

ON SHEFFER SYMMETRIC FUNCTIONS IN THREE-VALUED LOGIC

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A Sheffer function is a function which can produce by superposition all functions of a considered set. In this paper we give an exact formula for the number of n -ary Sheffer symmetric functions in three-valued logic.

1. Preliminaries

The set of k -valued logical functions, i.e. the union of all the sets $\{f \mid E_k^n \rightarrow E_k\}$ for $n \in \omega$, where $E_k = \{0, 1, \dots, k-1\}$, is denoted by P_k .

A *Sheffer function* in P_k is a function which can produce by superposition (cf. [8,18]) all functions of P_k . A typical example of such a function is the NAND operation in the algebra of logic. In the combinatorial circuits it corresponds to a logical primitive which can construct all circuits solely by using it. A variation of the definition of Sheffer functions is the concept of “Sheffer with constants” which assumes that one can freely utilize constant-valued functions, besides a primitive function, at function composition. This is considered as a more suitable assumption in real combinatorial circuit design, since the constant-valued functions are usually obtained with no extra cost.

We will briefly present results about Sheffer functions in P_2 .

The number of Sheffer functions of n variables in P_2 is $2^{2^n-2} - 2^{2^{n-1}} - 1$ (cf. [5, 7, 18]) while the number of n -ary symmetric Sheffer functions of P_2 is 2^{n-1} for n even and $2^{n-1} - 2^{(n-1)/2}$ otherwise [23]. When n is large, 25% of the (symmetric) functions are Sheffer functions.

The number of n -variable Sheffer functions with constants in P_2 is $2^{2^n} - 2^{n+1} + n + 2 - \Psi(n)$, where $\Psi(n)$ denotes the number of monotone increasing functions of n variables [7]. The number of symmetric Sheffer functions with constants in P_2 is $2^{n+1} - n - 4$ ($n \geq 2$) [21]. When n is large, almost all (symmetric) Boolean functions are Sheffer with constants.

Sheffer functions in P_3 have been reported in the literature (cf. [12, 13, 17, 18, 25]).

Let $W_n(k)$ be the number of n -ary Sheffer functions in P_k . The following values were found by Foxley [4], Martin [12] and Wheeler [25]:

n	2	3	4	5
$W_n(3)$	3774	2110663244298	$\sim 1.3105 \times 10^{38}$	$\sim 2.5834 \times 10^{115}$

Throughout the paper we denote $\binom{n}{i}$ by $C(n, i)$.

Theorem 1 (Wheeler [25]). *The number of n -ary ($n \geq 2$) Sheffer functions in P_3 is:*

$$8 \cdot 3^{3^n-3} - 2 \cdot 3^{3^{n-1}-1} - 2^{2^n-1} 3^{3^n-2^n} - 6 \prod_{i=1}^{n-1} (2^{2^i} + 1)^{C(n, i)} + 6 \prod_{i=1}^{n-1} (2^{2^i-1} + 1)^{C(n, i)}.$$

In this paper we establish a formula for the number of symmetric n -ary Sheffer functions in P_3 .

2. Symmetric functions

A function $f: E_k^n \rightarrow E_k$ is said to be *symmetric* if

$$f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

holds for all $x_1, \dots, x_n \in E_k$ and every permutation π of $\{1, \dots, n\}$.

Any function of P_k can be generated from symmetric functions of P_k , because the symmetric function $f(x, y) = \max(x, y) + 1 \pmod k$ is a Sheffer function in P_k [24]. Thus symmetric functions can be used for representation of k -valued logical functions.

The symmetry of functions usually simplifies the synthesis of switching functions. Moreover, symmetric functions have algebraic properties which make it desirable to treat them as a separate class. Simplification, synthesis and identification problems for symmetric functions have been subject of investigations (cf. [1, 10, 14]).

For each assignment $x = (x_1, \dots, x_n)$ we define an a -vector $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{k-1}]$ ($\alpha_0 + \alpha_1 + \dots + \alpha_{k-1} = n$) associated with this assignment, where α_i is the number of i 's in the assignment. A function is symmetric if and only if the values of the function for all the assignments having the same a -vector are the same [10]. Therefore, if α is an a -vector for x then we can define $f(\alpha) = f(x)$.

A symmetric function is determined by its values on all a -vectors. The number of all a -vectors is $C(n + k - 1, k - 1)$.

Let $s_n(X)$ denote the number of n -ary symmetric functions of the set X . Then the following theorem follows immediately from the above considerations.

Theorem 2 [10]. $s_n(P_k) = k^{C(n+k-1, k-1)}$.

Therefore $s_n(P_3) = 3^{C(n+2, 2)}$.

We use the notation of functions preserving a relation (cf. [18]) to describe some subsets of P_3 . An h -ary relation ρ on E_k , $h \geq 1$, is a subset of E_k^h whose elements

are written as columns. Then for $a_1, \dots, a_h \in E_k^n$ we define

$$(a_1, \dots, a_h)^T \in \varrho \Leftrightarrow (a_{1j}, \dots, a_{hj})^T \in \varrho \text{ for all } j, 1 \leq j \leq n,$$

where $a_i = (a_{i1}, \dots, a_{in})$ for $i = 1, \dots, h$.

The relation ϱ is written as a matrix whose columns are elements of the relation ϱ .

For instance, let

$$\begin{aligned} h = 2, & & k = 3, & & \varrho = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \\ a_1 = (0, 0, 1, 2), & & a_2 = (1, 1, 2, 0), & & a_3 = (0, 1, 2, 0). \end{aligned}$$

Then

$$(a_1, a_2)^T = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \in \varrho \text{ while } (a_1, a_3) = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \notin \varrho.$$

The set of functions *preserving* ϱ (denoted by $\text{Pol } \varrho$) is defined by

$$\begin{aligned} \text{Pol } \varrho = \{f \mid (a_1, \dots, a_h)^T \in \varrho \Rightarrow (f(a_1), \dots, f(a_h))^T \in \varrho \\ \forall a_j \in E_k^n, 1 \leq i \leq h\}. \end{aligned}$$

We present the numbers of symmetric functions for some subsets of P_3 (maximal sets of P_3 , see [8]) which we will use for determining the number of Sheffer symmetric functions in P_3 .

The set of selfdual functions in P_3 is defined in the following way:

$$S = \text{Pol} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \{f \mid f(x_1 + 1, \dots, x_n + 1) = f(x_1, \dots, x_n) + 1 \pmod{3}\}.$$

Lemma 1. $s_n(S) = 3^{C(n+2,2)/3}$ for $n \not\equiv 0 \pmod{3}$ and 0 otherwise.

Proof. For $n \equiv 0 \pmod{3}$ there is no n -ary selfdual function, because from

$$a = f([\frac{1}{3}n, \frac{1}{3}n, \frac{1}{3}n]) = f(0, \dots, 0, 1, \dots, 1, 2, \dots, 2)$$

(each $\frac{1}{3}n$ times) and definition of S we get

$$a + 1 = f(1, \dots, 1, 2, \dots, 2, 0, \dots, 0) = f([\frac{1}{3}n, \frac{1}{3}n, \frac{1}{3}n]).$$

This is a contradiction. Therefore $s_n(S) = 0$ for $n \equiv 0 \pmod{3}$.

If $n \not\equiv 0 \pmod{3}$ then

$$f([\alpha_1, \alpha_2, \alpha_0]) = f([\alpha_0, \alpha_1, \alpha_2]) + 1 \pmod{3}$$

and

$$f([\alpha_2, \alpha_0, \alpha_1]) = f([\alpha_0, \alpha_1, \alpha_2]) + 2 \pmod{3}$$

and the last three a -vectors are different. Hence, the function f is determined by values of $\frac{1}{3}C(n+2, 2)$ vectors. \square

Note that the corresponding formula for the number of selfdual symmetric functions in P_2 is established in [1].

The sets T_{ij} , $\{i, j\} \subseteq \{0, 1, 2\}$ are defined as follows:

$$T_{01} = \text{Pol}(0\ 1), \quad T_{02} = \text{Pol}(0\ 2), \quad T_{12} = \text{Pol}(1\ 2),$$

i.e.

$$T_{ij} = \{f \mid \{x_1, \dots, x_n\} \subseteq \{i, j\} \Rightarrow f(x_1, \dots, x_n) \in \{i, j\}\}.$$

Lemma 2. $s_n(T_{01}) = s_n(T_{02}) = s_n(T_{12}) = 2^{n+1}3^{C(n+2,2)-n-1}$.

Proof. The result for T_{01} follows from the observation that the $n+1$ vectors $[\alpha_0, \alpha_1, 0]$ ($\alpha_0 + \alpha_1 = n$) have either 0 or 1 as values while the remaining vectors have three possible values. \square

For each set $T_i = \text{Pol}(i)$, $i \in \{0, 1, 2\}$, $f(i, \dots, i) = i$ is satisfied. Thus we have verified the following result.

Lemma 3. $s_n(T_0) = s_n(T_1) = s_n(T_2) = 3^{C(n+2,2)-1}$.

Consider the following three sets.

$$U_0 = \text{Pol}\begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{pmatrix}, \quad U_1 = \text{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix}, \quad U_2 = \text{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

Lemma 4. $s_n(U_0) = s_n(U_1) = s_n(U_2) = \prod_{\alpha_0=0}^n (2^{n-\alpha_0+1} + 1) = \prod_{i=1}^{n+1} (2^i + 1)$.

Proof. Let $\{i, j, m\} = \{0, 1, 2\}$. Then the set U_m can be defined in the following way: $f(x_1, \dots, x_n) \in U_m$ if and only if for any numbers $1 \leq p_1 < p_2 < \dots < p_s \leq n$ the following condition is satisfied: for all n -tuples (a_1, \dots, a_n) , where $a_r = m \Leftrightarrow r = p_t$ for some t , $1 \leq t \leq s$ ($1 \leq r \leq n$) the function $f(x_1, \dots, x_n)$ has either the values i and j or only the value m . For given numbers p_1, \dots, p_s there are 2^{n-s} vectors satisfying condition $a_r = m \Leftrightarrow r = p_t$ for some t . Thus there are 2^{n-s} possibilities if the values are i or j and only one if the value is m . For given s there are $C(n, s)$ s -tuples p_1, \dots, p_s which leads to the terms with \prod in the Wheeler expression for the number of Sheffer functions in P_3 .

Consider symmetric functions $f \in U_0$. For given α_0 there are $n - \alpha_0 + 1$ vectors $[\alpha_0, \alpha_1, \alpha_2]$ ($\alpha_1 + \alpha_2 = n - \alpha_0$). It follows from the definition of U_0 that the value of function for each of these vectors is either from $\{1, 2\}$ or 0. Thus there are $2^{n-\alpha_0+1} + 1$ possibilities for values of $f([\alpha_0, \alpha_1, \alpha_2])$ for given α_0 . For distinct α_0 values of f are independently defined. The assertions of the lemma follow obviously from this and from the similarity of the sets U_0 , U_1 and U_2 . \square

3. Sheffer symmetric functions in P_3

Symmetric Sheffer functions of P_3 were first investigated by Swift [22]. He showed that 90 symmetric functions of two variables (among 729 possible functions) are Sheffer functions in P_3 . We will find this number for arbitrary n .

We say that a function from P_3 has the form $/a, b, c/$ if $f(0, \dots, 0) = a, f(1, \dots, 1) = b$ and $f(2, \dots, 2) = c$.

On the basis of the functional completeness theorem for P_3 [8] a function in P_3 is a Sheffer function in P_3 if and only if it is not contained in any of 18 P_3 -maximal sets (defined in [8]). Using only this criterion is not useful for finding the number of Sheffer functions. In [9, 12] the following criterion for Sheffer functions is proved.

Theorem 3 [9, 12]. *The function $f \in P_3$ is a Sheffer function in P_3 if and only if one of the following conditions is satisfied:*

- (1) $f \notin S$ and f has the form either $/1, 2, 0/$ or $/2, 0, 1/$.
- (2) f has one of the forms $/1, 0, 0/, /1, 0, 1/, /2, 0, 0/, /2, 2, 0/, /1, 2, 1/, /2, 2, 1/$ and f is not in the sets $T_{01} \cup U_0, T_{01} \cup U_1, T_{02} \cup U_0, T_{02} \cup U_2, T_{12} \cup U_1$ and $T_{12} \cup U_2$ respectively.

Let A_1 and A_2 be the sets of functions $f \notin S$ such that f has the form $/1, 2, 0/$ for $f \in A_1$ and $/2, 0, 1/$ for $f \in A_2$.

Suppose $\{i, j, m\} = \{0, 1, 2\}$. By x^r we denote x, \dots, x (r times). Let B_{ij} be the set of functions f such that $f \notin T_{ij} \cup U_i$ and f has the form defined by $f(i^n) = j, f(j^n) = i$ and $f(m^n) = i$.

We denote by $\tau_n^{ij}(X)$ (and simply by $\tau_n(X)$ when it does not depend on i and j) the number of n -ary symmetric functions of the set X having the form defined by $f(i^n) = j, f(j^n) = i$ and $f(m^n) = i$.

Lemma 5. $\tau_n^{ij}(T_{ij} \cap U_i) = \prod_{r=1}^{n-1} (2^r + 1)$.

$$\tau_n(P_3) = s_n(P_3)/3^3, \quad \tau_n(S) = s_n(S)/3,$$

$$\tau_n^{ij}(T_{ij}) = s_n(T_{ij})/(2^{2^3}), \quad \tau_n^{ij}(U_i) = s_n(U_i)/((2^1 + 1)(2^{n+1} + 1)).$$

Proof. We prove only the first and the last assertion since the remaining results follow similarly from $f(i^n) = j, f(j^n) = i$ and $f(m^n) = i$ and the corresponding definitions immediately.

Consider the first assertion. It is clear that two vectors a and b satisfying

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 0 & 1 & 2 & j & m \\ 0 & 1 & 2 & m & j \end{pmatrix}$$

contain the same number of i 's. Thus a symmetric function f belongs to the set U_i if and only if for each two vectors having the same number of i 's either $f(a) = f(b) = i$

or $\{f(a), f(b)\} \subseteq \{j, m\}$ is satisfied. In other words, a symmetric function f is in the set U_i if and only if for each r, t, u with $0 \leq u < t \leq r \leq n$ the set $\{f(i^{n-r}, j^t, m^{r-t}), f(i^{n-r}, j^u, m^{r-u})\}$ is equal to either $\{i\}$ or $\{j, m\}$. Suppose f is a symmetric function of the set $T_{ij} \cap U_i$ satisfying $f(i^n) = j, f(j^n) = i$ and $f(m^n) = i$. Then $f(j^r, m^{n-r}) = i$ for $1 \leq r \leq n-1$ due to $f \in U_i$ and $f(j^n) = i$. For each $r, 1 \leq r \leq n-1, f(i^{n-r}, j^r)$ is equal to i or j since $f \in T_{ij}$. If $f(i^{n-r}, j^r) = i$ then $f(i^{n-r}, j^t, m^{r-t}) = i$ for each $t, 0 \leq t \leq r$, because $f \in U_i$. If $f(i^{n-r}, j^r) = j$ then from $f \in U_i$ we have $f(i^{n-r}, j^t, m^{r-t}) \in \{j, m\}$ for $0 \leq t < r$. Thus for given r there are $2^r + 1$ possible choices of values in vectors with $(n-r)$ i 's. Bearing in mind the independence of the values of f for different i we prove the assertion.

We turn our attention to the last assertion. $f(i^n) = j$ is one of $2^1 + 1 = 3$ possible choices for the value. $f \in U_i$ and $f(j^n) = i$ imply $f(j^r, m^{n-r}) = i$ for all $r, 0 \leq r \leq n$. This is one of the $2^{n+1} + 1$ possible choices for values of these $n + 1$ vectors. \square

Now we can state our main result.

Theorem 4. *The number of n -ary symmetric Sheffer functions of P_3 is*

$$8 \cdot 3^{C(n+2,2)-3} - 2^n 3^{C(n+2,2)-n-1} - (2^{n+1} - 4) \prod_{i=1}^{n-1} (2^i + 1)$$

for $n \equiv 0 \pmod{3}$,

$$8 \cdot 3^{C(n+2,2)-3} - 2^n 3^{C(n+2,2)-n-1} - (2^{n+1} - 4) \prod_{i=1}^{n-1} (2^i + 1) - 2 \cdot 3^{C(n+2,2)/3-1}$$

for $n \not\equiv 0 \pmod{3}$.

Proof. Let $\sigma(n)$ be the number of n -ary Sheffer symmetric functions. Then we have

$$\sigma(n) = s_n(A_1) + s_n(A_2) + s_n(B_{01}) + s_n(B_{10}) + s_n(B_{02}) + s_n(B_{20}) + s_n(B_{12}) + s_n(B_{21}).$$

From the definitions of the above sets it follows:

$$s_n(A_1) = s_n(A_2) = \tau_n(P_3) - \tau_n(S),$$

$$s_n(B_{ij}) = \tau_n(P_3) - \tau_n^{ij}(T_{ij}) - \tau_n^{ij}(U_i) + \tau_n^{ij}(T_{ij} \cap U_i)$$

for $0 \leq i, j \leq 2, i \neq j$.

The assertion of Theorem 4 follows from this remark and Lemmas 1-5 obviously. \square

In Table 1 we give the numbers $s_n(P_3)$ and $\sigma(n)$ for $n \leq 5$.

When n is large, $\sigma(n)/s_n(P_3) \rightarrow 8/27 = 0.296296\dots$. Note that this ratio for the set of all functions in P_3 is also $8/27$ as remarked in [25].

Table 1.

n	0	1	2	3	4	5
$s_n(P_3)$	3	27	729	59049	14348907	10460353203
$\sigma(n)$	0	0	90 [22]	11484	3302802	3041829126

4. Concluding remarks

We give some brief results about Sheffer functions in P_k .

Let $Q_n(k) = k^{-k^n} W_n(k)$. $Q_n(k)$ denotes the proportion of n -ary Sheffer operations in P_k . Graham [6] noted that the proportion of functions f with two variables satisfying $f(x, x) \neq x$ for every $x \in \{0, 1, \dots, k-1\}$ (i.e. preserving no singleton subset of $\{0, 1, \dots, k-1\}$) is at most $(1-1/k)^k$. Bajramov [2] studied a similar question for a fixed $k > 2$ and $n \rightarrow \infty$, obtaining the results that $Q_n(k)$ is asymptotically equal to $(1-1/k)^k$ as $n \rightarrow \infty$. Davies [3] proved that $Q_n(k) \rightarrow 1/e$ as $k \rightarrow \infty$ (where $e = 2.718281 \dots$). In other words, for a large k , the probability that an n -ary operation on $\{0, 1, \dots, k-1\}$ picked at random be Sheffer is close to $1/e = 0.367879 \dots$. In fact, Davies proved much more by showing that, as $k \rightarrow \infty$, for a randomly selected n -ary function f in P_k we can decide with a probability of almost 1 whether f is Sheffer by testing the conditions $f(x, \dots, x) \neq x$ for all $x \in \{0, 1, \dots, k-1\}$ only.

Muzio [13] has shown that 15 201 out of 19 683 two-place functions from P_3 are complete with constants. Lowesmith [11] has found that 333 two-place symmetric functions among 729 possible functions are complete with constants. It still remains an open problem about the number of Sheffer with constants n -ary and symmetric n -ary functions in P_3 .

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