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IMMOBILIZING A SHAPE

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ABSTRACT

Let shape P be any simply-connected set in the plane, bounded by a Jordan curve, that is not a circular disk. We say that a set of points I on the boundary of P immobilize the shape if any rigid motion of P in the plane causes at least one point of I to penetrate the interior of P. We prove that four points always suffice to immobilize any shape. For a large class of shapes, which includes polygons without parallel edges, three points are sufficient to immobilize. An $O(n \log n)$ algorithm is given that finds a set 3 points that immobilize a given polygon without parallel edges. The algorithm becomes linear for convex polygons. Some results are generalized for d-dimensional polytopes, where $2d$ points are always sufficient and sometimes necessary to immobilize.

Keywords: Immobilization, polygons, Jordan curve, polytopes, algorithms.

1. Introduction

A set points I is said to immobilize a planar shape P if any rigid motion of P in the plane forces at least one point of I to penetrate the interior of P. Any minimal I contains only points belonging to the boundary of P. The disk is excluded from consideration since any number of points on its boundary leave it free to rotate. By a *shape* we mean a set bounded by a Jordan curve; a Jordan curve is a homeomorphic image of a circle—a continuous curve without self-intersections that separates the (nonempty) interior and the exterior regions of the corresponding shape.

Problems of immobilization of planar shapes were introduced by W. Kuperberg[8] and later reported by O'Rourke[14] where a number of open questions were presented:

- Do four points always suffice to immobilize any shape? Any convex shape?

- Find all the classes of convex shapes for which three points do not suffice.
- Do three points suffice for all smooth convex shapes?
- Design an algorithm finding a set of points immobilizing a given polygon.
- Extend to three (and higher) dimensions.

A partial answer to the first of these questions can be obtained using some results from grasping. For shapes P with smooth boundary, except on some finite number of points, Mishra, Schwartz and Sharir[9] and independently Markenscoff, Ni and Papadimitriou[9] studied the problem of closure grasp, i.e. ability to respond to any external force torque by applying appropriate forces at the grasp points. They proved that there exists so-called force-torque closure grasp using a minimal set S of four finger points. From a discussion by Mishra and Silver[12] it follows that any rigid velocity of P causes at least one of the points of S to have an instantaneous velocity strictly directed towards the interior of P. In consequence S immobilizes P. It was also proved[11] that the set S may be found in $O(n)$ time. More recently, Montejano and Urrutia,[13] using methods from differential geometry, proved that any smooth shape may be immobilized using three points. Finally, Czyzowicz, Stojmenovic and Szymacha[3] gave a linear-time algorithm that checks whether n given points immobilize a given polygon. The ideas of using the inscribed circle and Voronoi diagram, exploited in this paper, were first used by Baker, Fortune and Grosse,[2] while the idea of normals to the boundary of a triangle meeting at a point is used by Markenscoff and Papadimitriou,[10] all in the context of an equilibrium grip. The problem of immobilization, which from all known variations of grasping, is studied here for any shape bounded by a Jordan curve.

A rigid motion of a shape P on the plane is a mapping M from the set $t \times P$ (t represents time) to the plane, continuous with respect to its first coordinate, such that for every pair of points u, $v \in P$ the distance between their images remains constant for all t and $M(0, u) = u$ for every element of P. A set of points I immobilizes the shape P if the only motion of P which does not allow the penetration of some element of I to the interior of P is the identity $M(t, u) = u$ for all t and u.

In Section 2 we study the problem of immobilization of a polygon. We first investigate immobilization of a triangle and then extend considerations to immobilizing convex and simple polygons. We prove that any polygon in the plane without parallel edges can always be immobilized using three points. We give a family of polygons that require four points to immobilize; any such polygon must always contain two parallel edges. This class includes polygons other than parallelograms[8] Given convex polygon P and three points on its boundary we give a criterion to check whether the points immobilize P. When the points are not located at the vertices of the polygon we have a similar criterion for the class of simple polygons. An $O(n \log n)$ algorithm is obtained to find a set of three points that immobilize a polygon without parallel edges. In the case of a convex polygon, our algorithm works in $O(n)$ time. In both cases, algorithms computing Voronoi diagrams for line segments are used.[6,7,1]

In Section 3 we deal with arbitrary shapes, i.e. all we assume here is that their boundaries are Jordan curves. We prove that four points suffice to immobilize any

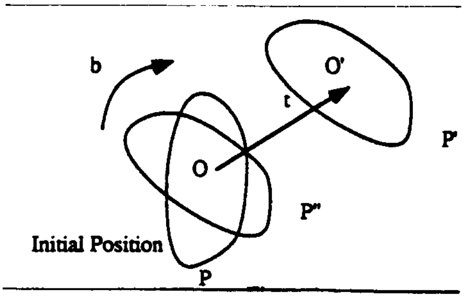


Fig. 1. Translation of a shape.

shape P. Our proof contains two cases depending on the position of the largest circle S inscribed in P. In the first case, when the center of S belongs to the interior of the convex hull of $S \cap P$, three points are sufficient to immobilize P. In the other, more difficult case, four points may be needed.

In Section 4 some results are extended to higher dimensions. We prove that any d-dimensional polytope P can be immobilized by a set containing at most $2d$ points. Moreover, if some set of $d+1$ vectors normal to faces of P is linearly independent, $d+1$ points suffice to immobilize P. From our proof it follows that for any number n between d and $2d$ there is a d-dimensional polytope P that require n points to immobilize it.

2. Immobizing a Polygon

2.1. Immobilizing a Triangle

In this section we study the problem of immobilizing triangles. We first prove a lemma that is valid for arbitrary shapes and that will be useful later on to obtain other results.

The following will prove useful:

Observation 1: *Any movement of a shape P which, after the time t_0, has moved from an initial position P to a new position $P' = M(t_0, P)$ can be rerouted using the following two actions:*

-a rotation $\beta(t_0, O, P)$ of P around any point O in the plane by an appropriate angle β, to obtain the interim position P''

-a translation $\tau(t_0, O, P'')$ of P'' to the destination position P' by a vector τ (τ is equal to the vector OO', where O' is the new posiiton of O). (See Fig. 1.)

The $M(t_0, P) = \tau(t_0, O, \beta(t_0, O, P))$. Since the movement is continuous, it is easy to see that both τ and β must be continuous functions of their first parameter. We may then assume that for any arbitrarily small value $\mu > 0$ we can always choose a time moment $t_0 > 0$ such that for $t_0 > t > 0$, the positions of $M(t, P)$ are such that corresponding values of β and $/\tau/$ are both less than μ.

We will use the following method to build a set of points I that immobilize P. Each point W from I belonging to the boundary of P restricts the movement of

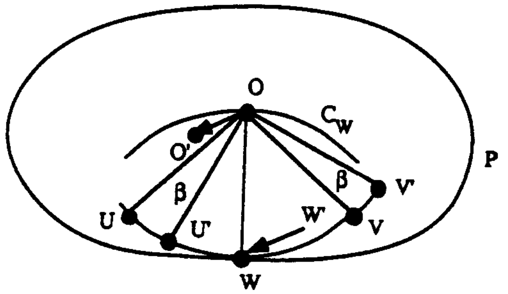


Fig. 2. Penetrating a circle sector.

P to avoid W penetrating P. If P' is the position of P after an arbitrarily small movement, the movement can be rerouted as indicated and W should not become an interior point of P'. Observe, however, that although positions P and P' of the shape disallow penetration, it is possible that W penetrates the intermediate position P''.

Consider a circle C with center O. Given two points U and V on C, let OUV be the region of the plane bounded by the line segments joining O to U and V and the arc UV of C in the counterclockwise direction from U to V. The region OUV will be called a circle sector of C.

Lemma 1 *Let P be a shape, OUV a circle sector of a circle C such that O is in the interior of P, and W be an interior point of the arc UV that is also a point on the boundary of P (see Fig. 2). If the circle sector OUV lies entirely inside P, then there is an $\epsilon > 0$ such that for any $0 < t < \epsilon$ any movement of $M(t, P)$ that moves O to a point O' inside the circle C_W centered at W with radius WO causes W to penetrate P.*

Proof. Suppose that $M(t, P)$ moves O to a point O' where O' is in the neighborhood of O and inside C_W. We can choose O' such that after rerouting the movement of P as indicated in Observation 1, with O being the center of rotation, both β and $/\tau/$ will be arbitrarily small. The ratoation by a small angle β moves arc OUV to a new arc $OU'V'$ (see Fig. 2) such that W is still an interior point of the new arc $U'V'$ and $OU'V$ (or alternatively OUV') is still entirely inside P. The translation by τ moves O to O'. However, it is easy to note that the same translation moves a point W' from the interior of the circular sector $OU'V'$ to the point W. This results in W penetrating P. □

Theorem 1 *Three points X, Y and Z immobilize a triangle T with vertices A, B and C if and only if the three orthogonal lines to the boundary of T at the points X, Y and Z are concurrent.*

Proof. We prove first the necessity of the condition. Clearly each of X, Y and Z must lie on different sides of T. Let X, Y and Z belong to sides BC, AC, and AB, respectively. Suppose that the three orthogonals at the points X, Y and Z do not meet at a single point. Let O be a point in the interior of the triangle determined by these orthogonals. Then the three OXB, OYC and OZA are all acute (or all

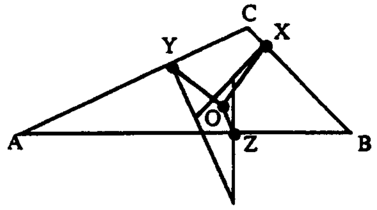


Fig. 3. Rotating without penetration.

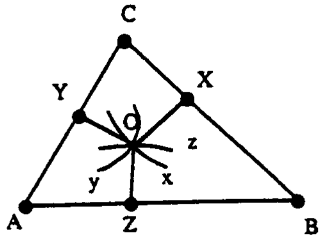


Fig. 4. Immobilizing a triangle.

obtuse) (see Fig. 3). Therefore the triangle T may be rotated counterclockwise (or clockwise) around O by an $\varepsilon > 0$ angle and the points X, Y and Z will remain outside the interior of T.

Suppose now that the three orthogonals intersect at a point O. To prove that X, Y and Z immobilize T, we show that any movement of T will force one of X, Y or Z to penetrate the interior of the image of T. We first consider the case when O is *inside* T (see Fig. 4).

We show that the point O cannot move anywhere from its initial position. Suppose that O moves to a point O' within a distance $\varepsilon > 0$ from O. Let C_X, C_Y and C_Z be circles containing O on their boundaries and centered at X, Y and Z respectively. The conditions of Lemma 1 are satisfied and O cannot move inside any of the circles x, y and z. But it is easy to note that any point from the neighborhood of O is inside at least one of the circles C_X, C_Y or C_Z; thus if $O' \neq O$ then at least one of the points X, Y or Z will penetrate T. Therefore $O = O'$ and the only allowable movement for T is a rotation around O. This is however impossible because in this case some interior points of T will move to X, Y and Z (causing the penetration of X, Y and Z).

Consider now the case when O is outside T. Suppose without loss of generality that the straight line passing through A and B separates O from C (see Fig. 5(a)).

Then OZ is the shortest of the segments OX, OY and OZ and OZ is completely outside T while OY and OX intersect T. Suppose that T can move to a new position T'. We reroute the movement by a rotation around O by an appropriate angle β and translation by corresponding vector τ. Let $T'' = A''B''C''$ be the rotated position of T. It is easy to show that X and Y are in the interior of T'' while Z is in the

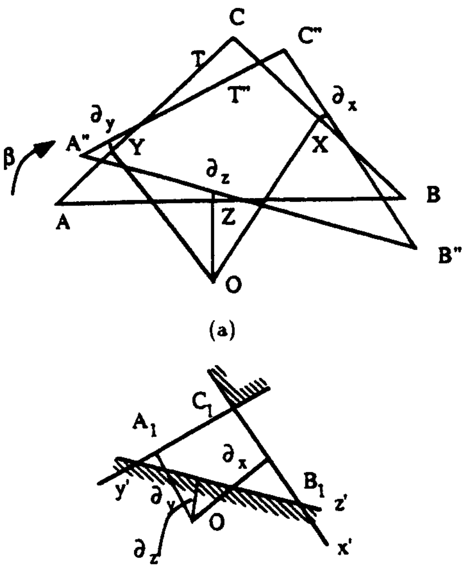


Fig. 5. Orthogonals intersect outside triangle.

exterior of T''. Therefore the translation of T'' to destination T', should be such that X and Y "escape" from T' while Z stays outside T'. Let ∂_x, ∂_y and ∂_z be the distances of X, Y and Z from $B''C'', A''C''$ and $A''B''$, respectively. Then $\partial_z < \partial_x$ and $\partial_z < \partial_y$ since OZ is the shortest of OZ, OY and OX ($\partial_x = /OX/(1-\cos\beta)$, and similarly for the other two distances). Let x', y' and z' be straight lines parallel to $B''C'', A''C''$ and $A''B''$ with distances ∂_x, ∂_y and ∂_z from O, respectively, such that O does not lie between any two corresponding parallel lines (see Fig. 5(b)). Observe that triangle $A_1B_1C_1$ in Fig. 5(b) is similar to ABC. X and Y can "escape" from T'' only if the translation vector τ brings O to a point O' lying in the half-planes determined by x' and y' that do not contain O. To keep Z outside T'', point O' must be located within the half-plane determined by z' containing O. However, as $\partial_z < \partial_x$ and $\partial_z < \partial_y$ the three mentioned half-planes have empty intersection.

The above argument holds also for the case when O is on the boundary of T.

□

Corollary 1 *Given two points X and Y on two different sides of a triangle T, it might not be possible to find a third point Z on the remaining side of T such that X, Y and Z immobilize T (see Fig. 6). This happens only for obtuse T.*

2.2. Immobilizing a Convex Polygon

We are now ready to give necessary and sufficient conditions under which three points immobilize a convex polygon.

Given a convex polygon P we say that three of its sides x, y and z *enclose* P if the triangle $T(z, y, x)$ determined by the three lines containing them contains P.

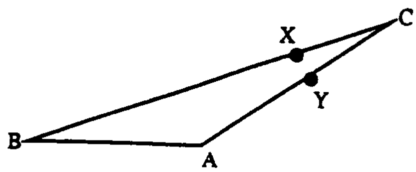


Fig. 6. No third point to immobilize.

Theorem 2 *A convex polygon P can be immobilized by three points X, Y and Z if and only if:*

- *each of them belongs to the interior of a different side, say x, y and z of P respectively such that x, y and z enclose P, and*
- *the orthogonals to x, y and z at the points X, Y and Z respectively meet at a common point.*

Proof. We prove first that none of X, Y and Z is a vertex of P. Suppose then that X is a vertex of P and that *any* triangle that containts X, Y and Z on its boundary contains P. Then it is easy to see that we can take a triangle T with sides a, b and c that contains P such that X, Y and Z are interior points of a, b and c respectively. Moreover, T can be chosen such that the orthogonals to a, b and c at X, Y and Z respectively do not meet. Then X, Y and Z do not immobilize T and therefore do not immobilize P (see Fig. 7(a)).

Suppose then that X, Y and Z are not vertices of P. It is clear that x, y and z must enclose P, otherwise we can translate it away (see Fig. 7(b)).

To prove the sufficiency of our conditions, observe that by Theorem 1, the points X, Y and Z immobilize $T(x, y, z)$ and therefore they also immobilize P. □

2.3. Immobilizing a Simple Polygon

We proceed now to study simple polygons. We start by noticing that using the same argument used in Theorem 1 the following result can be proven:

Lemma 2 *Let x, y and z be three edges of a polygon P and X, Y and Z be interior points of x, y and z respectively. Then if X, Y and Z immobilize P the orthogonals at X, Y and Z to their respective sides x, y and z meet at a common point.*

Assign to each edge x of P the halfplane containing x on its boundary and containing the points from the interior of P that are in the proximity of an interior point of x. We now say that the sides x, y and z of P form a *triangular triple* of P if the intersection of three halfplanes assigned to x, y and z is nonempty and bounded (i.e. a triangle).

Theorem 3 *A polygon P can be immobilized by three points X, Y and Z different from vertices of P if and only if:*

- *the orthogonals at the points X, Y and Z to its respective sides x, y and z meet at a common point, and*

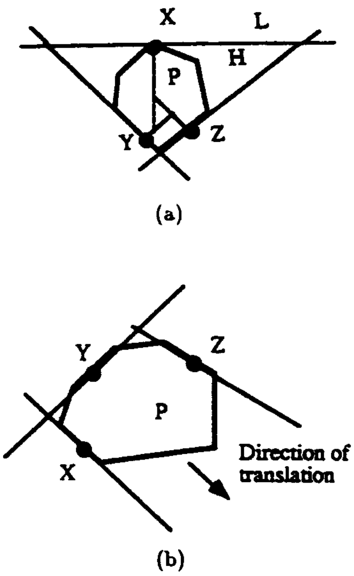


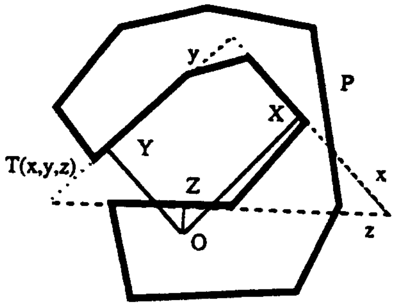
Fig. 7. Three non-immobilizing points.

- *x, y and z form a triangular triple of P.*

Proof. Lemma 2 proves the necessity of the first condition. Suppose that x, y and z do not enclose P. This may happen for one of two reasons: either the intersection of the halfplanes assigned to x, y and z forms an unbounded region and then, as in the convex case (see Fig. 7(a)), P may be translated away, or this intersection is empty. In the latter case, if the orthogonals meet inside $T(x, y, z)$ P may be rotated around the point of their intersection. In the remaining nontrivial case tne orthogonals meet at a point O that is outside $T(x, y, z)$ (as indicated on Fig. 8(a)). A similar analysis to that performed in the proof of Theorem 1 (refer to Fig. 5(b)) leads now to a different conclusion: any translation vector OU, where U is in the interior of the triangle $T(x', y', z')$ (see Fig. 8(b)) sets the points X, Y, and Z outside P' (the new position of P). It is easy to see that this may be done for any small value of the rotation $\beta > 0$ and so that the corresponding translation $\tau(\beta)$ is a continuous function of β. As a consequence there exists a continuous movement that is the composition of the rotation around O and the translation by vector OU that does not cause any X, Y, or Z to penetrate P.

The sufficiency of both condition follows exactly as in the proof of Theorem 2. □

From the proof of Theorem 3 follows an interesting example (see Fig. 9) of a polygon Q without parallel sides, and three points X, Y and Z on its boundary which "trap" the polygon without immobilizing it, that is we can always move Q a bit without penetrating it with X, Y or Z, however we cannot move Q to another position far enough from X, Y and X without penetrating it by at least one of X, Y or Z. It is interesting to notice that the only possible movement of Q is such that



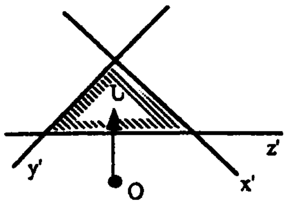


Fig. 8. Immobilizing a polygon.

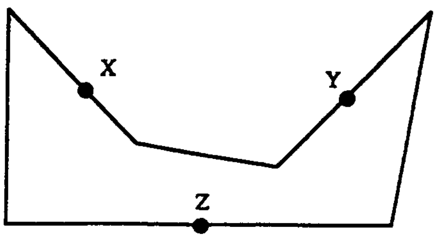


Fig. 9. Trapping without immobilizing.

the three points X, Y and Z "slide" on its boundary, that is *all of them remain on the boundary* of Q. Notice that in such a case the three orthogonals to the boundary of the polygon at the three concerned points meet at a common point, and the extensions of the three sides containing these points also meet at a common point.

Obviously, any convex polygon P needs at least three points to immobilize it. We will soon see that three points will suffice, also for any simple polygon P, when there are no two parallel sides in P. Before that we have to turn our attention to polygons which may be immobilized using two points only. Clearly, at least one of these two points will have to be located at the reflex vertex of P.

Theorem 4 *Two points X and Y immobilize a simple polygon P if and only if the line segment XY joining them forms an angle at least $\pi|2$ with four adjacent sides of P and if two of these four sides are parallel they must lie on opposite sides of the line containing XY.*

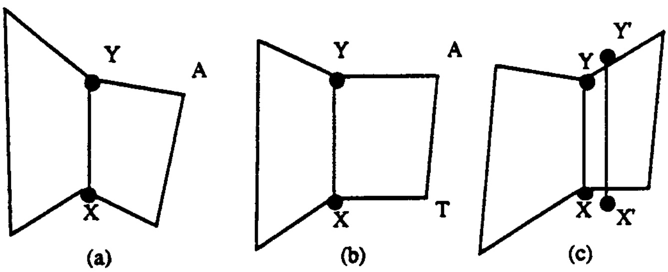


Fig. 10. Rotating around X.

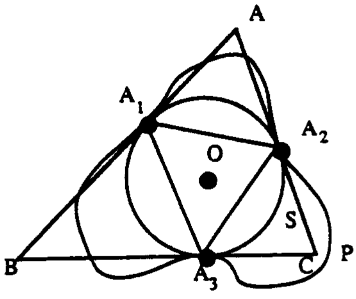


Fig. 11. Immobilizing a polygon without parallel edges.

Proof. To prove the necessity observe that if one of the four angles (say XYA in Fig. 10(a)) is $< \pi|2$ then P may rotate around X. If the two sides on the same side of XY are parallel (as XT and YA in Fig. 10(b)) then P may be translated perpendicularly to XY.

Sufficiency follows from the fact that for any pair of points X' and Y' not in the interior of P, X' on an ε the neighborhood of X and Y' in an ε neighborhood of Y, we have $/X'Y'/ > /XY/$ (see Fig. 10(c)). □

Theorem 5 *Any polygon without parallel edges can be immobilized using three points.*

Proof. Let S be the largest circle contained inside P and let O be the center of S. If among the points at which S touches P we cannot choose three points A_1, A_2, and A_3 such that O is in the interior of the triangle $A_1A_2A_3$ then S must touch P at the endpoints of a diameter of S. Since P has no parallel edges it is easy to see by Theorem 4 that these two points immobilize P. In the case that S touches P

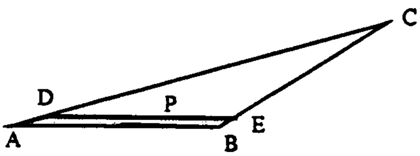


Fig. 12. Four immobilizing points needed.

in three points A_1, A_2 and A_3 such that O is an interior point of triangle $A_1A_2A_3$ (see Fig. 11), let T be the triangle defined by the tangents to S at A_1, A_2 and A_3. Clearly A_1, A_2 and A_3 immobilize T and hence P. □

It is not true, however, that all polygons can be immobilized using three points. For example, any parallelepiped with four vertices cannot be immobilized with three points. In fact examples given by Kuperberg[8] might suggest that each convex figure needing four points to immobilize is an intersection of two objects each one being either a strip or a disk. We show, however, that there are convex polygons other than parallelepipeds that also cannot be immobilized with three points.

Theorem 6 *For every $n > 3$ there are convex polygons with n vertices for which exactly four points are needed to immobilize them.*

Proof. An example of a quadrilateral, not a parallelepiped, for which four points are needed to immobilize it can be obtained as follows: Consider a triangle T with vertices A, B and C such that the angle at B is obtuse. Then the quadrilateral P with vertices A, B, E and D, such that D and E are on the edges BC and DE of T and DE is parallel to AB and close enough to it, cannot be immobilized by using three points (see Fig. 12).

To prove this result, we first notice that if three points X, Y and Z were to immobilize P, then by Theorem 2, the three sides of P containing them would have to enclose P. Then these sides would be the segments AB, AD and BE. It is easy to verify, however, that if the segment DE is close enough to AB condition (b) of Theorem 2 is not satisfied.

To prove that for every $n > 3$ there are polygons that cannot be immobilized with three points, it is sufficient to notice that we can substitute the side AD in P by a convex chain of edges close enough to AD and the same argument holds. □

Corollary 2 *Any polygon P can be immobilized with at most four points.*

The proof of this corollary follows from the results of Mishra, Schwartz, Sharir and Silver[11,12]. We give now an elementary proof of this fact. .

Proof. Consider a circle S contained in P with the largest possible radius. If S intersects the boundary of P in three or more points $X_1, \ldots, X_n$ such that the center of S is contained in the convex closure of $X_1, \ldots, X_n$. In this case it follows using arguments similar to those in Theorem 1 that $X_1, \ldots, X_n$ immobilize the convex polygon bounded by the tangents to S at $X_1, \ldots, X_n$ and therefore they also immobilize P. Suppose then that S intersects the boundary of P in two points U and V. It is easy to see that U and V must be diametrically opposed in S. Two cases arise:

- At least one of U and V is a vertex of P.
- U and V belong to the interior of two parallel sides u and v of P

Suppose, without loss of generality, that UV is a vertical segment. Let us take any horizontal line H that does not intersect any vertex of P and intersects the interior of UV. Let L and R be the leftmost and the rightmost intersection points of the boundary of P with H (see Fig. 13). It is now straightforward to verity that U, V, L and R immobilize P. □

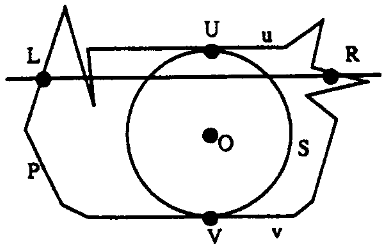


Fig. 13. Four immobilizing points needed.

Theorem 7 *Let P be a polygon without parallel edges. In $O(n \log n)$ time ($O(n)$ time if P is convex) we can find a set of three points immobilizing P.*

Proof. As P does not have parallel edges, the largest circle C inscribed in P touches its boundary in three points X, Y and Z that by Theorem 3 immobilize it. The center of circle C is a vertex of a Voronoi diagram $V(P)$ of the set of edges of P. $V(P)$ may be constructed in $O(n \log n)$ time[10,11,13] ($O(n)$ time if P is convex[12]). Our result now follows. □

A similar argument can be used to find a set of four points that immobilize P for the case when P contains parallel edges.

3. Immobilizing a Shape in the Plane

Given a shape P, a circle S contained in P is called a locally largest inscribed circle of P if there is no other circle C contained in P that contains S.

A shape P is called a type-3 shape if it contains a locally largest inscribed circle S that touches the boundary of P in three points A_1, A_2 and A_3 such that the center O of S is an interior point of triangle A_1, A_2, A_3 (see Fig. 11).

If S is not a type-3 shape then it is easy to prove that any locally largest circle S is a diameter circle, i.e S intersects the boundary of P only in the endpoints of one or two diameters of S (an example is the intersection of two disks; see also figures by O'Rourke[14]). In such cases we call P a *diameter shape.*

To be more precise, we say that a circle touches a set of points if:

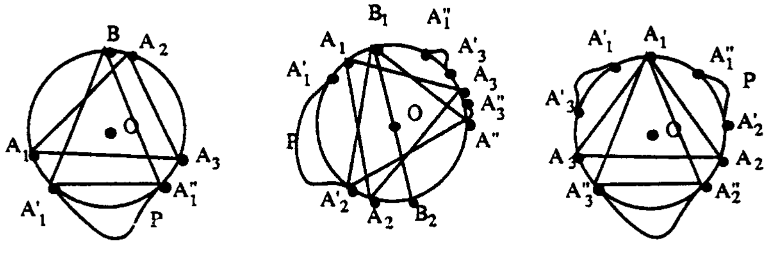
- no point from the set lies inside the circle, and
- the interior of every larger concentric circle contains a point from the set.

3.1. Immobilizing a Type-3 Shape

In this section we prove the following result:

Theorem 8 *If P is a type-3 shape then P has an immobilizing set consisting of three points.*

Proof. Since P is a type-3 shape, it contains a locally largest circle S that intersects the boundary of P in three points A_1, A_2 and A_3 such that O is an interior point of triangle $A_1A_2A_3$. We show first how to find a set H of three points (not necessarily A_1, A_2, A_3) on the boundaries of S and P, such that O belongs to the



(a) (b) (c)

Fig. 14. Preventing rotation.

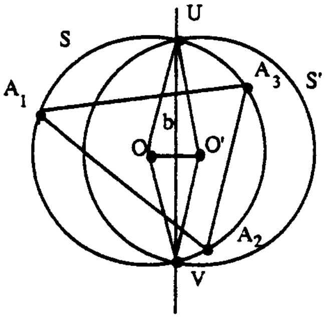


Fig. 15. Immobilizing type-3 shape.

interior of the triangle determined by them and such that any rotation around O causes at least one of them to penetrate P, next we prove that these points disallow any other movement as well.

Notice first that we may assume that each of A_1, A_2, A_3 belongs to some maximal arc, which is not a point, from the intersection of the boundaries of S and P. Three cases arise:

- A_1, A_2 and A_3 belong to the same maximal arc with end points A_1' and A_1''. Then we take as $H = \{A_1', A_1'', B\}$ were B is the point in the middle of the arc $A_1'A_1''$. Clearly H as chosen does not allow rotations of P (see Fig. 14(a)).

- A_1 and A_2 belong to the same arc $A'A''$ of S and A_3 to a different arc $A_1'A_1''$ of S. Suppose w.lo.g. that arc $A'A''$ covers less than half of the boundary of S. Then $A'A''$ contains a point B_1 such that the point on S diametrically opposed to B_1 lies on the arc $A_2'A_2''$ containing A_3. Choose $H = \{B_1, A_1', A_1''\}$. It now follows that B_1, A_1' and A_1'' also disallow rotations of P without one of A_1' or A_1'' penetrating P (see Fig. 14(b)).

- Each $A_i, i = 1, 2, 3$ belongs to a separate arc $A_i'A_i''$ of the intersection of S with the boundary of P. Assume each of $A_i'A_i''$ covers less than half of the boundary of S, otherwise we can use our three points using the same arguments as in (i). Following the counterclockwise orientation, choose three points of H as follows: the last point of some arc, the first point of the next arc and any interior point of the third arc (see Fig. 14(c)).

Next we prove that H indeed immobilizes P. Suppose that P can be moved to a new position in which O has moved slightly to a new position O'. Let S' be the new position of S, and let S and S' intersect in points U and V; U and V lie on the bisector b of the segment OO' (see Fig. 15) which separates the boundary of S into two halves, one being inside and one being outside S', respectively. When O' is near O, OO' lies completely inside triangle $A_1A_2A_3$. In that case b intersects triangle $A_1A_2A_3$, therefore splitting points A_1, A_2 and A_3. Then there is at least one point of A_1, A_2 and A_3 on each side of b. Therefore at least one of them is in the interior of S', and that one penetrates S. □

3.2. Diameter Shapes

Let P be a diameter shape, S a diameter circle of P with center O and U and V be touching points of S to the boundary of P. Suppose, for simplicity, that UV is vertical, both U and V with x-coordinate equal to O, and V is below U.

For $\partial > 0$ we consider a ∂-internal of P consisting of two continuous pieces, upper P_u and lower P_l, containing U and V, respectively (see Fig. 19) such that each point on them has x-coordinate between -∂ and ∂; each P_u and P_l is further subdivided into left and right portion by U or V respectively. The choice of ∂ will be discussed below.

In order to immobilize P, we can restrict the analysis to a neighborhood of U and V only; clearly any set that will immobilize restricted shape will also immobilize P.

Left $f(A)$ be the radius of the largest circle centered at A, which touches both P_u and P_l. As A must be equidistant from P_u and P_l, f is defined only for some points between P_u and P_l. There are two cases:

- for every $\varepsilon > 0$ there exists a point A, such that $/AO/ < \varepsilon$, f is defined for A and $f(A) < f(O) = /OU/$. In order words, S cannot move from its original position O, without intersecting the exterior of P;

- there exists $\varepsilon > 0$ such that $f(A) = f(O)$ for any center A for which f is defined and such that $/AO/ < \varepsilon$. Intuitively in this case S may slide inside P in some ε neighborhood of O; we refer to such a shape as a tube.

3.2.1. Immobilizing diameter non-tube shapes

We show how to immobilize any diameter shape P that belong to the case a) of the previous section; tubes will be studied in the next section.

Theorem 9 *Four points always suffice to immobilize any diameter non-tube shape.*

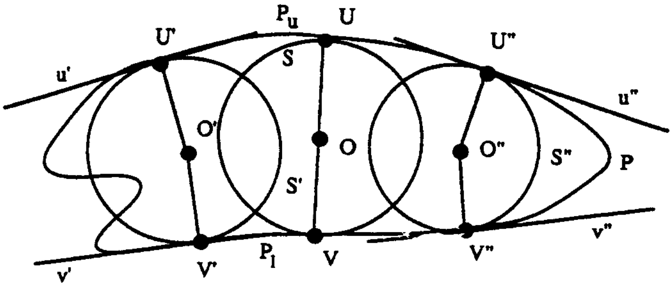


Fig. 16. Immobilizing diameter non-tube shapes.

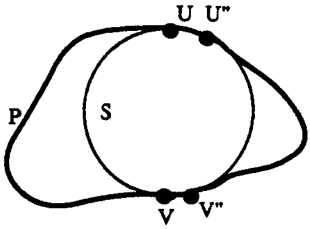


Fig. 17. Circular arc on shape.

Proof. To immobilize the shape P, we choose a set I of four points $U', V'U''$, and V'' on P, one on each left and right portion of P_u and P_l, such that U' and V' and V' (U' and V'') are touching points of an inscribed circle S' (S'') centered at O' (O'', respectively) with P_u and P_l, where O' and O'' are in the neighborhood of O and lying on opposite sides of the line UV (as shown in Fig. 16). By u', v', u'', and v'' we denote the four tangents to S' and S'' (they are not necessarily tangents to P). We will show that for any $\varepsilon > 0$ we can choose $\partial > 0$ such that the following properties are satisfied on ∂-intervals:

(i) the slopes of the tangents u', v', u'', and v'', (the angles the tangents form with x-axis) are between -ε and ε. The choice of any $\varepsilon < \pi/4$ will suffice for our purpose;

(ii) the slope of $O'O''$ is between -ε and ε. The choice of any $\varepsilon < \pi/4$ will also suffice;

(iii) the two tangents u' and v' at the touching points U' and V' of S' with P_u and P_l intersect to the left of UV (or, in other words, the angle $U'O'V'$ in the polygon $UU'O'V'V$ is $> \pi$); We refer to this angle as the *critical* angle at O'. Analogously the two tangents u'' and v'' intersect to the right of UV.

However, if $S \cap P$ contains many points in any neighborhood of U or V (or both) some special cases may arise. Observe first that these points must be situated on one side only (say right) of OU (see Fig. 17). Otherwise P would be a type-3 shape.

In this case we replace S'' and O'' by the diameter circle S and its center O; as U'' and V'' we take two points in the right neighborhood of U and V, respectively. This is true also if U or V contain many points in their left neighborhood or if in

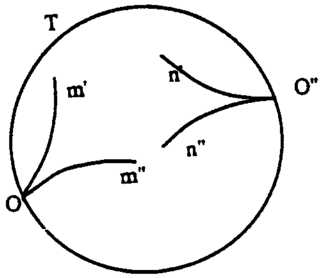


Fig. 18. Region of center movement.

the neighborhood of only one point among U and V we have many points from $S \cap P$.

Thus in the sequel we may assume that P and S do not share many points in a neighborhood of U or V. Assuming that (i), (ii), and (iii) are valid, we will now complete the proof of Theorem 9. The proof that the four points immobilize P uses the fact that any movement of P preserves the distance between O' and O''. Consider possible movements of points O' and O''. Let m' and m'' be two arcs starting at O' such that the tangents at O' to these arcs are parallel to u' and v', respectively. Applying Lemma 1 (and property (iii)) twice for center O' and interior points U' and V', respectively, we get as possible movement of O' the region limited by m' and m'' and the line UV. Similarly the movement of O'' is only within the region limited by n' and n'' (see Fig. 18). Consider now the circle T with diameter $O'O''$. Since the slope $O'O''$ (according to (ii), and the slopes of four arcs at O' and O'' (according to (i)) are between ε and -ε, the possible movements of both O' and O'' are within the circle T. Clearly any such movement will decrease the distance between O' and O''. However, any movement of the shape P must preserve the distance. This is a contradiction, and Theorem 9 is proved.

To complete the above proof of Theorem 9 we need to prove properties (i), (ii), and (iii) for appropriate choice of S' and S''.

First we prove the following lemmas.

Lemma 3 *The set of centers O' of inscribed circles of P_u and P_l forms a continuous curve (homeomorphic image of an interval) in the neighborhood of point O (we call it the center curve).*

Proof. Consider a circle R with radius $r < /OU/$ centered at O. Let the diameter circle S and the vertical lines $x = -\partial$ and $x = \partial$ intersect at points A, B, C, and D, as illustrated in Fig. 19. The angles AOU, UOD, BOV, and VOC are all equal to ε, where $\partial = /OU/ \sin \varepsilon$. Let A', B', C', and D' be intersections of OA, OB, OC and OD with R (see Fig. 19), and let X be any point on P_u. It is easy to see that the distance $/XY/$ is a monotone increasing function on Y when Y scans from A' to B'. Since this is valid for any point X on P_u, the distance from Y to P_u is a monotone increasing function when going from A' to B'. Similarly the distance from Y to P_l is a monotone decreasing function when Y scans from A' to B'. Since A' is obviously closer to P_u than to P_l, and vice versa for B', there

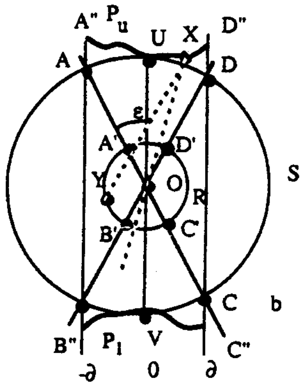


Fig. 19. Continuous curve of centers.

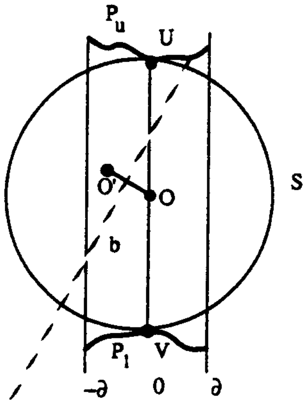


Fig. 20. Slope OO' is horizontal.

exists exactly one point Y on the arc $A'B'$ that is equidistant from P_u and P_l; Y is the center of an inscribed circle for P_u and P_l. Analogously there exists exactly one such center on the arc $D'C'$. Therefore the set of centers contains exactly one point to the left and one point to the right of O at any given "small" distance from O. The distances from P_u (and P_l) are continuous functions in the plane; if the sequence of centers of inscribed circles converges towards a point Y on the arc $A'B'$ then Y must also be equidistant from P_u and P_l. This is sufficient to claim that the set of centers is a continuous curve. □

Lemma 4 *Suppose that P_u and P_l do not share any arc with their diameter circle S left (right) to UV. Given and ε such that $\pi/4 > \varepsilon > 0$, we can choose $\partial > 0$ so that in the ∂-interval the slope OO' is between -ε and ε for any center O' of an inscribed circle of P_u and P_l that is sufficiently close to O.*

Proof. Choose $\partial < /OU/ \sin\varepsilon$. Suppose that there exists a center O' of an inscribed circle for P_u and P_l with the slope that is outside the interval $[-\varepsilon, \varepsilon]$ within any neighborhood of O; suppose that an infinite number of them have slopes that are smaller than -ε and are to the left of UV (the other three cases can be discussed analogously); see Fig. 20. Since the angle $O'OU$ is acute, $/O'U/ < /OU/$ when O' is close enough to O. On the other hand, as $\partial < /OU/ \sin\varepsilon$, b does not intersect P_l, where b is bisector of segment OO'. Thus for any point X in $P_l /O'X/ > OX/ \geq /OV/ = /OU/$. It follows that O' is strictly closer to P_u than to P_l, which is a contradiction. □

According to our earlier discussion we may assume that U and V are the only common points of P and S in the neighborhood of U and V. Let A'', B'', C'', and D'' be (first) intersections of P_u and P_l with the verticle lines $x = -\partial$ and $x = \partial$ (i.e. the endpoints of P_u and P_l; see Fig. 20). Distances from O to these points are greater than $/OU/$; let the closest one be at distance $/OU/ + \varepsilon'$, where $\varepsilon' > 0$. Then we may restrict the neighborhood of O to at most ε' (i.e. $/OO'/ < \varepsilon', /OO''/ < \varepsilon'$). This will assure that the chosen inscribed circles with centers O' and O'' really touch P (i.e. do not intersect P at one of A'', B'', C'' or D'').

From Lemma 3 it follows that two points O' and O'' can be chosen at any distance $< \min(\partial, \varepsilon')$ from O. Since the slopes OO' and OO'' are between -ε and ε (Lemma 4), the slope $O'O''$ is also between -ε and ε, where ε can be chosen arbitrarity ($\pi/4 > \varepsilon > 0$). This assures property (ii). Next, it is easy to show that for any point X on P_u or O_l the angle between $O'X$ and OU is also within $[-\varepsilon, \varepsilon]$ for any such choice of O'. From this it follows that the slope of any tangent to the inscribed circle centered at O' (or, analogously, O'') is within $[-\varepsilon, \varepsilon]$, since such a tangent at home point X from P is perpendicular to $O'X$. This confirms property (i).

To verify property (iii) we prove first the following two lemmas.

Lemma 5 *If the inscribed circle of P_u and P_l is a diameter circle for each center O' belonging to a (closed) interval on the center curve then the diameter of the circle is the same for all these centers (i.e. the shape is a tube).*

Proof. We show first that any such circle S' does not share with P_u or P_l an infinite set I of points in the neighborhood of U' or V'. Suppose that, to the contrary, it does so for a center O'. Then it is easy to show that the centers lying on the same side of the diameter $U'V''$ as I are not centers of diameter circles. Now from Lemma 3 and Lemma 4 (this lemma can be applied to any point O' instead of O) it follows that the center curve is smooth since there exists the tangent to the curve at any center O' and the tangent is normal to the diameter $U'V'$.

Next, we show that at least one of P_u and P_l is also a smooth curve. Let O'' approach O' on the center curve. Observe that $U'V'$ and $U''V''$ cannot intersect; indeed if they do, say $O''U''$ and $U'O'$ intersect at point T then $/O''U'/ + /O'U''/ \geq /O''U'' + /O'U'/ = /O''T/ + /TU''/ + /O'T/ + /TU'/ > /O''U'/ + /O'U''/$, which is a contradiction. So U'' must then approach U' and V'' must approach V' (note that the distance is a continuous function). Suppose that, when O'' moves towards O' the segment $O''O'$ becomes horizontal but, say, $V''V'$ is not. Then the bisector b of $V'V''$ leaves both O'' and O' on the same side. If V' is on the same side of b as O' and O'' then $/O''V'/ < /O''V''/$; otherwise $/O'V''/ < O'V'/$, which is in both cases a contradiction. Thus $V''V'$ is also becoming horizontal, i.e. the center

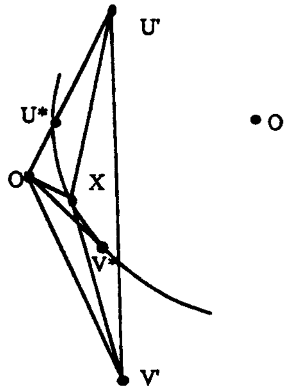


Fig. 21. Decreasing radius of inscribed circle.

curve and P_l (in this case) have parallel tangents at O' and V', respectively. In case $V'' = V'$ it may be shown that V' corresponds to any center on interval between O' and O'' on the center curve; the tangent at V' does not exist but the corresponding tangents for P_u are well defined (i.e. P_u is then smooth).

Now we map the center curve continuously to an interval I of a straight line (say, x-axis). Suppose that P_l is smooth. Construct $f(x)$ in the following way: for every point $x_O \in I$ corresponding to O' in the center curve let $f(x_O) = /O'V'/$. It may be shown that $f(x)$ is a smooth curve with always horizontal tangent. But if $f(x) = 0$ for each x belonging to an interval, applying integral gives $f(x) = c$ (constant). Therefore the diameter is constant on the interval. □

Lemma 6 *If the critical angle at the center O' is $< \pi$ then the radius of the inscribed circle decreases in the neighborhood of O' when O' approaches O.*

Proof. Let a function f be defined as follows: $f(/OO'/) = /O'U'/$, i.e. the argument is the distance between O and O' (O' is unique, see the proof of Lemma 3.1) and the value is the radius of inscribed circle at O'. Suppose that the critical angle at O' is smaller than π. Therefore at least one of the angles $OO'U'$ and $OO'V'$, say angle $OO'U'$, is smaller than $\pi/2$. Consider a circle C centered at O with radius r such that $r < /OO'/$ but r is greater than the distance from O to the segment $O'U'$ (see Fig. 22). Let U^* be the intersection of C and $O'U'$. U^* is closer to P_u than to P_l as the bisector of $O'U^*$ is almost horizontal. If C intersects $O'V'$ then V^* is defined analogously. Otherwise let V^* be such that $O'V^*$ is the lower tangent to C (see Fig. 21). $O'V^*$ is almost vertical; therefore its bisector is almost horizontal and thus V^* is closer to P_l than P_u. Therefore the arc U^*V^* contains a point X that is equidistant from P_u and P_l. Radius r can be chosen (increased) such that V^* and X fall inside triangle $O'U'V'$. Then at least one of the angles $O'XU'$ or $O'XV'$, say $O'XV'$, is greater than $\pi/2$. Hence $/XV'/ < /O'V'/$ and $f(/OX/) < f(/OO'/)$. This means that the function f decreases in the neighborhood of O' when O' approaches O. □

To verify property (iii), suppose that the critical angle at O' is not greater than π for any point O' on the center curve which is near O. It is easy to prove that the set of centers O' for which the critical angle is $= \pi(< \pi)$ is a closed (open, respectively) set on the center curve, consisting of the union of possibly infinite number of closed (or open) intervals, respectively. A coording to Lemma 5, the diameter is a constant function within any of the closed intervals of centers O' with the critical angle equal to π.

Since the diameter $/OU/$ is a local maximum, $f(/OO'/ \leq f(/OO/) = f(O)$. If the critical angle at O' is smaller than π then $f(/OO'/)$ decreases when O' moves toward O (Lemma 6); if, on the other hand, the critical angle is equal to π, O' belongs to a closed interval with such critical angles and remains a constant function (Lemma 5). This is possible only if $f(/OO'/) = f(O)$, i.e. when the diameter is a constant function and all critical angles are equal to π. But then the shape is a tube, which contradicts the type of shape studied. Hence there exists a center O' such that the critical angle at O' is greater than π, and property (iii) is verified.

Therefore properties (i), (ii), and (iii) may be satisfied and this completes the proof of Theorem 9. □

3.2.2. Immobilizing tubes

In this section we will prove that four points will always suffice to immobilize any tube. The idea of the proof is to choose first two immobilizing points U and V in the intersection of the diameter circle S and the tube. Then we show that the only possible motion of the shape P, without one of these two points penetrating the interior of P, is the "sliding" of P between U and V. U and V remain then in the boundary of P and the points from the center curve move to O. Each point in the plane moves along a smooth curve, determined by the center curve. Two additional points will be chosen, each one to prevent the motion in one of two possible directions of the sliding. This will be easy to do when during such motion some points from the boundary of P must move to the interior of P. If this does not happen we will prove that no point of P can move to the exterior of P, otherwise P would change its area. In the remaining case, all the points from the boundary of P during the motion must stay in the boundary of the original position of P. We prove in Lemma 3.5 that this may happen only when P is a circle.

Lemma 7 *Suppose that during the motion of P every point from the boundary of P moves to a point belonging to the boundary of the initial position of P. This is possible only when P is a circle.*

Proof. Take the circumcircle of P, i.e. the minimum radius circle C enclosing P, centered at O. Observe that O cannot move during the motion, otherwise the maximum distance from O', the new position of O, to $P \cap C$ would be greater than the radius of C. As at each instance of time the motion is an isometry this is clearly impossible. Therefore the motion preserving the boundary of P must be a rotation around O. Under this motion, a point from $P \cap C$ traces an arc of C and the whole circle is traced when the motion is repeated a few times. Thus $P = C$. □

Theorem 10 *Four points always suffice to immobilize any tube.*

Proof. First we show that any movement of P that respects U and V as immobilizing points maps two other points U' and V' to U and V (respectively)

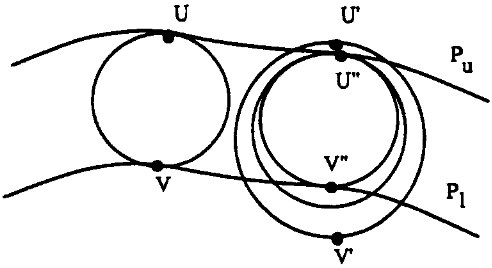


Fig. 22. Immobilizing tubes.

such that $U'V'$ are also touching points of another diameter circle of P (i.e. the shape is "sliding" along UV). To prove this fact, neither U' nor V' can be an interior point of P, since in that case U or V will penetrate P. Thus U' and V' are outside P or on the boundary of P. Suppose that at least one of them lies outside P. The circle S' that has $U'V'$ as diameter intersects in this case both P_u and P_l. A continuous decrease in the diameter of S' (first keeping the same center and decreasing until S' just touches P_u or P_l; then keeping the same touching point and moving the center of S' closer to the point until S' touches the other curve; see Fig. 22) gives another inscribed circle S'' that touches P in two points U'' and V'' such that $/U''V''/ < /U'V'/ = /UV/$. This is a contradiction because we get an inscribed circle with diameter less than $r = /UV/$.

Therefore only points U' and V' from the boundary of P move to points U and V such that $U'V'$ is the diameter of an incribed circle of $P(/U'V'/ = /UV/)$. Let T be a point from the boundary of P, and let T' be the point that moves to T. Since the distances are preserved by any given movement, it follows that $/T'U'/ = /TU/$, and $/T'V'/ = /TV/$. Thus at any given time τ the point T' that moves to T is uniquely determined, and we may write $T' = f(T, \tau)$ where τ stands for time with $f(T, O) = T$ (the cases $\tau < 0$ and $\tau > 0$ correspond to the movements in one or another direction). $f(T, \tau)$ is clearly a continuous curve. It is easy to show that $f(T, \tau)$ is a smooth curve, determined by the center curve which is smooth (see the proof of Lemma 5) and the tangent to it. For each of the two directions we will show that there is a point from the boundary of P that penetrates the interior of P. Choosing one such point to immobilize P will prevent the whole movement in the direction, and four points that immobilize P will be found. Let $\tau > 0$ (the case $\tau < 0$ is considered analogously). We partition all points from the boundary of P into four disjoint classes:

- Points T for which there exists $\varepsilon(T) > 0$ such that $f(T, T)$ is the boundary point on P for any τ in the interval $0 \leq \tau \leq \varepsilon(T)$ (points that move along the boundary of P).

- Points T for which $f(T, \tau)$ assumes values of both some interior and some exterior points from P in any interval $0 \leq \tau \leq \varepsilon(T)$, no matter how small $\varepsilon(T)$ is chosen. Such a point T oscillates between the interior and the exterior of P.

- Points T, that do not belong to class (1), for which there exists $\varepsilon(T) > 0$ such that $f(T,\tau)$ is not an interior point of P for any τ in the interval $0 \leq \tau \leq \varepsilon(T)$.
- Points T, that do not belong to (1), for which there exists $\varepsilon(T) > 0$ such that $f(T,\tau)$ is not an exterior point of P for any τ in the interval $0 \leq \tau \leq \varepsilon(T) > 0$.

There are two cases:
- class (2) or (4) is nonempty. Therefore there exists a point T on the boundary of P such that, no matter how small is the movement (i.e. no matter how small is the time $\varepsilon(T) > 0$), there exists T' from the interior of P which moves to T in time $0 < \varepsilon'(T) \leq \varepsilon(T)$. This means that T penetrates P during the movement, and can thus be chosen to immobilize P in a given direction (note that T may or may not belong to the tube part of P).
- classes (2) and (4) are empty. We will show that then class (3) is also empty. Suppose that, to the contrary, it is not, and that T belongs to class (3). Then for any sufficiently small $\varepsilon(T) > O$ there exists T' from the exterior of P which moves to T in time $0 < \varepsilon'(T) \leq \varepsilon(T)$. However, the area of P is an invariant of any movement. Thus some points from the interior of P must move to the boundary of P (to keep the same area) in the same time $\varepsilon'(T)$. Therefore one of the classes (2) or (4) is nonempty, which is a contradiction. Thus all points belong to class (1), i.e. any point from the boundary of P moves to the point from the boundary of P. By Lemma 7 that is possible only if P is a circle. This completes the proof for the tube shapes. □

In consequence we have the following major result.

Theorem 11 *Four points always suffice to immobilize any shape which is not a circle.*

Proof. Follows from Theorems 3, 4, and 5. □

The reader may check that four points suffice also to immobilize any shape P with holes (except concentric rings).

4. Generalizations to Higher Dimensions

In this section we will generalize some results on immobilization of polygons in the plane to the high-dimensional case.

Consider the largest inscribed sphere S (centered at O) of a given d-dimensional polytope P. Suppose S touches P in points $A_1, A_2, \ldots, A_t$. Let $T = CH(A_1, A_2, \ldots, A_t)$ denote the convex hull of these points.

Lemma 8 $O \in CH(A_1, A_2, \ldots, A_t)$.

Proof. If O is located outside T, O must be a vertex of $CH(O, A_1, A_2, \ldots, A_t)$, and therefore there exists a $(d-1)$-dimensional hyperplane C passing through O such that $A_l, A_2, \ldots, A_t$ are all on the same side of C (and not on C). Let OX be a vector normal to C such that all angles $XOA_i (1 \leq i \leq t)$ are obtuse. Then, when we move point O in the direction OX it will be a center of an inscribed sphere larger than S. □

Lemma 9 *Let $A_i A_2 \ldots A_{d+1}$ be a d-dimensional simplex containing O in its interior. Then $\{A_1, A_2, \ldots, A_{d+1}\}$ immobilizes P.*

Proof. For any motion keeping O in place, the final position of this movement may be described as a composition of $d-1$ rotations around O. Some points among $A_1, A_2, \ldots, A_{d+1}$ (all those which move at all) will then penetrate the interior of P. Therefore, any possible motion must O to a new position $O' \neq O$, and S moves to S'. Let b be the $(d-1)$-dimensional hyperplane that is the bisector of OO'. Because $A_1A_2 \ldots A_{d+1}$ contains O, when O is close enough to O', on each side of b there are some points among $\{A_1, A_2, \ldots, A_{d+1}\}$. All points of S that lie on the opposite side of b than O are then inside S', the new position of S, (once more when O and O' are close enough), and thus penetrate P. For $d = 2$ consult Fig. 16 and the corresponding part of the proof of Theorem 8. □

Now we will turn our attention to the question of the upper bound for the number of points necessary to immobilize a polytope. Before we pass to the general d-dimensional case, let us consider, as a more intuitive illustration, the case of 3-dimensional polyhedra.

Theorem 12 *Six points suffice to immobilize any three-dimensional polyhedron.*

Proof. By Lemma reflem:4.1, O is inside or on the boundary of $CH(A_1, A_2, \ldots, A_t)$. Let m be the minimum number such that there exists an m-dimensional simplex T' with $m+1$ vertices taken from $\{A_1, A_2, \ldots, A_t\}$ containing point O in its interior. Let these points be named $A_1, A_2, \ldots, A_{m+1}$. Thus $T' = CH(A_1, A_2, \ldots, A_{m+1})$. Consider the following cases:

Case 1) $m = 1$. Then O is in the interior of a segment, say, A_1A_2, and A_1A_2 is a diameter of S. We will include A_1 and A_2 in the set of points to immobilize P. A_1A_2 is the minimal distance between corresponding faces conatining A_1 and A_2 (there may be, in the case of non-convex polyhedron, several faces containing A_1 or A_2). This distance is exactly the distance between parallel planes that are tangent to S at A_1 and A_2, respectively. The points from these planes that are in the neighborhood of A_1 or A_2 are inside or on the boundary of P, and thus the only motion (if any) that does not cause A_1 or A_2 to penetrate P must be the motion within the plane normal to A_1A_2. In other words, for any point $p \in P$, its motion remains within the plane containing p and normal to A_1A_2. P intersects any such plane in a simple polygon, and that polygon can be immobilized in that plane with four points (Corollary 2). Thus P can be immobilized with six points.

Case 2) $m = 2$. Then O is inside a triangle, say, $A_1A_2A_3$. We include A_1, A_2, and A_3 in the set of points to immobilize P. Consider the tangent planes to S at A_1, A_2, and A_3. P is obviously the superset of these planes in the neighborhood of touching points and will have restricted motion as the figure that is formed by the three tangent planes. The only possible motions of P are now translations along the line normal to the plane $A_1A_2A_3$. The translations can be prevented by choosing two more points, one for each direction of translation, thus giving a total a five points for immobilization.

Case 3) $m = 3$. O is the interior point of the tetrahedron, say, $A_1A_2A_3A_4$. A_1, A_2, A_3, and A_4 will then immobilize P by Lemma 9. So, in this case four points suffice to immobilize P. □

Theorem 13 $2d$ *points are always sufficient and sometimes necessary to immobi-*

lize a given d-dimensional polytope P.

Proof. The proof is by induction on d. It is aready proved for $d = 2$ and $d = 3$. For $d = 1$ it is trivially sufficient to immobilize a segment on a line with two points. Suppose that the statement is true for any dimension smaller than d. We prove that the statement is then true for dimension d as well.

According to Lemma 8, the center O of the largest incribed sphere S must be located inside or on the boundary of $CH(A_1, A_2, \ldots, A_t)$. Let m be the minimal number $m \geq 1$ such that there exists a m-dimensional simplex T' with $m + 1$ vertices taken from $\{A_1, A_2, \ldots, A_t\}$ containing point O in its interior. Let these points be named $A_1, A_2, \ldots, A_m + 1$. Thus $T' = CH(A_1, A_2, \ldots, A_{m+1})$. Consider the following cases:

Case 1) $m = d$. Then by Lemma 9, $\{A_1, \ldots, A_{d+1}\}$ immobilizes P.

Case 2) $1 \leq m < d$. Include the points $A_1, A_2, \ldots, A_{m+1}$ in the set to immobilize P.

Analogously as in Theorem 12, there is no motion of P within the m-dimensional space determined by the points $A_1, A_2, \ldots, A_{m+1}$. Thus each of the possible motions, so far, must be within a $(d - m)$-dimensional space that is orthogonal to the above m-dimensional one. Since $d - m < d$, by the induction hypothesis, this motion can be prevented by $2(d-m)$ addition points. Therefore $2(d-m)+m+1 = 2d - m + 1 \leq 2d$ points suffice to immobilize P.

The necessity follows from the obvious fact that a d-dimensional cube (or parallelepiped) requires $2d$ points to immobilize it. □

The following theorem is a generalization of Theorem 5.

Theorem 14 *Let P be a polytope in d-dimensional space. If there does not exist a linearly dependent set of d vectors $v_1, v_2, \ldots, v_d$, such that each v_i is orthogonal to some face of P then P may be immobilized with $d + 1$ points.*

Proof. Let m and T' be defined as in the proof of Theorem 13. Vectors $OA_1, OA_2, \ldots, OA_{m+1}$ then form a linearly dependent set of $m + 1$ vectors ($m + 1$ vectors in m-dimensional space). These vectors are indeed normals to some faces of P. According to the condition of the theorem it follows that $m + 1 > d$. Therefore $m = d$ and the result follows from Lemma 9. □

Corollary 3 *Any d-dimensional simple polytope needs at least d points to immobilize it.*

Proof. The proof is obvious by noting that in d dimensions for any $d-1$ points there exists an axes of rotation keeping these $d - 1$ points in place. □

The reader may verify that there exits d-dimensional non-convex simple polytopes for which d points suffice to immobilize. From Lemma 9 (the conditions of the lemma are satisfied given a random polytope) and Corollary 3 follows

Corollary 4 *Expected number of points necessary to immobilize a simple d-dimensional polytope is equal to d or $d + 1$.*

In the case of convex p, however, d points will not be sufficient to immobilize P. The region delimited by the hyperplanes tangent to P at these d points must be unbounded and P may be translated away (similarly as in Fig. 7(a) for the planar case). As a consequence we have

Corollary 5 *Expected number of points necessary to immobilize a d-dimensional convex polytope is equal to* $d+1$.

5. Conclusions and Open Problems

In this paper we studied the problems of immobilization of two types of figures: polygons (polytopes) and planar sets bounded by a Jordan curve. A number of interesting open problems follow from this work.

Theorem 3 gives a characterization of immobilization of a polygon by three points not located at its vertices. An interesting question is to extend this characterization to cover the placement of immobilization points anywhere on the boundary of the polygon.

Theorem 7 gives an $O(n \log n)$ algorithm finding three points immobilizing a given polygon having no parallel sides. However, this algorithm may output four immobilization points for some polygons having parallel sides, which may actually require only three points. For convex polygons, we can find out whether four points are actually needed and eventually output the optimal solution but it will take an $O(n^3)$ time following Theorem 2. It is an open problem to reduce the complexity of the algorithm finding the optimal number of immobilization points. For the case of non-convex polygons, it remains an open problem to recognize those that need four points to immobilize them. The problem of finding the optimal immobilizing set may be solved also by giving first the full answer to the question stated by Kuperberg[8] about the characterization of the class of polygons (convex polygons) needing four points to immobilize. The result from Theorem 6 is not a complete solution of this problem.

It seems that the Theorem 2 (and Theorem 3) may be fully extended to the case of d-dimensional polytope P. In particular, $d+1$ points should immobilize a convex polytope if and only if the $(d-1)$-dimensional hyperplanes tangent to P in these points enclose P, and the lines orthogonal to the hyperplanes at the points of immobization meet at a common point. Another extension to higher dimension was suggested by Kuperberg[8] where instead of using points, immobilization by lines, planes, etc. may be considered.

For the case of arbitrary shapes, an extension to higher dimensions may be considered an interesting area of further research. For planar shape with holes (ring is excluded), where each hole is bounded by a Jordan curve, we conjecture that three points should be always sufficient to immobilize it. Moreover, two points should be sufficient in most cases. This is obvious for polygonal shapes where the two points are placed at endpoints of the diameter of a hole.

Finally, we would like to recall a challenging question asked by Kuperberg which was not addressed in this paper. Say a set C of points not in the interior of P *captures* P if P cannot be moved to infinity without at least one point of C becoming internal to P at some time. Is the minimum number of points needed to capture P always the same as the minimum number of points needed to immobilize it? The answer is negative for general shapes (a shape of the form of letter H is an example) but remains open for convex shapes.

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