# GENERATING AND COUNTING HEXAGONAL SYSTEMS

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## ABSTRACT

An algorithm for generating and counting nonisomorphic hexagonal systems of the given perimeter is presented which enables us to determine the number of nonisomorphic hexagonal systems of the perimeter n up to $n = 46$. We also determine the number of nonisomorphic hexagonal systems with h hexagons for $h \leq 11$.

## 1. DEFINITION AND CHARACTERIZATION

In [1] a characterization of hexagonal systems is given using the words over the alphabet $V = \{0,1,2,3,4,5\}$, and a function f which maps the set of all the finite oriented paths of the hexagonal grid H (Fig. 1) into the set $V^* = \bigcup_{n\geq 0} V^n$ of all the words over the alphabet V. It is known that the Euclidean plane can be tiled with the regular congruent hexagons. In this way an infinite hexagons grid H is obtained (Fig. 1). This grid is a plane realization of an infinite planar cubic graph *H* with all faces bounded by exactly six edges. Another plane realization of the grid *H* is given in Fig. 2. A hexagonal system is defined as a part of the grid H consisting of all vertices and edges belonging to some circuit C or to its interior. It means that a hexagonal system is a hex-mino in the sence of Lunnon ([4]) but without holes. The following theorem was proved which gives a sufficient and necessary condition under which the equality

$$p = f(P)$$

concerning an arbitrary word $p = x_1 x_2 \dots x_n \in V^n$, $n \geq 6$, and a closed oriented path P without repeated vertices in the hexagonal tesselation H is valid:

THEOREM. *The equation*

$$p = f(P)$$

*is satisfied if and only if each of the following three relationships hold:*

*(i)* $\forall\ k \in \{1,2,\dots,n-1\}$ $x_{k+1} = x_k \pm 1 \pmod 6$ *where we operate with the elements of the set* V *as with integers,* $n \geq 6$.

*(ii)* $\ell_0(p) = \ell_3(p) \wedge \ell_1(p) = \ell_4(p) \wedge \ell_2(p) = \ell_5(p)$.

*(iii)* *For any nonempty subword* r *of* p *diferent from* p *it holds that:*

$$\ell_0(r) \neq \ell_3(r) \vee \ell_1(r) \neq \ell_4(r) \vee \ell_2(r) \neq \ell_5(r),$$

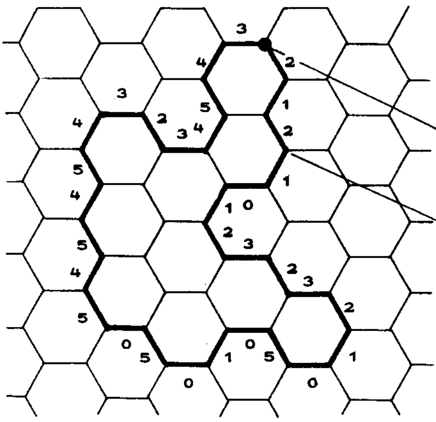
*where* $\ell_j(p)$ *is the number of appearances of the letter* j *in the string* $p \in V^* = \bigcup_{n \geq 0} V^n$.

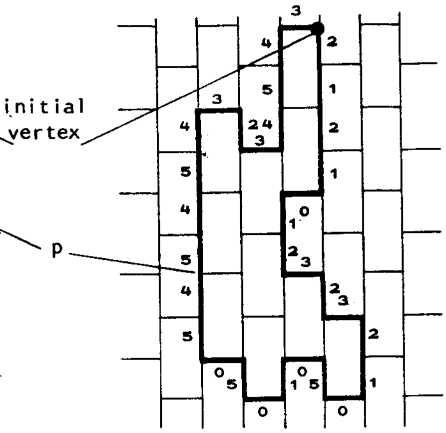
PROOF. Here we give a new proof of the theorem, different from that given in [1].

Necessity.

(i) Follows from the properties of the hexagonal grid.

(ii) In order to prove that $\ell_0(p) = \ell_3(p)$ we consider the grid H´ represented in Fig. 2, equivalent to the grid H (The corresponding graphs are isomorphic). The statement follows from the fact that the sum of vectors lying on the orien-





$f(P) = p = 34543234545450501050123232101212$

Figure 1.

Figure 2.

ted circuit (boundary of the considered hexagonal system) equals 0, and that the same is true for the sum of their ortogonal projections onto a line paralel to the vector 0. Similarly, it can be proved that

$$\ell_1(p) = \ell_4(p) \text{ and } \ell_2(p) = \ell_5(p).$$

(iii) Follows from the fact that there is no repetition of vertices in P.

*Sufficiency.*

Bearing in mind the properties of the grid H, we conclude that for any word satisfying conditions (i), (ii) and (iii), a path P can be constructed such that p = f(P).

## 2. ISOMORPHISM OF HEXAGONAL SYSTEMS

Using the theorem we can determine for any word p over the alphabet $V = \{0,1,2,3,4,5\}$ whether it represents a hexagonal system (in fact the boundary of a hexagonal system) or not.

But, the same hexagonal system can be represented by different words. Now, we shall define an equivalence relation $\rho$ in the set $V_1^* \subset V^*$ of words satisfying conditions (i), (ii), (iii) of the Theorem, in such a way that all the words from the same equivalence class determine the same hexagonal system up to isomorphism, while the elements from different equivalence classes determine nonisomorphic hexagonal systems.

We use the following notations and definitions:

$S$ is the set of all hexagonal systems,

$\pi$ is the congruence relation in the set $S$,

$\alpha$, $\beta$, $\sigma$: $V_1^* \to V_1^*$ are the following functions:

$$\sigma(x_1x_2...,x_n) = \sigma(x_1)\ (x_2)...\sigma(x_n),$$

where

$\sigma: V \to V$ is an arbitrary element of the permutation group generated by

$$\sigma_a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 0 & 5 & 4 \end{pmatrix}$$

$$\alpha(x_1x_2...x_n) = x_2x_3...x_nx_1, \quad \beta(x_1x_2...x_n) = x_nx_{n-1}...x_2x_1.$$

$\rho$ is the binary relation in the set $V_1^*$ defined as follows:

$$\forall\ p,q \in V_1^* \quad p\ \rho\ q \Leftrightarrow q = (\alpha^k\sigma)(p) \vee q=(\alpha^k\sigma\beta)(p)$$

for some function $\sigma$ and for some nonnegative integer $k$.

The function $f$ in the natural way induces the bijections

$$F : S/\pi \to V_1^*/\rho$$

where $S/\pi$ is the set of all nonisomorphic hexagonal systems, and $V_1^*/\rho$ is the set of all nonequivalent words satisfying conditions (i), (ii) and (iii).

## 3. THE ALGORITHM AND COMPUTATIONAL RESULTS

From the given characterization, it follows that all nonisomorphic hexagonal systems can be obtained by generating all the nonequivalent words which satisfy conditions (i), (ii) and (iii).

We give an algorithm for determination of all nonequivalent words with length m where $m = 1,2,\ldots,n$. The obtained nonequivalent words we divide into two classes: symmetric and nonsymmetric words. A given word p is symmetric if $p = (\alpha^k\sigma\beta)(p)$ for some $\sigma$ and k. Let $S_m$ and $\bar{S}_m$ denote the number of nonequivalent symmetric and nonsymmetric words of the length m respectively. From $\sigma_b|\sigma_a| = \sigma_a^5|\sigma_b|$ and $\sigma_b^2 = e$ it follows that the element $\sigma$ of permutation group generated by $\sigma_a$ and $\sigma_b$ is in the set $G = \{e,\sigma_a,\sigma_a^2,\sigma_a^3,\sigma_a^4,\sigma_a^5,\sigma_b,\sigma_a\sigma_b,\sigma_a^2\sigma_b,\sigma_a^4\sigma_b,\sigma_a^5\sigma_b\}$.

For each obtained nonequivalent word we may determine the number h of hexagons of the corresponding hexagonal system. We consider coordinate system with unit vectors 0 and 2. Coordinates of elements 0,1,2,3,4,5 in this system are:

$$d^1(0) = 1,\ d^1(1) = 1,\ d^1(2) = 0,\ d^1(3) = -1,$$
$$d^1(4) = -1,\ d^1(5) = 0,\ d^2(0) = 0,\ d^2(1) = 1,$$
$$d^2(2) = 1,\ d^2(3) = 0,\ d^2(4) = -1,\ d^2(5) = -1.$$

The number of hexagonsh for the word $x_1x_2\ldots x_m$ we may obtain by following procedure:

*Step 1.* Let $h = 0$, $u_2 = d^1(x_1)$, $v_2 = d^2(x_1)$.

*Step 2.* For each i, $2 \le i \le m-1$, let $u_1 = u_2$,

$$v_1 = v_2,\ u_2 = u_2 + d^1(x_i),\ \ v_2 = v_2 + d^2(x_i),$$

$$h = h + u_1 v_2 - u_2 v_1.$$

*Step 3.* Let $h = |h/6|$.

By $S_{m,i}$ ($\bar{S}_{m,i}$) we denote the number of nonequivalent symmetric (nonsymmetric) words of the length m with i hexagons of the corresponding hexagonal system.

By $p_{j,m}$ we denote the word $x_j x_{j+1} \ldots x_m$, $1 \leq j \leq m$. Now, we can describe the algorithm in the following way.

*Step 1.* Read n. Let $S_i=0, \bar{S}_i=0, S_{i,j}=0, \bar{S}_{i,j}=0$, for $1 \leq i,j \leq n$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$ and $m = 4$.

*Step 2.* $m = m + 1$, $x_m = x_{m-1} + 1 \pmod 6$.

*Step 3.* If $\ell_0(p_{j,m}) = \ell_3(p_{j,m}) \wedge \ell_1(p_{j,m}) = \ell_4(p_{j,m}) \wedge \ell_2(p_{j,m}) = {}_5(p_{j,m})$ for some $j > 1$ then go to step 4. If this condition is satisfied only for $j = 1$ then go to step 5. Otherwise, (i.e. if this condition is not satisfied for each j, $1 \leq j \leq m$) if $m > n$ then go to step 2 else (i.e. if $m = n$) go to step 4.

*Step 4.* If $x_m - x_{m-1} = 1 \pmod 6$ then put $x_m = x_{m-1} + 5 \pmod 6$ and go to step 3. In opposite case, if $m = 5$ then write $S_i$, $\bar{S}_i$, $S_{i,j}$ and $\bar{S}_{i,j}$ for $1 \leq i,j \leq n$ and finite the program. If $m > 5$ then put $m = m - 1$ and go to step 4 again.

*Step 5.* The word $p = x_1 x_2 \ldots x_m$ represents a hexagonal system with perimeter $m \leq n$. But, some of obtained words are equivalent. All words, which are equivalent to the considered word p, define the same class in $V_1^*/\rho$. For given word p, equivalent words are: $(\alpha^k\sigma)(p)$ and $(\alpha^k\sigma\beta)(p)$, where $\sigma \in G$, $0 \le k \le m$.

We consider only the equivalent words beginning by 0121 (because each class of equivalent words of the length $n > 6$ contains a word with prefix 0121) and sort them by using the lexicographic order. We choose the first word p′ in this lexicographic order as a represent of this class. Hence, if considered word p is equal to represent p′ of the corresponding class of words then p represents a nonisomorphic hexagonal system. Therefore, we write p in this case and if p is symmetric word then put $S_m = S_m + 1$, $S_{m,h} = S_{m,h} + 1$ and in the opposite case put $S_m = \bar{S}_m + 1$ and $\bar{S}_{m,h} = \bar{S}_{m,h} + 1$. In both of cases (p represents and p not represents a nonisomorphic hexagonal system) go to step 4 after checking in step 5.

The presented algorithm is based on lexicographic order of words $x_1x_2\ldots x_m$ $(4 \le m \le n)$. We show the trace of the program for $n = 8$. We not obtain any hexagonal system, because the word 012345 representing a hexagonal system is omitted.

correcto

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Step      1:  0121
Step      2:  01212
Steps   3,2:  01213
Steps   3,2:  012134
Steps   3,2:  0121345
Steps   3,4:  01212343
Steps   3,4:  0121232
Steps   3,2:  01212323
Steps   3.4:  01212321
Steps   3,4:  012121
Steps   3,2:  0121212
Steps   3,2:  01212123
Steps   3,4:  01212121
Steps   3,4:  0121210
Steps   3,2:  01212101
Steps   3,4:  01212105
Steps   3,4:  01212
Steps   3,4:  012101
Steps   3,2:  0121012
Steps   3,2:  01210123
Steps   3,4:  01210121
Steps   3,4:  0121010
Steps   3,2:  01210101
Steps   3,4:  01210105
Steps   3,4:  012105
Steps   3,2:  0121050
Steps   3,2:  01210501
Steps   3,4:  01210505
Steps   3,4:  0121054
Steps   3,2:  01210545
Steps   3,4:  01210543
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We denote by $S_n$ ($S_h$) the number of nonisomorphic hexagonal systems of the perimeter n (with h hexagons). The number $S_h$ are obtained using the fact that $n \le 4h + 2$. The numerical results are given in Table 1 and Table 2. In the Table 3 is the number $S_{n,h}$ of all nonisomorphic hexagonal systems with perimeter n and h hexagons in their interior.

| $n$ | $S_n$ | $n$ | $S_n$ | $n$ | $S_n$ |
|---|---|---|---|---|---|
| 6 | 1 | 20 | 14 | 34 | 18714 |
| 8 | 0 | 22 | 50 | 36 | 53793 |
| 10 | 1 | 24 | 97 | 38 | 162565 |
| 12 | 1 | 26 | 312 | 40 | 482416 |
| 14 | 3 | 28 | 744 | 42 | 1467094 |
| 16 | 2 | 30 | 2291 | 44 | 4436536 |
| 18 | 12 | 32 | 6186 | 46 | 13594266 |

Table 1.

| $h$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_h$ | 1 | 1 | 3 | 7 | 22 | 81 | 331 | 1435 | 6505 | 30086 | 141229 |

Table 2.

| h \ n | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | $S_h$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 4 | 0 | 0 | 0 | 0 | 1 | 1 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 6 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 22 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 4 | 14 | 24 | 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 81 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 10 | 25 | 68 | 106 | 118 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 331 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 9 | 21 | 67 | 144 | 329 | 453 | 411 | 0 | 0 | 0 | 0 | 0 | 0 | 1435 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 15 | 55 | 154 | 396 | 825 | 1601 | 1966 | 1489 | 0 | 0 | 0 | 0 | 6505 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 9 | 42 | 123 | 416 | 1018 | 2340 | 4518 | 7652 | 8395 | 5572 | 0 | 0 | 30086 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 26 | 100 | 333 | 1008 | 2811 | 6405 | 13415 | 24020 | 36109 | 35885 | 21115 | 141229 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 13 | 61 | 279 | 878 | 2713 | 7375 | 18306 | 38727 | 74985 | 124119 | 168318 | |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 35 | 187 | 708 | 2459 | 7253 | 20119 | 50405 | 114326 | 226837 | 408785 | |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 16 | 120 | 533 | 2001 | 6594 | 20321 | 55124 | 141268 | 328592 | 691933 | |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 70 | 347 | 1570 | 5612 | 18396 | 55857 | 155656 | 395860 | 947291 | |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 30 | 223 | 1121 | 4501 | 16234 | 51691 | 156434 | 436698 | 1132642 | |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 117 | 763 | 3414 | 13286 | 46166 | 148430 | 440491 | 1236839 | |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 53 | 471 | 2437 | 10587 | 39143 | 133713 | 424429 | 1262442 | |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 18 | 256 | 1647 | 7885 | 32042 | 116648 | 388180 | 1223950 | |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 129 | 1009 | 5726 | 25050 | 97607 | 345834 | 1140529 | |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 47 | 587 | 3838 | 18876 | 79472 | 296025 | 1029521 | |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 290 | 2467 | 13652 | 62027 | 247989 | 903415 | |
| 23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 126 | 1448 | 9349 | 47167 | 200545 | 771061 | |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 43 | 789 | 6124 | 34324 | 157989 | 643859 | |
| 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 13 | 377 | 3735 | 24118 | 120524 | 522218 | |
| 26 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 156 | 2146 | 16114 | 89288 | 414305 | |
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 53 | 1105 | 10313 | 63732 | 319258 | |
| 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 509 | 6214 | 43833 | 240218 | |
| 29 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 207 | 3456 | 29029 | 174603 | |
| 30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 68 | 1799 | 18198 | 123790 | |
| 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 20 | 825 | 10886 | 84207 | |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 344 | 6042 | 55451 | |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 121 | 3129 | 34739 | |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 39 | 1461 | 20811 | |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 616 | 11719 | |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 228 | 6155 | |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 72 | 2979 | |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 20 | 1324 | |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 525 | |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 186 | |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 57 | |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| $S_n$ | 1 | 0 | 1 | 1 | 3 | 2 | 12 | 14 | 50 | 97 | 312 | 744 | 2291 | 6186 | 18714 | 53793 | 162565 | 482416 | 1467094 | 4436536 | 13594266 | |

Table 3.

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