A few formulas

Euler

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Summations, geometric series

$$\sum_{k=n_1}^{\infty} a^k = \frac{a^{n_1}}{1-a} \quad |a|<1 \qquad \sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a} \quad \begin{matrix} a \neq 1 \\ n_2 \geq n_1 \end{matrix}$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad |a|<1 \qquad \sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a} \quad \begin{matrix} a \neq 1 \\ n_1 \geq 0 \end{matrix}$$

Even and odd parts

$$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t) \qquad x_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

$$x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] \qquad x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$$

Convolutions

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$h(t)$ response for differential equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$h(t) = \sum_{k=0}^{N-1} A_k e^{s_k t} u(t) + \sum_{k=0}^{M-N} B_k \frac{d^k \delta(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t) \qquad h'(t) = \sum_{k=0}^{N-1} A_k e^{s_k t} u(t) \ \text{(simpl. sys.)}$$

$h[n]$ response for difference equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{N-1} A_k \alpha_k^n u[n] + \sum_{k=0}^{M-N} B_k \delta[n-k]$$

$$\sum_{k=0}^{N} a_k y[n-k] = x[n] \qquad h'[n] = \sum_{k=0}^{N-1} A_k \alpha_k^n u[n] \ \text{(simpl. sys.)}$$

LTI systems and eigenfunctions

$$e^{st} \xrightarrow{LTI(cont.)} H(s)e^{st}$$

$$z^n \xrightarrow{LTI(discr.)} H(z)z^n$$

$$e^{j\omega t} \xrightarrow{LTI(cont.)} H(j\omega)e^{j\omega t}$$

$$e^{j\omega n} \xrightarrow{LTI(discr.)} H(e^{j\omega})e^{j\omega n}$$

$$\cos(\omega t) \xrightarrow{LTI(cont.)} |H(j\omega)|\cos(\omega t + \angle H(j\omega))$$

$$\cos(\omega n) \xrightarrow{LTI(discr.)} |H(e^{j\omega})|\cos(\omega n + \angle H(e^{j\omega}))$$

Standard first and second order low-pass systems, continuous time

$$H(j\omega) = \frac{1}{1 + j\omega\tau} \qquad H(j\omega) = \frac{\omega_n{}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2}$$

Standard first and second order recursive systems, discrete time

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad |a|<1$$

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}} \qquad 0 \leq r < 1,\ 0 \leq \theta \leq \pi$$

Continuous time sampling

$$x_p(t) = x(t) \times p(t) \qquad x_d[n] = x(nT)$$

$$X_p(j\omega) = f_S \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \qquad \omega_s = \frac{2\pi}{T} = 2\pi f_s$$

$$X_d(e^{j\omega}) = X_p(j\omega f_s)$$

$$H_0(j\omega) = e^{-j\pi\omega/\omega_s}\, 2\sin(\pi\omega/\omega_s)/\omega \ \text{(sample \& hold)}$$

Other formulas

$$A\cos(\phi+\theta) = A\sin(\phi+\theta+\pi/2) = B\cos\phi - C\sin\phi$$

$$(B = A\cos\theta \quad C = A\sin\theta \quad A^2 = \sqrt{B^2+C^2} \quad \theta = \tan^{-1}\left(\frac{C}{B}\right))$$

$$ae^{j\phi} + a^* e^{-j\phi} = 2\,\mathrm{Re}\{ae^{j\phi}\} = 2|a|\cos(\phi + \angle a)$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + c$$

$$\frac{d\,\mathrm{atan}(x)}{dx} = \frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$$

Properties – Continuous time Fourier series (C.T.F.S.)

Definitions:

$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ $\qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$a_0 = \frac{1}{T}\int_T x(t)dt$

$x(t)$ periodic with period T sec.,

Fundam. angular frequency $\omega_0 = 2\pi f_0 = 2\pi/T$ rad./sec.

$x(t) \overset{C.T.F.S.}{\longleftrightarrow} a_k \qquad y(t) \overset{C.T.F.S.}{\longleftrightarrow} b_k$

If $x(t) \overset{LTI}{\longrightarrow} y(t)$ then $b_k = a_k H(j\omega)\big|_{\omega=k\omega_0}$

Linearity: $Ax(t)+By(t) \overset{C.T.F.S.}{\longleftrightarrow} A\,a_k + B\,b_k$

Shifting: $x(t-t_0) \overset{C.T.F.S.}{\longleftrightarrow} e^{-jk\omega_0 t_0}\,a_k$

Scaling: $x(\alpha t) \overset{C.T.F.S.}{\longleftrightarrow} a_k$

($\alpha > 0$, period T/α)

Flipping: $x(-t) \overset{C.T.F.S.}{\longleftrightarrow} a_{-k}$

Conjugate: $x^*(t) \overset{C.T.F.S.}{\longleftrightarrow} a^*_{-k}$

$x^*(-t) \overset{C.T.F.S.}{\longleftrightarrow} a^*_k$

Symmetries:

if $x(t)$ is real: $a_k = a^*_{-k}$, $|a_k| = |a_{-k}|$, $\angle a_k = -\angle a_{-k}$

$x(t)$ real and even : a_k real and even $a_k = a_{-k}$

$x(t)$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$

Periodic convolution:

$\int_T x(\tau)y(t-\tau)d\tau \overset{C.T.F.S.}{\longleftrightarrow} T\,a_k\,b_k$

Modulation: $x(t)y(t) \overset{C.T.F.S.}{\longleftrightarrow} a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

$e^{jm\omega_0 t}\,x(t) \overset{C.T.F.S.}{\longleftrightarrow} a_{k-m}$

Differentiation: $\frac{dx(t)}{dt} \overset{C.T.F.S.}{\longleftrightarrow} jk\omega_0 a_k$

Integration: $\int_{\tau=-\infty}^{t} x(\tau)d\tau \overset{C.T.F.S.}{\longleftrightarrow} \frac{a_k}{jk\omega_0}$ (if $a_0 = 0$)

Parseval: $\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Table of continuous time Fourier series (C.T.F.S.)

$x(t)$ periodic, period $T = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$ sec.	Fourier series coefficients a_k
$e^{j\omega_0 t}$	$a_1 = 1$ $a_k = 0$ elsewhere
$\cos(\omega_0 t)$	$a_1, a_{-1} = 1/2$ $a_k = 0$ elsewhere
$\sin(\omega_0 t)$	$a_1, a_{-1} = 1/(2j)$ $a_k = 0$ elsewhere
$\begin{cases} 1 & \|t\| < T_1 \\ 0 & T_1 < \|t\| < T/2 \end{cases}$ (periodic T)	$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$ $a_k = \frac{2T_1}{T} = \frac{T_1\omega_0}{\pi}$ $k = 0$
1	$a_0 = 1$ $a_k = 0$ elsewhere
$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$a_k = \frac{1}{T}$

Properties – Discrete time Fourier series (D.T.F.S.)

Definitions:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j(k\frac{2\pi}{N})n} \qquad x[n] = \sum_{k=<N>} a_k e^{j(k\frac{2\pi}{N})n}$$

$$a_0 = \frac{1}{N}\sum_{n=<N>} x[n]$$

$x[n]$ periodic with period N samples (fundamental angular frequency $\omega_0 = \frac{2\pi}{N}$ rad./sample)

$$x[n] \xleftrightarrow{D.T.F.S.} a_k \qquad y[n] \xleftrightarrow{D.T.F.S.} b_k$$

If $x[n] \xrightarrow{LTI} y[n]$ then $b_k = a_k H(e^{j\omega})\Big|_{\omega = k\frac{2\pi}{N}}$

Periodicity: $x[n] \xleftrightarrow{D.T.F.S.} a_k = a_{k+N}$

Linearity: $Ax[n] + By[n] \xleftrightarrow{D.T.F.S.} Aa_k + Bb_k$

Shifting: $x[n-n_0] \xleftrightarrow{D.T.F.S.} e^{-jk\frac{2\pi}{N}n_0} a_k$

Flipping: $x[-n] \xleftrightarrow{D.T.F.S.} a_{-k}$

Conjugate: $x^*[n] \xleftrightarrow{D.T.F.S.} a^*_{-k}$

$x^*[-n] \xleftrightarrow{D.T.F.S.} a^*_k$

Symmetries:

if $x[n]$ is real : $a_k = a^*_{-k}$, $|a_k| = |a_{-k}|$, $\angle a_k = -\angle a_{-k}$

$x[n]$ real and even : a_k real and even $a_k = a_{-k}$

$x[n]$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$

Periodic convolution:

$$\sum_{m=<N>} x[m]y[n-m] \xleftrightarrow{D.T.F.S.} N\, a_k\, b_k$$

Modulation: $x[n]y[n] \xleftrightarrow{D.T.F.S.} \sum_{l=<N>} a_l b_{k-l}$

$$e^{jm\frac{2\pi}{N}n} x[n] \xleftrightarrow{D.T.F.S.} a_{k-m}$$

Accumulation : $\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{D.T.F.S.} \frac{1}{\left(1-e^{-jk\frac{2\pi}{N}}\right)} a_k$

(if $a_0 = 0$)

Parseval: $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

Duality : if $x[n] \xleftrightarrow{DTFS} a_k$ then $a[n] \xleftrightarrow{DTFS} \frac{1}{N} x_{-k}$

Table of discrete time Fourier series (D.T.F.S.)

$x[n]$ periodic, period N samples	Fourier series coefficients a_k (periodic with period N)
$e^{j\omega_0 n}$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1 \quad k = m, m \pm N, m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\cos(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1/2$ $k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\sin(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1/(2j)$ $k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\begin{cases} 1 & \|n\| \le N_1 \\ 0 & N_1 < \|n\| \le N/2 \end{cases}$ (periodic N , N even)	$a_k = \dfrac{\sin\left(\frac{2\pi}{N}k(N_1 + 1/2)\right)}{N\sin(\frac{\pi}{N}k)}$ $k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = (2N_1+1)/N \;\; k = 0, \pm N, \pm 2N, \ldots$
1	$a_k = 1 \quad k = 0, \pm N, \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\sum_{m=-\infty}^{\infty} \delta[n-mN]$	$a_k = \frac{1}{N}$

Properties – Continuous time Fourier transform (C.T.F.T.)

Definitions:

$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$

ω in rad./sec.

$X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi\, a_k\delta(\omega-k\omega_0)$ if $x(t)$ periodic

$x(t)\overset{CTFT}{\longleftrightarrow}X(j\omega) \quad y(t)\overset{CTFT}{\longleftrightarrow}Y(j\omega)$

Linearity: $ax(t)+by(t)\overset{CTFT}{\longleftrightarrow}aX(j\omega)+bY(j\omega)$

Shifting: $x(t-t_0)\overset{CTFT}{\longleftrightarrow}e^{-j\omega t_0}X(j\omega)$

Scaling: $x(at)\overset{CTFT}{\longleftrightarrow}\frac{1}{|a|}X(\frac{j\omega}{a})$

Flipping: $x(-t)\overset{CTFT}{\longleftrightarrow}X(-j\omega)$

Conjugate: $x^*(t)\overset{CTFT}{\longleftrightarrow}X^*(-j\omega)$

$x^*(-t)\overset{CTFT}{\longleftrightarrow}X^*(j\omega)$

Symmetries:

if $x(t)$ is real : $X(j\omega)=X^*(-j\omega)$,

$|X(j\omega)|=|X(-j\omega)|,\ \angle X(j\omega)=-\angle X(-j\omega)$

$x(t)$ real and even : $X(j\omega)$ real and even $X(j\omega)=X(-j\omega)$

$x(t)$ real and odd: $X(j\omega)$ imag., odd $X(j\omega)=-X(-j\omega)$

Convolution:

$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau \overset{CTFT}{\leftrightarrow} X(j\omega)Y(j\omega)$

Modulation:

$x(t)y(t)\overset{CTFT}{\leftrightarrow}\frac{1}{2\pi}X(j\omega)*Y(j\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta$

$e^{j\omega_0 t}x(t)\overset{CTFT}{\longleftrightarrow}X(j(\omega-\omega_0))$

$\cos(\omega_0 t)x(t)\overset{CTFT}{\longleftrightarrow}\frac{1}{2}X(j(\omega-\omega_0))+\frac{1}{2}X(j(\omega+\omega_0))$

Differentiation: $\frac{dx(t)}{dt}\overset{CTFT}{\longleftrightarrow}j\omega\, X(j\omega)$

Integration: $\int_{-\infty}^{t}x(\tau)d\tau\overset{CTFT}{\longleftrightarrow}\frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$

Differentiation in freq.: $tx(t)\overset{CTFT}{\longleftrightarrow}j\frac{dX(j\omega)}{d\omega}$

Integration in freq.:

$-\frac{1}{jt}x(t)+\pi x(0)\delta(t)\overset{CTFT}{\longleftrightarrow}\int_{-\infty}^{\omega}X(j\eta)d\eta$

Parseval: $\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$

Duality : if $x(t)\overset{CTFT}{\longleftrightarrow}X(j\omega)$ then

$X(t)\overset{CTFT}{\longleftrightarrow}2\pi\, x(-j\omega)$

Table of continuous time Fourier transforms (C.T.F.T.)

signal $x(t)$ typ. aperiodic	$X(j\omega)$ (ω in rad./sec.)
if $x(t)$ is periodic, with period $T=\frac{2\pi}{\omega_0}=\frac{1}{f_0}$ sec.	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta(\omega-k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\begin{cases}1 & \|t\|<T_1\\ 0 & T_1<\|t\|<T/2\end{cases}$ (periodic T)	$2\sum_{k=-\infty}^{+\infty}\frac{\sin(k\omega_0T_1)}{k}\delta(\omega-k\omega_0)$ $k\neq 0$; $\frac{4\pi T_1}{T}\delta(\omega)=2T_1\omega_0\delta(\omega)$ $k=0$
1	$2\pi\delta(\omega)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_s\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)$ $\omega_s=\frac{2\pi}{T}$
$\begin{cases}1 & \|t\|<T_1\\ 0 & \|t\|>T_1\end{cases}$	$\frac{2\sin(\omega T_1)}{\omega}$
$\frac{\sin(Wt)}{\pi t}$ $W>0$	$\begin{cases}1 & \|\omega\|\le W\\ 0 & \|\omega\|>W\end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
$e^{-at}u(t)$ $\mathrm{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
$-e^{-at}u(-t)$ $\mathrm{Re}\{a\}<0$	$\frac{1}{a+j\omega}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ $\mathrm{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^n}$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$ $\mathrm{Re}\{a\}<0$	$\frac{1}{(a+j\omega)^n}$
$e^{-at}\sin(\omega_0 t)u(t)$ $a>0$ $\omega_0\ge 0$ a,ω_0 real	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)u(t)$ $a>0$ $\omega_0\ge 0$ a,ω_0 real	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$
$-e^{-at}\sin(\omega_0 t)u(-t)$ $a<0$ $\omega_0\ge 0$ a,ω_0 real	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$
$-e^{-at}\cos(\omega_0 t)u(-t)$ $a<0$ $\omega_0\ge 0$ a,ω_0 real	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$

Properties – Discrete time Fourier transform (D.T.F.T.)

Definitions:

$x[n] = x(nT) = x(t)\big|_{t=nT}$, where $T = 1/f_s = 2\pi/\omega_s$ is the sampling period in sec., and n is an integer, results in:

$$X_p(j\omega) = f_s \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)), \qquad X(e^{j\omega}) = X_p(j\omega f_s)$$

where $X(j\omega)$ is the original CTFT of $x(t)$, and $X(e^{j\omega})$ is the DTFT of $x[n]$ defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Periodicity: $x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$

Linearity: $ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$

Shifting: $x[n - n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$ $\quad n_0$ integer

Expansion, insertion of zeros:

$x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$ where k is a positive integer

$x_{(k)}[n] = x[n/k]$ if n is a multiple of k

$x_{(k)}[n] = 0$ elsewhere

Flipping: $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$

Conjugate: $x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega})$

$x^*[-n] \xleftrightarrow{DTFT} X^*(e^{j\omega})$

Symmetries:

if $x[n]$ is real : $X(e^{j\omega}) = X^*(e^{-j\omega})$,

$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$

$x[n]$ real and even : $X(e^{j\omega})$ real, even $X(e^{j\omega}) = X(e^{-j\omega})$

$x[n]$ real, odd $X(e^{j\omega})$ imag., odd $X(e^{j\omega}) = -X(e^{-j\omega})$

Convolution:

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \overset{DTFT}{\leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$$

Modulation: $x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

$e^{j\omega_0 n}x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

Accumulation:

$$\sum_{m=-\infty}^{n} x[m] \overset{DTFT}{\leftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - m2\pi)$$

Differentiation in freq.: $nx[n] \xleftrightarrow{DTFT} j\dfrac{dX(e^{j\omega})}{d\omega}$

Parseval: $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega$

Duality : If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ then $X(t) \xleftrightarrow{CTFS} x_{-k}$

Table of discrete time Fourier transforms (D.T.F.T.)

signal $x[n]$ typ. aperiodic	$X(e^{j\omega})$ (periodic 2π, ω in rad./sample)
if $x[n]$ is periodic, with period N samples	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\frac{2\pi}{N})$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$
$\cos(\omega_0 n)$	$\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
$\sin(\omega_0 n)$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi) - \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
1	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - l2\pi)$
$\sum_{m=-\infty}^{\infty} \delta[n - mN]$	$\frac{2\pi}{N} \sum_{m=-\infty}^{+\infty} \delta(\omega - m\frac{2\pi}{N})$
$\begin{cases} 1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1 \end{cases}$	$\sin(\omega(N_1 + 1/2)) / \sin(\omega/2)$
$\frac{\sin(Wn)}{\pi n}$ $\quad 0 < W < \pi$	$\begin{cases} 1 & 0 \le \lvert\omega\rvert \le W \\ 0 & W < \lvert\omega\rvert \le \pi \end{cases}$ period. 2π
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$a^n u[n]$ $\quad \lvert a\rvert < 1$	$1/(1 - ae^{-j\omega})$
$-a^n u[-n-1]$ $\quad \lvert a\rvert > 1$	$1/(1 - ae^{-j\omega})$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$ $\quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$
$\frac{-(n+r-1)!}{n!(r-1)!} a^n u[-n-1]$ $\quad \lvert a\rvert > 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$
$r^n \sin(\omega_0 n)u[n]$ $0 \le r < 1$ $\quad 0 \le \omega_0 \le \pi$	$\frac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$r^n \cos(\omega_0 n)u[n]$ $0 \le r < 1$ $\quad 0 \le \omega_0 \le \pi$	$\frac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \sin(\omega_0 n)u[-n-1]$ $r > 1$ $\quad 0 \le \omega_0 \le \pi$	$\frac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \cos(\omega_0 n)u[-n-1]$ $r > 1$ $\quad 0 \le \omega_0 \le \pi$	$\frac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$

Properties – bilateral (two-sided) Laplace transform

Definitions:

$$X(s)=\int_{-\infty}^{+\infty}x(t)e^{-st}dt \qquad x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$

$$x(t)\overset{LT}{\longleftrightarrow}X(s)\quad ROC_x \qquad y(t)\overset{LT}{\longleftrightarrow}Y(s)\quad ROC_y$$

Linearity: $ax(t)+by(t)\overset{LT}{\longleftrightarrow}aX(s)+bY(s)$

$ROC_x \cap ROC_y$

Shifting: $x(t-t_0)\overset{LT}{\longleftrightarrow}e^{-st_0}X(s)$ $\quad ROC_x$ unchanged

Scaling: $x(at)\overset{LT}{\longleftrightarrow}\frac{1}{|a|}X(\frac{s}{a})$

ROC_x dilated factor $|a|$ or compressed factor $\frac{1}{|a|}$,

and ROC_x inversed if $a<0$

Flipping: $x(-t)\overset{LT}{\longleftrightarrow}X(-s)$ $\quad ROC_x$ inversed

Conjugate: $x^*(t)\overset{LT}{\longleftrightarrow}X^*(s^*)$ $\quad ROC_x$ unchanged

Symmetry: if $x(t)$ real : $X(s)=X^*(s^*)$,

$|X(s)|=|X(s^*)|$

Convolution:

$$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\overset{LT}{\leftrightarrow}X(s)Y(s)$$

$ROC_x \cap ROC_y$

Modulation: $e^{s_0t}x(t)\overset{LT}{\longleftrightarrow}X(s-s_0)$

ROC_x shifted to right by $\mathrm{Re}\{s_0\}$

Differentiation: $\frac{dx(t)}{dt}\overset{LT}{\longleftrightarrow}s\,X(s)$ $\quad ROC_x$ unchanged

Integration: $\int_{-\infty}^{t}x(\tau)d\tau\overset{LT}{\longleftrightarrow}\frac{1}{s}X(s)$

$ROC_x\cap\left(\mathrm{Re}\{s\}>0\right)$

Differentiation in freq.: $-tx(t)\overset{LT}{\longleftrightarrow}\frac{dX(s)}{ds}$

ROC_x unchanged

Table of bilateral (two-sided) Laplace transforms

Signal $x(t)$	Laplace transform $X(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$1/s$	$\mathrm{Re}\{s\}>0$
$-u(-t)$	$1/s$	$\mathrm{Re}\{s\}<0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>-a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<-a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}\{s\}>-a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}\{s\}<-a$
$e^{-at}\sin(\omega_0 t)u(t)$ $\omega_0\ge 0 \quad a,\omega_0$ real	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$
$e^{-at}\cos(\omega_0 t)u(t)$ $\omega_0\ge 0 \quad a,\omega_0$ real	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$
$-e^{-at}\sin(\omega_0 t)u(-t)$ $\omega_0\ge 0 \quad a,\omega_0$ real	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}<-a$
$-e^{-at}\cos(\omega_0 t)u(-t)$ $\omega_0\ge 0 \quad a,\omega_0$ real	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}<-a$