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NOTE

A Parametrization of Digital Planes by Least-Squares Fits and Generalizations

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In this paper we prove that digital plane segments and their least-squares plane fit are in one-to-one correspondence, which gives a simple representation of a digital plane segment by its base description and coefficients of the least-squares plane fit. This leads to a constant space representation of digital rectangles in space. The method used is generalized and modified for constant space representation of sets which may consists of digital surface segments of different kinds.

1. INTRODUCTION

Consider a plane α in the three-dimensional Euclidean space with the equation $z = Ax + By + C$. The plane α will be digitized using the digitization scheme in which the first digital points (points with integer coordinates, often referred to as pixels) below a given plane are taken. Obviously, it is equivalent to translating the plane by -0.5 in the vertical direction and rounding.

So, the associated set of digital points for the plane α, called a digital plane, is defined as

$$P(\alpha) = \{(i, j, \lfloor Ai + Bj + C \rfloor), i \text{ and } j \text{ are integers}\},$$

where $\lfloor u \rfloor$ is the greatest integer not bigger than u. In general, we will be dealing with finite subsets of $P(\alpha)$, or more precisely, with digital plane segments which are obtained by digitizing parts of planes whose projections on the xy-plane are bounded regions, called bases of digital plane segments. So, if the plane α is digitized and if a region Q in the xy-plane is given, then the digital plane segment $P(\alpha, Q)$ (more precisely, digital plane segment with the base Q) is defined as

$$P(\alpha, Q) = \{(i, j, \lfloor Ai + Bj + C \rfloor), \\ (i, j) \in Q \text{ where } i \text{ and } j \text{ are integers}\}.$$

For practical reasons, at first the bases of digital plane segments are assumed to be rectangles. In this case, digital plane segments are called digital quadrangles. If

$$Q = Q(p, q, r, s) = \{(i, j), \\ p \le i < q, r \le j < s, i \text{ and } j \text{ are integers}\},$$

then digital plane segment $P(\alpha, Q)$ will be denoted by $P(\alpha, p, q, r, s)$. For convenience and without loss of generality, we can assume that p and r are equal to zero ($p = r = 0$), while q and s are integers, say m and n, respectively. Under the previous assumptions, the digital quadrangle $(P(\alpha, p, q, r, s) = P(\alpha, 0, m, 0, n)$ will be denoted by $P_{m,n}(\alpha)$. The major contribution of this paper is to give a constant space representation for digital plane segments which have the constant space representation of their bases. The representation consists of the base representation plus three coefficients of the least squares plane fit corresponding to the

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observed digital plane segment. For example, the representation of digital rectangles in space requires seven numbers.

The idea for using the least-squares fitting techniques for representations of digital objects was proposed in [4]. In [5], it was proved that the least-squares line fit uniquely determines the digital line on a segment. Therefore they obtained a new representation of digital line segments by four numbers. This representation is an alternative to the well known representation of digital lines by adjacent pairs [3] and to the one suggested in [2].

While constant space representations for digital lines exist in literature, currently no such representation is known for digital plane segments. Let us mention here that if a plane segment is digitized, then for a trivial representation of its digitization can be used: the plane equation and the base description. However, this trivial method has several deficiencies. For illustration, we give two of them:

—in real world image processing an equation of digitized surface is usually unknown. For example: for a "digital pyramid" can be assumed that its sides are digital plane segments, but the equations of the planes containing its original sides still remain unknown;

—there are infinitely many plane segments whose digitization (on the same base) gives the same digital plane segments. So an "one-to-one" mapping between digital plane segments and their representations can be useful.

In Section 3, it is proved that rectangular digital plane segments on the fixed base and their least-squares plane fit are in "one-to-one" correspondence. Determination of the least-squares fitting plane for a given set of points is a linear problem and so, it is easily solvable. Unfortunately, the determination of the least squares surface fit for some other surfaces is usually a nonlinear problem and consequently this is a problem of high computational complexity. In Sections 4 and 5, it is given another (not least-squares fitting) coding scheme which enables unique coding of digital surface segments from certain sets of digital surface segments. Those sets can be consisting of the digital surface segments which are digitizations of surface segments of different kinds. The conditions, which are proposed to be satisfied by the coded sets, are (in practice) not restrictions.

2. PRELIMINARIES

Suppose that we are given a finite set T of pixels in the three-dimensional Euclidian space (denoted as R^3). The least-squares plane fit for T is a plane which minimizes the sum of the squares of the vertical distances to all points in T. The method for determining such planes is well-known from statistics, e.g., [1]. If T is given by $\{(x_i, y_i, z_i), i = 1, 2, \ldots, t\}$ and if the equation of its least squares plane fit is $z = ax + by + c$, then the function $F(a, b, c) = \sum_{i=1}^{n}(ax_i + by_i + c - z_i)^2$ should be minimized.

By solving the equations $\partial F/\partial a = 0$, $\partial F/\partial b = 0$, $\partial F/\partial c = 0$, i.e.,

$$\begin{aligned} a\sum_{i=1}^{t} x_i^2 + b\sum_{i=1}^{t} x_i y_i + c\sum_{i=1}^{t} x_i &= \sum_{i=1}^{t} x_i z_i \\ a\sum_{i=1}^{t} x_i y_i + b\sum_{i=1}^{t} y_i^2 + c\sum_{i=1}^{t} y_i &= \sum_{i=1}^{t} y_i z_i \\ a\sum_{i=1}^{t} x_i + b\sum_{i=1}^{t} y_i + c\sum_{i=1}^{t} 1 &= \sum_{i=1}^{t} z_i \end{aligned} \tag{1}$$

one can obtain the coefficients a, b and c of the least-squares plane fit. The unique solution exists whenever the determinant of system (1) is nonzero (for example, if all pixels belong to a straight line then the solution is not unique). If T is a digital quadrangle $P_{m,n}(\alpha)$, then T consists of $m \cdot n$ digital points $(i, j, z(i, j))$, satisfying $0 \le i < m$, $0 \le j < n$ and $z(i, j) = \lfloor Ai + Bj + C \rfloor$. So, the coefficients of above system $\sum_{i=1}^{t} 1$, $\sum_{i=1}^{t} x_i$, $\sum_{i=1}^{t} y_i$, $\sum_{i=1}^{t} x_i^2$, $\sum_{i=1}^{t} y_i^2$, and $\sum_{i=1}^{t} x_i \cdot y_i$ can be easily determined. When these values are entered in the system of linear equations (1) this system becomes:

$$\begin{aligned} &\frac{nm(m-1)(2m-1)}{6} \cdot a + \frac{nm(n-1)(m-1)}{4} \cdot b + \frac{nm(m-1)}{2} \cdot c = \sum_{(i,j,z(i,j)) \in P_{m,n}(\alpha)} iz(i,j) \\ &\frac{nm(n-1)(m-1)}{4} \cdot a + \frac{nm(2n-1)(n-1)}{6} \cdot b + \frac{nm(n-1)}{2} \cdot c = \sum_{(i,j,z(i,j)) \in P_{m,n}(\alpha)} jz(i,j) \\ &\frac{nm(m-1)}{2} \cdot a + \frac{nm(n-1)}{2} \cdot b + nm \cdot c = \sum_{(i,j,z(i,j)) \in P_{m,n}(\alpha)} z(i,j). \end{aligned} \tag{2}$$

The previous system has the determinant equal to $(n^2 - 1)(m^2 - 1)n^3m^3/144$, and so, for $n > 1$ and $m > 1$ the coefficients a, b and c are uniquely determined. These coefficients a, b and c may be identified with the least squares plane fit $L_{m,n}(\alpha) : z = ax + by + c$ for a given digital quadrangle $P_{m,n}(\alpha)$.

As a consequence of the previous observation, we have the following lemma.

LEMMA 1. *If the digital quadrangle $P_{m,n}(\alpha)$ is given, then its least-squares plane fit $L_{m,n}(\alpha)$ is uniquely determined whenever $n > 1$ and $m > 1$.*

In the rest of the paper, the conditions $n > 1$ and $m > 1$ will not be mentioned but assumed.

3. ONE-TO-ONE CORRESPONDENCE BETWEEN LEAST-SQUARES PLANE FIT AND DIGITAL PLANES

In the previous section it is shown that if a digital quadrangle $P_{m,n}(\alpha)$ is given, then its least-squares plane fit, denoted as $L_{m,n}(\alpha)$, can be determined uniquely. A key question is whether there exist two different digital quadrangles, with the same corresponding least-squares plane fit. The answer is no, and this implies that the digital quadrangles and their least-squares plane fits are in one-to-one correspondence.

THEOREM 1. *Let $P_{m,n}(\alpha)$ and $P_{m,n}(\alpha')$ be two digital quadrangles and let $L_{m,n}(\alpha)$ and $L_{m,n}(\alpha')$ be their corresponding least-squares plane fits. Then $P_{m,n}(\alpha) = P_{m,n}(\alpha')$ $\Leftrightarrow L_{m,n}(\alpha) = L_{m,n}(\alpha')$.*

Proof. Let the planes α and α' be given by the equations $z = Ax + By + C$ and $z = A'x + B'y + C'$, respectively. Then their corresponding digital quadrangles are $P_{m,n}(\alpha) = \{(i, j, \lfloor Ai + Bj + C \rfloor), 0 \le i < m, 0 \le j < n\} = \{(i, j, z(i, j)), 0 \le i < m, 0 \le j < n\}$ and $P_{m,n}(\alpha') = \{(i, j, \lfloor A'i + B'j + C' \rfloor), 0 \le i < m, 0 \le i < m, 0 \le j < n\} = \{(i, j, z'(i, j)), 0 \le i < m, 0 \le j < n\}$. Also, let the planes $L_{m,n}(\alpha)$ and $L_{m,n}(\alpha')$ have the equations $z = ax + by + c$ and $z = a'x + b'y + c'$, respectively. The implication $P_{m,n}(\alpha) = P_{m,n}(\alpha') \Rightarrow L_{m,n}(\alpha) = L_{m,n}(\alpha')$ follows from Lemma 1.

The opposite direction will be proved by a contradiction.

Suppose that $P_{m,n}(\alpha)$ and $P_{m,n}(\alpha')$ are two different digital quadrangles with the same associated least squares plane fits $L_{m,n}(\alpha)$ and $L_{m,n}(\alpha')$, i.e., $a = a'$ and $b = b'$ and $c = c'$. Since a, b and c, as well as a', b' and c' are solutions of system (2), it follows that

$$\sum_{(i,j,z(i,j))\in P_{m,n}(\alpha)} z(i,j) = \sum_{(i,j,z'(i,j))\in P_{m,n}(\alpha')} z'(i,j) \tag{3}$$

$$\sum_{(i,j,z(i,j))\in P_{m,n}(\alpha)} iz(i,j) = \sum_{(i,j,z'(i,j))\in P_{m,n}(\alpha')} iz'(i,j) \tag{4}$$

$$\sum_{(i,j,z(i,j))\in P_{m,n}(\alpha)} jz(i,j) = \sum_{(i,j,z'(i,j))\in P_{m,n}(\alpha')} jz'(i,j) \tag{5}$$

Without loss of generality, we may assume that $z(i, j)$ and $z'(i, j)$ are strictly positive integers (for $0 \le i < m$ and $0 \le j < n$), otherwise, both planes α and α' can be translated by a positive integer in the vertical direction, until all z-coordinates become positive. This process will increase all quantities (3)–(5) for both planes by an equal amount.

Let E denote the set of all digital points (x, y, z) lying below the plane α and above the xy-plane, satisfying $0 \le x < m$ and $0 \le y < n$, while E' denotes the set of all digital points (x, y, z) lying below the plane α' and above xy-plane, also satisfying $0 \le x < m$ and $0 \le y < n$ (x, y and z are integers).

In order to make a contradiction we start by interpretations of equalities (3), (4), and (5).

—Equation (3) implies that the number of digital points belonging to E and the number of digital points belonging to E' is the same. Moreover, we can write that the cardinality of the set $E \setminus E'$ and the cardinality of the set $E' \setminus E$ is the same and different from zero, i.e., $\#(E \setminus E') = \#(E' \setminus E) \neq 0$(the inequality follows because $P_{m,n}(\alpha) \neq P_{m,n}(\alpha')$).

—Equation (4) implies that the sum of x-coordinates of all digital points from E, coincides with the sum of all digital points from E'. Namely, suppose that each digital point (x, y, z) receives "weight" x. The sum of weights of digital points from E is $\sum_{(x,y,z)\in E} x$ (there are $z(i, j)$ points with weight i, for fixed i and j). Similarly, the sum of weights of digital points from E' is equal to $\sum_{(x,y,z)\in E'} x$. Further, after removing common points from both sides, we have

$$\sum_{(x,y,z)\in E\setminus E'} x = \sum_{(x,y,z)\in E'\setminus E} x.$$

—Analogously to the previous item one can derive (from (5)),

$$\sum_{(x,y,z)\in E\setminus E'} y = \sum_{(x,y,z)\in E'\setminus E} y.$$

The projection of the intersection of the planes α and α' on the xy-plane is the straight line $(A' - A)x + (B' - B)y = C - C'$. This straight line separates the projections of the points from $E \setminus E'$ and $E' \setminus E$ on the xy-plane. Without loss of generality, suppose that the digital points from $E \setminus E'$ satisfy the inequality $(A' - A)x + (B' - B)y > C - C'$, while these from $E' \setminus E$, satisfy the opposite inequality $(A' - A)x + (B' - B)y < C - C'$. The contradiction $S > S$ can be made as follows:

$$\begin{aligned} S &= (A' - A) \cdot \sum_{(x,y,z)\in E\setminus E'} x + (B' - B) \cdot \sum_{(x,y,z)\in E\setminus E'} y \\ &= \sum_{(x,y,z)\in E\setminus E'} (A' - A)x + (B' - B)y \\ &> \sum_{(x,y,z)\in E\setminus E'} C - C' = (C - C') \cdot \#(E\setminus E') \\ &= (C - C') \cdot \#(E'\setminus E) = \sum_{(x,y,z)\in E'\setminus E} (C - C') \\ &> \sum_{(x,y,z)\in E'\setminus E} (A' - A)x + (B' - B)y = (A' - A) \\ &\cdot \sum_{(x,y,z)\in E'\setminus E} x + (B' - B) \cdot \sum_{(x,y,z)\in E'\setminus E} y = S. \quad \blacksquare \end{aligned}$$

4. CODING SCHEME FOR SETS CONTAINING DIGITAL SURFACE SEGMENTS OF DIFFERENT KINDS

In the previous section a constant space representation for digital quadrangles is described. If a part of a plane α :

$z = Ax + By + C$ laying above a rectangle in the xy-plane is digitized, then the obtained digital quadrangle $P_{m,n}(\alpha)$ is equal to $\{(i, j, \lfloor Ai + Bj + C \rfloor, 0 \leq i < m, 0 \leq j < n, i$ and j are integers$\}$ where the lines $x_1 = 0, x_2 = m - 1, y_1 = 0$ and $y_2 = n - 1$ determine (in the xy-plane) the mentioned rectangle. The equation of the least-squares plane fit $z = ax + by + c$ can be derived by solving the minimization problem

$$\min_{(a,b,c)\in R^3} F(a, b, c) = \min_{(a,b,c)\in R^3} \sum_{\substack{0\leq i<m \\ 0\leq j<n}} (ai + bj + c - \lfloor Ai + Bi + C \rfloor)^2, \tag{6}$$

or equivalently, the system of equalities $\partial F/\partial a = 0$ and $\partial F/\partial b = 0$ and $\partial F/\partial c = 0$. This problem is a linear problem and for given m, n and $\alpha : z = Ax + By + C$ it can be solved easily.

Theorem 1 proves that digital plane segments and their least-squares plane fits are in one-to-one correspondence. This result enables us to define a constant space representation (coding scheme) for digital plane segments. If the least-squares fitting technique is applied for a continuous surface $z = f(x, y)$, different from the plane segments, then problem (6) becomes

$$\min_{\bar{f}(x,y)} \sum_{\substack{0\leq i<m \\ 0\leq j<n}} (\bar{f}(i, j)) - \lfloor f(i, j) \rfloor)^2, \tag{7}$$

where $\bar{f}(x, y)$ is the required least-squares function fit for the function $z = f(x, y)$.

Unfortunately, problem (7) is usually a nonlinear problem which requires complex numerical techniques for the solution. So, the least square fitting technique for the representation of 3D-digital surfaces is not suitable whenever the problem is of high computational complexity. In the rest of this section, a constant space coding scheme for a set of digital surface segments, which may consists of digital surface segments of different kinds, will be described. The digital surface segment is defined as a result of transforming a continuous surface segment by a particular digitization process. The digitization method used is the one applied for digital plane segments.

If a surface ω is the graph of a continuous function $z = f(x, y)$ for $(x, y) \in Q$, then a digital surface segment ($S(\omega, Q)$, with the base Q, will be defined as

$$S(\omega, Q) = \{(i, j, \lfloor f(i, j) \rfloor), \quad \text{where } i \text{ and } j \text{ are integers and } (i, j) \in Q\}.$$

Let us mention that if a surface ω, which is being digitized, is not representable as the graph of some explicitly given function $z = f(x, y)$, then it should be separated into several subsurfaces. For example, a sphere $(x - a)^2 + (y - b)^2 = (z - c)^2$ will be separated into two surfaces, called half-spheres, $z = c + \sqrt{(x - a)^2 + (y - b)^2}$ and $z = c - \sqrt{(x - a)^2 + (y - b)^2}$.

As in the previous section, the region Q (base of the observed digital surface segment) is assumed to be a rectangle in the xy-plane with the sides parallel to the coordinate axes. $S(\omega, p, q, r, s)$ will denote a digital surface segment having $(q - p) \cdot (s - r)$ pixels and the base equal to $Q(p, q, r, s)$.

The family Λ of sets of digital surface segments for which our coding scheme can be applied is defined by the following two conditions:

(C1) A set $\mathscr{S}$ from the family Λ is a set containing only digitization of surfaces which are graphs of continuous function $z = f(x, y)$ on rectangular bases.

(C2) Let ω_1 and ω_2 be two continuous surfaces whose digitizations, taken on rectangular bases, belong to the set $\mathscr{S}$. If E_1 denotes the set of all digital points lying below the surface ω_1, while the set of all digital points lying below the surface ω_2 is denoted by E_2, then for the projections of $E_1 \backslash E_2$ and $E_2 \backslash E_1$ on the xy-plane there exists a conic curve $A + Bx + Cy + Dxy + Ex^2 + Fy^2 = 0$, which separates these projections in the $xy -$ plane.

It is easy to conclude that the family Λ is nonempty. The sets from the family Λ which are the most interesting for practical reasons are: the set of digital half-sphere segments, the set of digital paraboloid segments, the set of digital plane segments, as well as unions of these sets.

THEOREM 2. *Let a set $\mathscr{S}$ from the family Λ be given. Then any digital surface segment $S(\omega, p, q, r, s) = \{(i, j, z(i, j) = \lfloor f(i, j) \rfloor), p \leq i < q, y \leq j < s$, i and j are integers$\}$ from the set $\mathscr{S}$, where surface ω is given by $= f(x, y)$, can be coded uniquely by* 10 *integer parameters* $(p, q, r, s, a, b, c, d, e, f)$ *where:*

- *p, q, r and s are parameters which define the rectangular base of $S(\omega, p, q, r, s)$;*
- $a = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} z(i, j)$;
- $b = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} i \cdot z(i, j)$;
- $c = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} j \cdot z(i, j)$;
- $d = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} i \cdot j \cdot z(i, j)$;
- $e = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} i^2 \cdot z(i, j)$;
- $f = \sum_{(i,j,z(i,j))\in S(\omega,p,q,r,s)} j^2 \cdot z(i, j)$.

Proof. Let the continuous surfaces ω_1 and ω_2 be given by the equations $z = z_1(x, y)$ and $z = z_2(x, y)$, respectively. Also, let $S(\omega_1, p_1, q_1, r_1, s_1) = \{i, j, \lfloor z_1(i, j) \rfloor), p_1 \leq i < q_1, r_1 \leq j < s_1$, i and j are integers$\}$ and $S(\omega_2, p_2, q_2, r_2, s_2) = \{i, j, \lfloor z_2(i, j) \rfloor), p_2 \leq i < q_2, r_2 \leq j < s_2$, i and j are integers} be two digital surface segments from $\mathcal{S}$, with the proposed codes $(p_1, q_1, r_1, s_1, a_1, b_1, c_1, d_1, e_1, f_1)$ and $(p_2, q_2, r_2, s_2, a_2, b_2, c_2, d_2, e_2, f_2)$. Then, the implication: $S(\omega_1, p_1, q_1, r_1, s_1) = S(\omega_2, p_2, q_2, r_2, s_2) \Rightarrow (p_1, q_1, r_1, s_1, a_1, b_1, c_1, d_1, e_1, f_1) = (p_2, q_2, r_2, s_2, a_2, b_2, c_2, d_2, e_2, f_2)$ follows from the definitions, while the opposite direction will be proved by a contradiction.

Suppose that $S(\omega_1, p_1, q_1, r_1, s_1)$ and $S(\omega_2, p_2, q_2, r_2, s_2)$ are two different digital surface segments having the same codes, i.e., $(p_1, q_1, r_1, s_1, a_1, b_1, c_1, d_1, e_1, f_1) = (p_2, q_2, r_2, s_2, a_2, b_2, c_2, d_2, e_2, f_2)$.

Similarly, as in the proof of Theorem 1 we can assume that the integers $\lfloor z_1(i, j)\rfloor$ and $\lfloor z_2(i, j)\rfloor$ are strictly positive integers and let us denote the set of all digital points, laying below the surface ω_1 and above the xy-plane, whose projections on the xy-plane belong to $Q(p_1 = p_2, q_1 = q_2, r_1 = r_2, s_1 = s_2)$ by E_1 and the set of all digital points lying below the surface ω_2 and above its base $Q(p_1 = p_2, q_1 = q_2, r_1 = r_2, s_1 = s_2)$ by E_2. So, $E_1 = \{(i, j, k), p_1 \le i < q_1, r_1 \le j < s_1, 0 < k \le z_1(i, j), i, j, k$ are integers} and $E_2 = \{(i, j, k), p_2 \le i < q_2, r_2 \le j < s_2, 0 < k \le z_2(i, j), i, j, k$ are integers}. From the equalities of the proposed codes for digital surface segments $S(\omega_1, p_1, q_1, r_1, s_1)$ and $S(\omega_2, p_2, q_2, r_2, s_2)$ it follows (analogously as in the proof of Theorem 1) that

$$\#(E_1\backslash E_2) = \#(E_2\backslash E_1); \tag{8}$$
$$\textstyle\sum_{(i,j,k)\in E_1\backslash E_2} i = \sum_{(i,j,k)\in E_2\backslash E_1} i; \tag{9}$$
$$\textstyle\sum_{(i,j,k)\in E_1\backslash E_2} j = \sum_{(i,j,k)\in E_2\backslash E_1} j; \tag{10}$$
$$\textstyle\sum_{(i,j,k)\in E_1\backslash E_2} ij = \sum_{(i,j,k)\in E_2\backslash E_1} ij; \tag{11}$$
$$\textstyle\sum_{(i,j,k)\in E_1\backslash E_2} i^2 = \sum_{(i,j,k)\in E_2\backslash E_1} i^2; \tag{12}$$
$$\textstyle\sum_{(i,j,k)\in E_1\backslash E_2} j^2 = \sum_{(i,j,k)\in E_2\backslash E_1} j^2. \tag{13}$$

By the assumption that for the projection (on the xy-plane) of $E_1\backslash E_2$ and $E_2\backslash E_1$ there exists a separating conic curve l having the equation $A + Bx + Cy + Dxy + Ex^2 + Fy^2 = 0$ we can conclude a contradiction. Namely, say that the projection of the points from $E_1\backslash E_2$ satisfy the inequality: $A + Bx + Cy + Dxy + Ex^2 + Fy^2 > 0$ while these from $E_2\backslash E_1$, satisfy the opposite inequality $A + Bx + Cy + Dxy + Ex^2 + Fy^2 < 0$. Now, we have ((8)–(13))

$$\begin{aligned} S = {} & A \cdot \sum_{(i,j,k)\in E_1\backslash E_2} 1 + B \cdot \sum_{(i,j,k)\in E_1\backslash E_2} i + C \cdot \sum_{(i,j,k)\in E_1\backslash E_2} j \\ & + D \cdot \sum_{(i,j,k)\in E_1\backslash E_2} ij + E \cdot \sum_{(i,j,k)\in E_1\backslash E_2} i^2 + F \cdot \sum_{(i,j,k)\in E_1\backslash E_2} j^2 \\ & > 0 > A \cdot \sum_{(i,j,k)\in E_2\backslash E_1} 1 + B \cdot \sum_{(i,j,k)\in E_2\backslash E_1} i + C \cdot \sum_{(i,j,k)\in E_2\backslash E_1} j \\ & + D \cdot \sum_{(i,j,k)\in E_2\backslash E_1} ij + E \cdot \sum_{(i,j,k)\in E_2\backslash E_1} i^2 \\ & + F \cdot \sum_{(i,j,k)\in E_2\backslash E_1} j^2 = S, \end{aligned}$$

so, $S > S$ is the contradiction. ∎

The following three corollaries are consequences of the previous theorem.

COROLLARY 1. *The set $\mathcal{E}$ of digital half-sphere segments with rectangular bases can be coded uniquely by nine parameters.*

Proof. The set $\mathcal{E}$ is from the family Λ, since the intersection of two spheres is a circle whose projection on the xy-plane is, in general, an ellipse which is a conic curve. The general equation of an ellipse in the xy-plane is $A + Bx + Cy + Ex^2 + Fy^2 = 0$, which implies that the parameter d from the code proposed by Theorem 2 can be omitted. Conditions (C1) and (C2) are obviously satisfied. ∎

COROLLARY 2. *The set of digital parabolic segments $\mathcal{P}$ with rectangular bases can be coded uniquely by 10 parameters.*

Proof. Since, a paraboloid has the equation of the form $z = \alpha x^2 + \beta x + \gamma y^2 + \delta y + \rho xy + \eta$, the intersection of two paraboloids is a curve whose projection on xy-plane satisfies the equation $A + Bx + Cy + Dxy + Ex^2 + Fy^2 = 0$ the statement follows. ∎

Let $\mathcal{L}$ denote the set of digital plane segments with rectangular bases. Then the following holds:

COROLLARY 3. *The sets $\mathcal{U}_1 = \mathcal{L} \cup \mathcal{E}$ and $\mathcal{U}_2 = \mathcal{L} \cup \mathcal{P}$ are codable by 9 and 10 parameters, respectively.*

Proof. Analogously to the previous one. ∎

5. COMMENTS AND CONCLUSION

In this paper, the least-squares fitting technique is applied for digital plane segments. It is shown that digital plane segments and their least squares plane fits are in one-to-one correspondence if the finite base for digital plane segments is fixed. This result enables us to define the first known constant space representation for digital plane segments with finite bases which can be represented by a finite number of parameters: parameters for base representation plus three parameters which are the coefficients of the least squares plane fit. In detail the situation was studied where bases are assumed to be rectangles in the xy-plane. This can easily be extended to the cases when bases are some other shapes representable by constant numbers of parameters.

Applications of the least squares fitting technique for some other digital surface segments are usually unsuitable because of the high computational complexity. However, the technique used for proving Theorem 1 could be generalized and applied for obtaining a constant space coding scheme for certain types of digital surface segments. As a result, a constant space representation is obtained for types of digital surface segments which are independent of the kind of digitized surface. For types which can be coded by the proposed code, it was assumed to satisfy two conditions which are (in practice) not strong restrictions. For such a scheme, six integers are enough, plus base representations.

The method described here for coding sets of digital surface segments can be modified and generalized in several ways. For example:

—bases are not necessarily assumed to be rectangles;

—condition (C2) can be replaced by condition (C2′) as follows:

—(C2′) Let ω_1 and ω_2 be two continuous surfaces whose digitizations, taken on rectangular bases, belong to the set $\mathcal{S}'$. If E_1 denotes the set of all digital points lying below the surface ω_1 while the set of all digital points lying below the surface ω_2 is denoted by E_2, then for the projections of $E_1 \backslash E_2$ and $E_2 \backslash E_1$, on the xy-plane there exists a curve having an equation of the fixed form

$$\sum_{(u,v)\in D} A_{uv} x^{p(u)} y^{q(v)} = 0,$$

(where D is a finite set, while $p(u)$ and $q(v)$ are not necessarily integers), which separates these projections in the xy-plane.

If the family of sets of digital surface segments satisfying conditions (C1) and (C2′) is denoted by Λ', then the following theorem holds.

THEOREM 3. *Let a set $\mathcal{S}'$ from the family Λ' be given, then any digital surface segment $S'(\omega, p, q, r, s) = \{(i, j, z(i, j) = \lfloor f(i, j) \rfloor), p \le i < q, r \le j < s$, i and j are integers$\}$ from the set $\mathcal{S}'$, where the surface ω is given by $z = f(x, y)$, can be coded uniquely by a constant number of parametrs.*

Proof. The required number of parameters is equal to $4 + \#D$. The first four parameters are used for base description, while the rest of the parameters are defined as

$$a_{uv} = \sum_{(i,j,z(i,j))} i^{p(u)} j^{q(v)} \text{ for } (u, v) \in D.$$

The proof of uniqueness of the proposed code is analogous to the one in Theorem 2. ■

We conclude the paper with a corollary which illustrates some possible applications of Theorem 3. Namely, in practice, the most often appearing digital surfaces are the surfaces of the second order. They are spheres, paraboloids, hyperboloids, ellipsoids and cones. They all can be determined by an equation of the form $Az^2 + Bz = Cx^2 + Dy^2 + Exy + Fx + Gy + H$. The intersection curve of any two of such surfaces has a projection on the xy-plane given by the equation

$$\sum_{\substack{0\le u\le 4\\ 0\le v\le 4}} A_{uv} x^u y^v = 0.$$

Let us note that in the terms used in condition (C2′) the set D is $D = \{(u, v), 0 \le u \le 4, 0 \le v \le 4\}$, where u and v are integers, $p(u) = u$, $t(v) = v$ and $\#D = 25$. Since conditions (C1) and (C2′) are satisfied, we have:

THEOREM 4. *Let $\mathcal{O}$ be a set which consists of digitized segments of surfaces of the second order. Then $\mathcal{O}$ can be coded uniquely by* 29 *parameters.*

Proof. Let a digital surface segment $S(\omega, p, q, r, s) = \{(i, j, z(i, j) = \lfloor f(i, j) \rfloor), p \le i < q, r \le j < s$, i and j are integers$\}$ belong to the set $\mathcal{O}$ and let ω be given by the equation $z = f(x, y)$. Then 29 parameters are enough for unique coding of $S'(\omega, p, q, r, s)$. Four parameters are necessary for the base descriptions and 25 parameters are defined as

$$a_{uv} = \sum_{(i,j,\lfloor f(i,j) \rfloor)\in S'(\omega,p,q,r,s)} i^u j^v,$$

where $(u, v) \in D$. ■

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