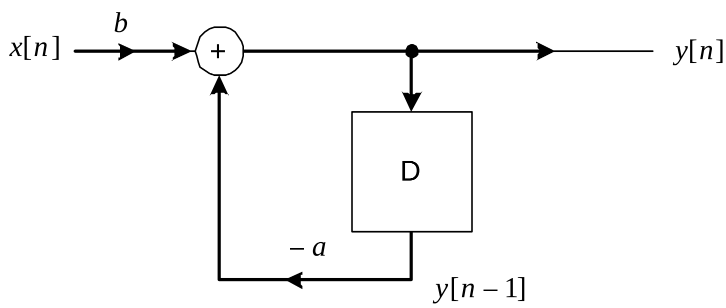
For example, the causal system described by the first-order difference equation is

$$y[n] + ay[n-1] = bx[n] . \qquad (2.49)$$

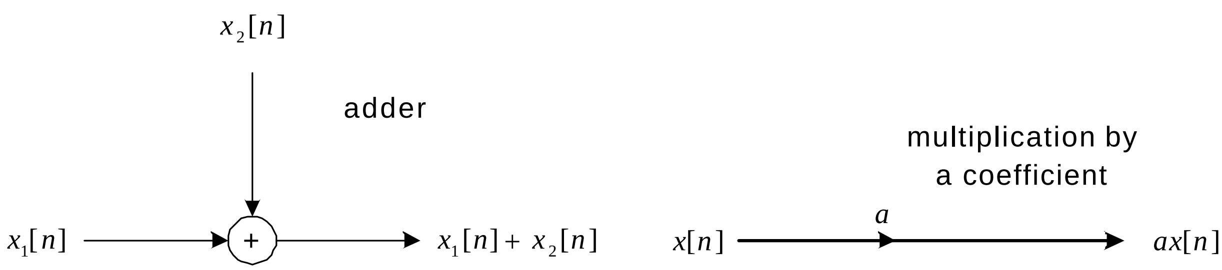
It can be rewritten as

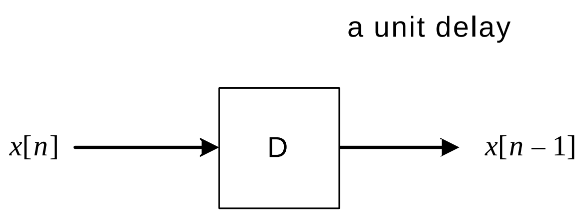
$$y[n] = -ay[n-1] + bx[n]$$

The block diagram representation for this discrete-time system is show:



Three elementary operations are required in the block diagram representation: **addition**, **multiplication** by a coefficient, and **delay:**





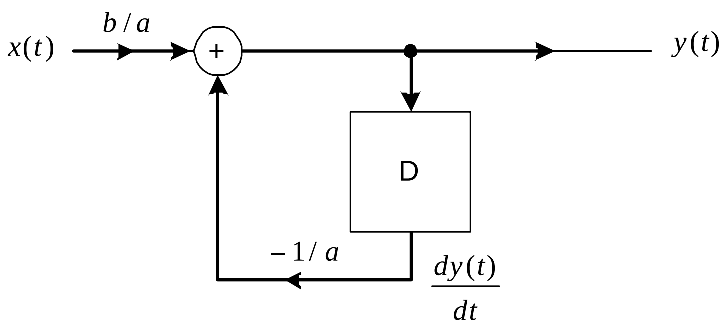
Consider the block diagram representation for continuous-time systems described by a first-order differential equation:

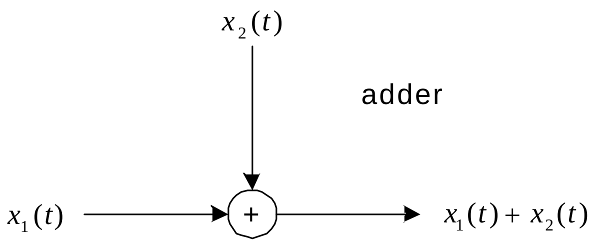
$$\frac{dy(t)}{dt} + ay(t) = bx(t)\,. \tag{2.48}$$

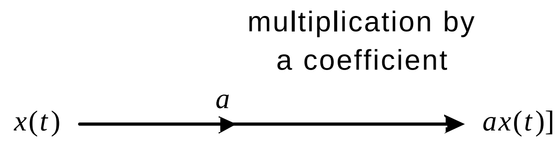
Eq. (2.48) can be rewritten as

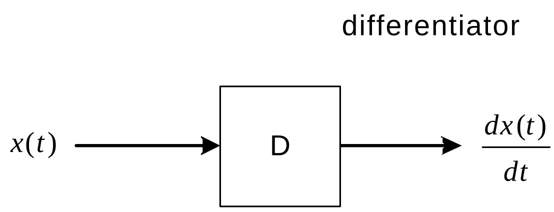
$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}bx(t)\,.$$

Similarly, the right-hand side involves three basic operations: **addition**, **multiplication** by a coefficient, and **differentiation:**









However, the above representation is not frequently used or the representation does not lead to practical implementation, since ***differentiators are both difficult to implemented and extremely sensitive to errors and noise***.

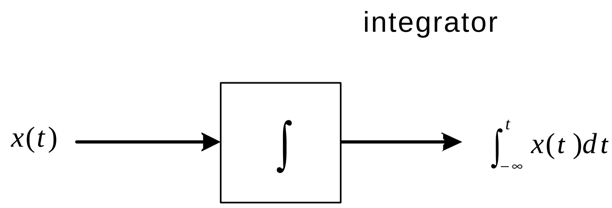
An alternative implementation is to used integrators rather than the differentiators. Eq. (2.48) can be rewritten as

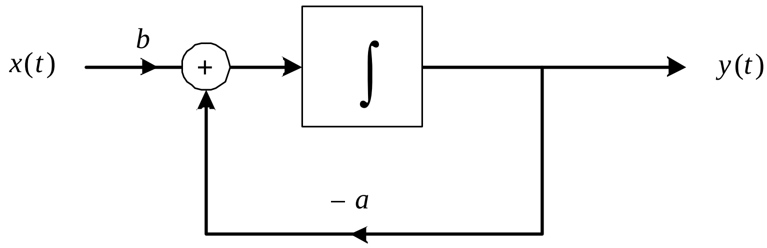
$$\frac{dy(t)}{dt} = bx(t) - ay(t)\,, \tag{2.49}$$

integrating from $-\infty$ to t, and assuming $y(-\infty) = 0$, then we obtain

$$y(t) = \int_{-\infty}^{t} \left[bx(t) - ay(t) \right] dt\ . \tag{2.50}$$

In this form, the system can be implemented using the adder and coefficient multiplier, together with an *integrator*, as shown in the figure below.





The integrator can be readily implemented using operational amplifiers, the above representations lead directly to analog implementations. This is the basis for both early analog computers and modern analog computation systems.

Eq. (2.50) can also express in the form

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(t) - ay(t)]dt \ , \tag{2.51}$$

where we consider integrating Eq. (2.50) from a finite point in time t_0. It makes clear the fact that the specification of $y(t)$ requires an initial condition, namely $y(t_0)$.

Any ***higher-order systems*** can be developed using the block diagram for the simplest first-order differential and difference equations.