On the Maximum Size of the Terms in the Realization of Symmetric Functions

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abstract

The symmetric functions of m-valued logic have a sum-product (i.e. max-min) representation whose terms are sums of fundamental symmetric functions (FSFs). These sums may be simplified if they contain adjacent SFSs. This naturally leads to the combinatorial problem of determining the maximum size $M(m,n)$ of adjacent-free sets of n-variable SFSs. In 1990 Muzio related $M(m,n)$ to a special graph $F(m,n)$. Continuing in this direction we give a simple closed formula for $M(m,n)$ and then deduce that for large m or large n the largest non-simplifiable set of n-variable SFSs consists of approximately one half of all possible FSFs proving thus also all the conjectures from Muzio's 1990 paper.

1 Introduction

Let m be an integer, $m \geq 2$ and $E_m := \{0, 1, \ldots, m-1\}$. For a positive integer n an n-ary *m-valued (logic) function* is a map $f : E_m^n \to E_m$ assigning a value from E_m to every n-tuple $(x_1, \ldots, x_n)$ over E_m. We denote such a function by $f(x_1, \ldots, x_n), f(\boldsymbol{x})$ or just f. The function f is *symmetric in the variables x_i and x_j* if

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n).$$

holds for all $x_1, \ldots, x_n \in E_m$. A function is *symmetric* (also called totally symmetric by some authors) if it is symmetric in each pair of its variables. The *base string* (*a-vector* by some authors) associated to $(x_1, \ldots, x_n) \in E_m^n$ is $\alpha_0 \ldots \alpha_{m-1}$, with $\alpha_i := |\{j : x_j = i\}|$ for all $0 \leq i \leq m-1$. (Clearly $\alpha_0 + \alpha_1 + \ldots + \alpha_{m-1} = n$.)

Every symmetric function f is clearly determined by a map f' from the set V_{mn} of base strings into E_m. For example, if $m = 3$ and $n = 2$ the value $f(1,2) = f(2,1)$ is determined by the value associated to the base string 011. A function is *decisive* if it assumes only the two values 0 (the least value) and $m-1$ (the greatest value).

In this paper we are interested in the following special symmetric functions. A *fundamental symmetric function* (FSF) is a decisive symmetric function $f_{\alpha_0 \ldots \alpha_{m-1}}$ which assumes $m-1$ only for the unique base string $\alpha_0 \ldots \alpha_{m-1}$. For example, if $m = 3$ and $n = 2$, $f_{011}(x_1, x_2) = 2$ if $x_1 = 1, x_2 = 2$ or $x_1 = 2, x_2 = 1$, and $f_{011}(x_1, x_2) = 0$ for the other values of x_1 and x_2. Clearly any decisive symmetric function can be expressed as a sum (disjunction) of FSFs.

The importance of the FSF's is that they can be easily used in the realization of an arbitrary symmetric function as described in [1, 4]. We explain this briefly.

Any symmetric function can be expressed by a normal form as

$$f(\boldsymbol{x}) = g_1(\boldsymbol{x}) + 2g_2(\boldsymbol{x}) + \ldots + (m-1)g_{m-1}(\boldsymbol{x}),$$

where each $g_i(\boldsymbol{x})$ is a decisive symmetric function, $x + y = \max(x, y)$ (sum, disjunction) and $xy = \min(x, y)$ (product, conjunction). Each of these decisive functions can be realized as a sum of FSF's. A detailed discussion of an algorithm to choose an appropriate realization for an arbitrary symmetric function can be found in [4].

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Usually each FSF is realized using some building blocks. To explain the building blocks we need a few notations. Define *simple symmetric functions* by

$$\begin{aligned}\tau_1 &:= x_1 + x_2 + \ldots + x_n,\\ \tau_2 &:= x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n,\\ &\vdots\\ \tau_n &:= x_1x_2\ldots x_n.\end{aligned}$$

Further define the *characteristic functions* by

$$c_i(x) := \begin{cases} m-1 & \text{if } x = i,\\ 0 & \text{if } x \neq i.\end{cases}$$

for $i = 0, 1, \ldots, m-1$. Finally put $C_{ij} := c_i(\tau_j)$. Now the n-variable functions C_{ij} are the primary building blocks to realize FSFs. It is known [1] that each FSF can be expressed as a product of C_{ij}s. For example, $f_{2032} = C_{06}C_{25}C_{23}C_{32}$. A disjunction of FSF's can be simplified whenever there are two adjacent (cf the definition in the next section) FSF's in the disjunction. For example, $f_{3022} + f_{2122} + f_{2032} = C_{06}C_{24}C_{23}C_{32}$ instead of the disjunction of the three terms. It is also known that no two non-adjacent FSF's in the disjunction can be combined to make a simpler term [3].

Muzio investigated the question of the maximum possible size k of a disjunction $f_1 + f_2 + \ldots + f_k$ of FSFs non-amenable to simplification. He estimated the size for the three and four valued cases and presented conjectures for the general m-valued case [3].

Inspired by [3] we investigate the maximum cardinality $M(m,n)$ of a set of n-variable mutually non-adjacent FSFs. We present a closed formula for $M(m,n)$ and prove all the conjectures from [3] including that the maximum size rapidly approaches one half of the total number of FSF's when n or m becomes large.

We approach this problem through bipartite graphs in the subsequent sections.

2 Preliminaries

For a given positive integer $m > 0$ and an integer $n \geq 0$ the set of *base strings* (cf Introduction) is

$$V_{mn} := \{\alpha_0 \ldots \alpha_{m-1} \mid \quad \alpha_i \geq 0 \text{ for all } 0 \leq i < m \text{ and } \alpha_0 + \ldots + \alpha_{m-1} = n\}.$$

For $n = 0$ $V_{m,0} = \{(0\ldots0)\}$ (a singleton). It is well known that

$$|V_{mn}| = \binom{m+n-1}{n} = \binom{m+n-1}{m-1}$$

and so the number $P(m,n)$ of FSF's in m values and n variables is given by:

Lemma 1. [3, 5]

$$P(m,n) = \binom{m+n-1}{m-1}.$$

Call two base strings $\alpha_0\ldots\alpha_{m-1}$ and $\beta_0\ldots\beta_{m-1}$ *adjacent* if $|\alpha_i - \beta_i| = |\alpha_{i+1} - \beta_{i+1}| = 1$ for some $0 \leq i < m-1$ while $\alpha_j = \beta_j$ otherwise. For example, the two strings 13022 and 12122 are adjacent since $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1 + 1$, $\alpha_2 = \beta_2 - 1$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$.

Define the *adjacency graph* $F(m,n)$ V_{mn} in the following way [3]. The vertex set is V_{mn}. Two vertices are adjacent if they are adjacent base strings. Fig. 1 (taken from [3]) shows the graphs $F(m,n)$ for small values of m and n.

Lemma 2. *The only pendent vertices (i.e. vertices of degree one) of* $F(m,n)$ *are* $0\ldots0n$ *and* $n0\ldots0$

Proof. Obviously the two vertices are only vertices that have only one adjacent base string. □

The *independence number* $\nu(G)$ of a graph G is the cardinality of the largest set of mutually non-adjacent vertices in G. Clearly

$$M(m,n) = \nu(F(m,n)).$$

3 Properties of the adjacency graph $F(m,n)$

The $m-1$-*dimensional grid* P_{n+1}^{m-1} is the graph with the set of vertices $E_{n+1}^{m-1} = \{0, 1, \ldots, n\}^{m-1}$ in which two vertices $(x_1, x_2, \ldots, x_{m-1})$ and $(y_1, y_2, \ldots, y_{m-1})$ are adjacent if and only if $\sum_{i=1}^{m-1}|x_i - y_i| = 1$. It is well known that P_{n+1}^{m-1} is a connected bipartite. First we show that $F(m,n)$ is isomorphic to a part of the grid P_{n+1}^{m-1}.

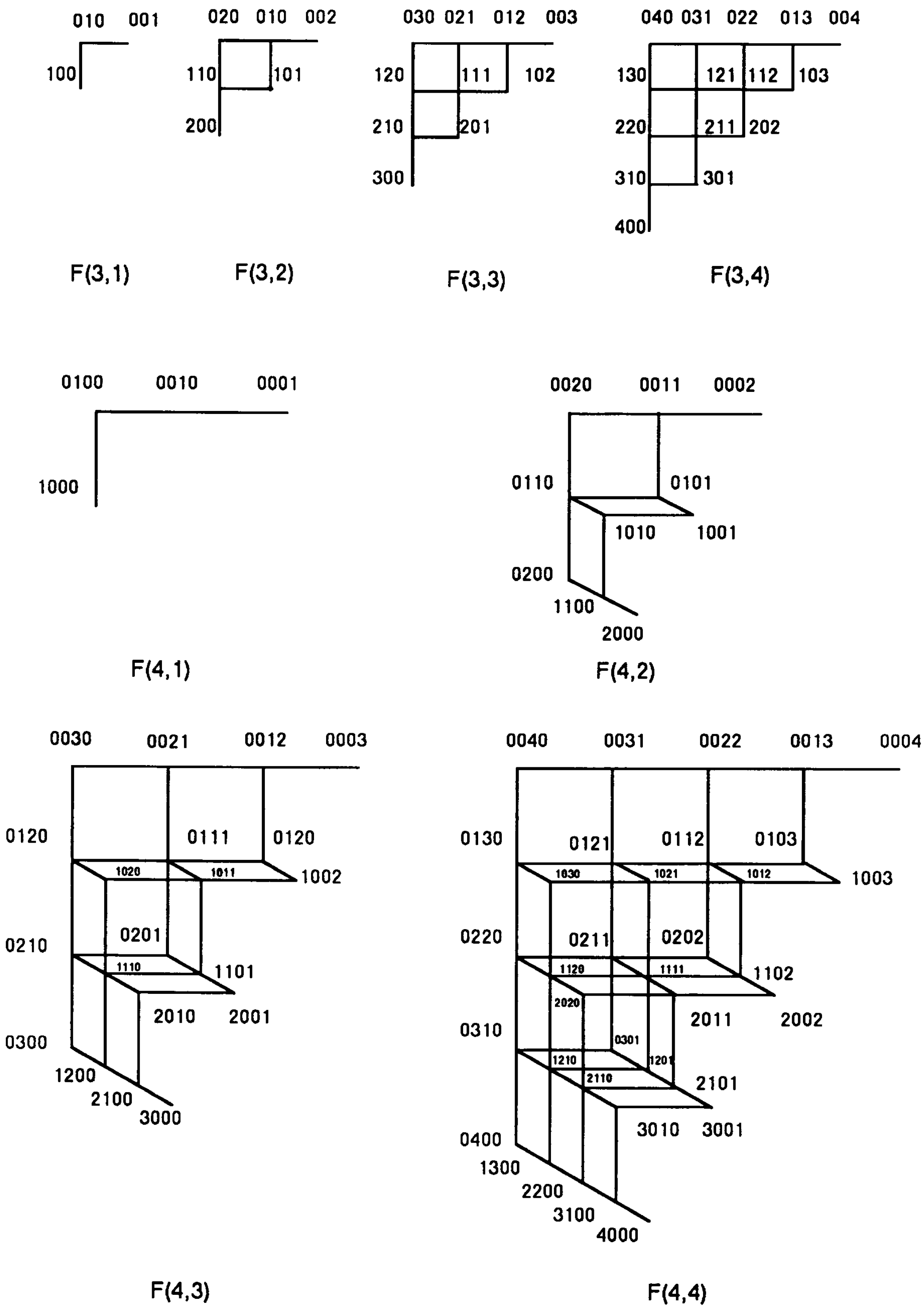

Figure 1: Adjacency graphs, $m = 3, 4$.

Lemma 3. *The graph $F(m,n)$ is a connected bipartite graph.*

Proof. Connectedness is easy to verify. Now we show that $F(m,n)$ is isomorphic to a subgraph of the grid P_{n+1}^{m-1}. For any base string $\alpha = \alpha_0 \dots \alpha_{m-1}$, define the partial sum string $g(\alpha) := s_1 s_2 \dots s_{m-1}$ by setting $s_1 := \alpha_0$, $s_2 := \alpha_0 + \alpha_1, \dots, s_{m-1} := \alpha_0 + \alpha_1 + \dots \alpha_{m-2}$. Clearly $0 \le s_1 \le \dots \le s_{m-1} \le n$ and so g maps $F(m,n)$ into P_{n+1}^{m-1}. Put $S_{mn} := g(V_{mn})$ and denote $G(m,n)$ the subgraph of P_{n+1}^{m-1} induced by S_{mn}. It is easy to check that map g is an isomorphism between the graphs $F(m,n)$ and $G(m,n)$. Since P_{n+1}^{m-1} is bipartite, $F(m,n)$ is also a bipartite. □

Note 4. $S(m,n)$ is the set of integer solutions of the inequalities $0 \le x_1 \le \dots \le x_{m-1} \le n$. In other words it is the set of the grid or lattice points in the polytope determined by the following m hyperplanes: $x_1 = 0, x_{m-1} = n$, $x_1 = x_2, \dots, x_{m-2} = x_{m-1}$. Fig. 2 indicates $G(3,n)$ and $G(4,n)$.

The vertices of a bipartite graph can be colored by two colors, say *black* and *white*, so that adjacent vertices have different colors. We can choose the colors so that the set of B_{mn} of all black vertices and the set W_{mn} of all white vertices of the graph $F(m,n)$ satisfy $|W_{mn}| \ge |B_{mn}|$.

The *distance* between two vertices u and v is the length (the number of edges) of the shortest path between them. Clearly, the vertices u and v of a bipartite graph are of the same color if and only if the distance between them is even. In the sequel we are interested in the independence number of $F(m,n)$. The case of $F(m,1)$ (one dimensional grid) suggests that we separate cases according whether m and n are even or odd.

Lemma 5. *The selfmap $\phi : V_{mn} \to V_{mn}$ defined by setting $\phi(\alpha_0, \dots, \alpha_{m-1}) := \alpha_{m-1} \dots \alpha_0$ is an automorphism of $F(m,n)$.*

Proof. Obvious. □

Note that since V_{mn} is a bipartite an automorphism ϕ maps either both W_{mn} and B_{mn} onto themselves or each onto the other.

Lemma 6. *If m is even and n odd then*

$$|B_{mn}| = |W_{mn}| = \frac{1}{2}\binom{m+n-1}{n}.$$

Proof. The only pendent vertices of $F(m,n)$ are $n0\dots0$ and $0\dots0n$. The distance between these two vertices is $(m-1)n$, so it is odd and the two vertices are of different colors. The automorphism ϕ of $F(m,n)$ from Lemma 5 satisfies $\phi(n0\dots0) = 0\dots0n$. Hence $\phi(B_{mn}) = W_{mn}$, i.e. $|B_{mn}| = |W_{mn}| = (1/2)|V_{mn}|$. □

For the independence number we need matchings. A *matching* of a graph G is a set of mutually disjoint edges of G. A *maximal matching* of G is a matching with the largest possible number of edges. A *perfect matching* of a graph G is a matching which covers all the vertices of G. In other words, it is a subset G' of its edges such that each vertex of G belongs exactly to one edge from G'. Hence, if an p-vertex graph G has a perfect matching M then p is even and M contains $p/2$ edges. Obviously, each perfect matching of G is a maximal matching of G.

Lemma 7. *If m is even and n odd then $F(m,n)$ has a perfect matching.*

Proof. By induction on $m+n$. The statement can be easily verified for $F(2,n)$ and $F(m,1)$. Suppose that it is true for $F(m-2,n)$ and $F(m,n-2)$, m even n odd. Partition the set V_{mn} of all vertices of $F(m,n)$ into three subsets:

$$\begin{aligned} A &= \{\alpha_0 \dots \alpha_{m-1} \in V_{mn} \mid \alpha_0 = \alpha_1 = 0\}, \\ B &= \{\alpha_0 \dots \alpha_{m-1} \in V_{mn} \mid \alpha_0 = 0 < \alpha_1 \text{ or } \alpha_0 = 1\}, \\ C &= \{\alpha_0 \dots \alpha_{m-1} \in V_{mn} \mid \alpha_0 > 1\}. \end{aligned}$$

Consider the subgraphs F_A, F_B and F_C of $F(m,n)$ induced by the subsets A, B and C, respectively. It is easy to see that F_A is isomorphic to $F(m-2,n)$, and F_C is isomorphic to $F(m,n-2)$. By the induction hypothesis both F_A and F_D have perfect matchings. On the other hand, the graph F_B also has a perfect matching consisting of the edges $\{0\alpha_1 \dots \alpha_{m-1}, 1(\alpha_1 - 1)\alpha_2 \dots \alpha_{m-1}\}$. Hence $F(m,n)$ has a perfect matching. □

Lemma 8. *Let G be a bipartite graph with color classes B and W such that $|B| \le |W|$. If G has a maximal matching of cardinality $|B|$ then $\nu(G) = |W|$.*

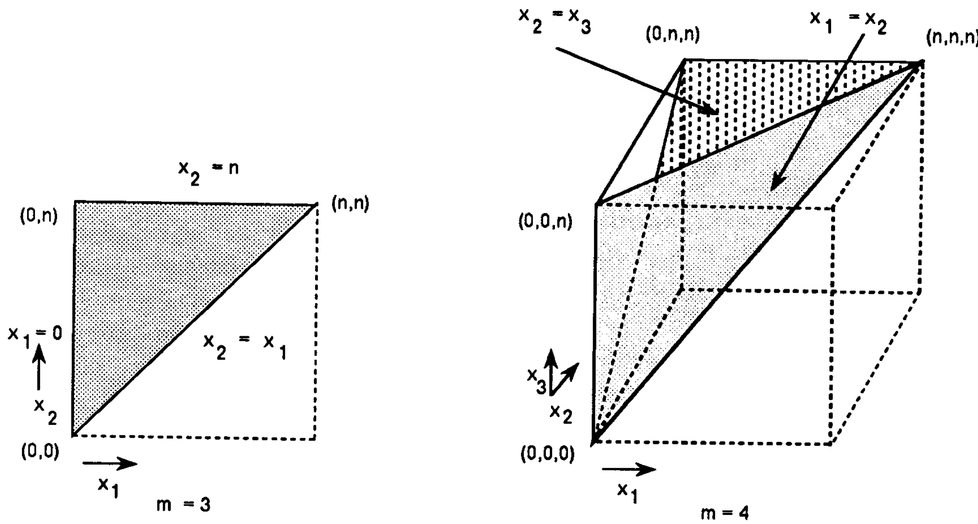

Figure 2: The polytopes for $G(3,n)$ and $G(4,n)$.

Proof. Suppose that the largest independent set of vertices of G is $C = R \cup T$, where $R \subset B$, $T \subset W$. If we replace each vertex $r \in R$ by the vertex $t \in T$ which is paired to r in the maximal matching, we obtain the independent set of the same cardinality which is a subset of W. On the other hand, W is obviously an independent set. □

Combining Lemmas 6–8 we have a part of our main result (the case of m even n odd) given as Theorem 12 in the next section. So in the sequel we consider the remaining cases, i.e. either m is odd or n is even.

Lemma 9. *If either m is odd or n is even then (i) $|W_{mn}| > |B_{mn}|$ and the two pending vertices belong to W_{mn}, and (ii) $|W_{mn}| = |W_{m,n-1}| + |W_{m-1,n}|$ and $|B_{mn}| = |B_{m,n-1}| + |B_{m-1,n}|$.*

Proof. By induction on $m + n$. The statement can be verified directly for $F(3,n)$ and $F(m,0)$ (notice that $F(m,0)$ is the trivial graph without edges).

Put $A := \{\alpha_0 \ldots \alpha_{m-1} \in V_{mn} | \alpha_0 = 0\}$ and $B := V_{mn} \setminus A$. The subgraph F_A of $F(m,n)$ induced by the set A is isomorphic to $F(m-1,n)$, while the subgraph F_B induced by the set B is isomorphic to $F(m,n-1)$. There are three cases.

(i) Let both m and n be even. Then by the induction hypothesis

$$|B_{m-1,n}| < |W_{m-1,n}| \qquad (1)$$

and the two pending vertices $0 \ldots 0n$ and $0n0 \ldots 0$ of F_B belong to the larger color class $W_{m-1,n}$. According to Lemma 6 we have

$$|B_{m,n-1}| = |W_{m,n-1}| \qquad (2)$$

in F_B. There is a path $0n0 \ldots 0$, $1(n-1)0 \ldots 0$, $\ldots$, $n0 \ldots 0$ of length n connecting $0n0 \ldots 0$ and $n0 \ldots 0$ in $F(m,n)$. Since we assume n even, both $0n0 \ldots 0$ and $n0 \ldots 0$ are in the same color class of $F(m,n)$. Both F_A and F_B are connected subgraphs of $F(m,n)$ and so their color classes are included in the two color classes of $F(m,n)$. Together we obtain that both $n0 \ldots 0$ and $0n0 \ldots 0$ are in the same color class of $F(m,n)$ which is W_{mn} due to (1). So we have

$$\begin{aligned} W_{mn} &= W_{m,n-1} \cup W_{m-1,n} \text{ and} \\ B_{mn} &= B_{m,n-1} \cup B_{m-1,n}. \end{aligned}$$

Hence we have

$$|W_{mn}| = |W_{m,n-1}| + |W_{m-1,n}| \text{ and} \qquad (3)$$

$$|B_{mn}| = |B_{m,n-1}| + |B_{m-1,n}|. \qquad (4)$$

Therefore $|W_{mn}| > |B_{mn}|$ from (1) and (2).

(ii) The proof is quite similar if both m and n are odd.

(iii) Let m be odd and n even. Then by the induction hypothesis we have (1) and

$$|B_{m,n-1}| < |W_{m,n-1}|. \quad (5)$$

The pending vertex $0 \ldots 0n$ is also a pending vertex of F_A. Since $F_A \cong F(m-1,n)$ by the induction hypothesis it belongs to the larger color class $W_{m-1,n}$ of F_A. Similarly, the vertex $n0 \ldots 0$ belongs to the larger color class $W_{m,n-1}$ of F_B. The distance between the two vertices (in $F(m,n)$) is $(m-1)n$. Since it is even, the two vertices belong to the same color class of $F(m,n)$. This class is $W_{m-1,n} \cup W_{m,n-1}$ and hence the larger color class of $F(m,n)$. So we conclude (3) and (4), and so $|W_{mn}| > |B_{mn}|$ from (1) and (5). □

Theorem 10. *The graph $F(m,n)$ has a maximal matching of cardinality $|B_{mn}|$.*

Proof. For m even and n odd, the statement follows from Lemma 6 and Theorem 12. Thus let either m be odd or n even. The proof will be by induction on $m+n$.

The statement can be easily verified for $F(m,1)$. Let A and B be as in the proof of previous lemma and consider the induced subgraphs F_A isomorphic to $F(m-1,n)$ and F_B isomorphic to $F(m,n-1)$. By the induction hypothesis, each of these two subgraphs has a maximal matching of cardinality $|B_{m-1,n}|$ and $|B_{m,n-1}|$, respectively. Since either m is odd or n is even, from Lemma 9(ii) it follows that

$$|B_{mn}| = |B_{m-1,n}| + |B_{m,n-1}|,$$

and the union of these maximal matchings of F_A and F_B is a maximal matching of $F(m,n)$ with cardinality $|B_{mn}|$. □

Corollary 11. *If either m is odd or n is even then*

$$M(m,n) = |W_{mn}|.$$

Proof. Follows from Lemma 8 and Theorem 10. □

4 The maximal size of a disjunction of FSFs

In [3] Muzio gave a constructive method to define an appropriate set FSFs that appears to be maximal, though this is not formally proved. On the basis of it, he estimated the respective formulas for $M(m,n)$ for $m=3$ and $m=4$. In the same paper he also stated three conjectures concerning $M(m,n)$ for arbitrary m (two recursive formulas and a closed formula for $M(m,n)$ for m even and n odd). Here we shall prove all his conjectures. Besides we shall present some other results including an exact formula for $M(m,n)$ for arbitrary nonnegative integers m and n.

Having in mind that $M(m,n)$ is equal to the independence number $\nu(F(m,n))$ of the adjacency graph $F(m,n)$, the following five statements follow immediately from the previous lemmas.

Theorem 12. *If m is even and n odd then*

$$M(m,n) = \frac{1}{2}P(m,n) = \frac{1}{2}\binom{n+m-1}{n}.$$

Lemma 13. *If m is even and n odd then*

$$\begin{aligned} M(m,n) &= P(m,n-1) + M(m-1,n) - M(m,n-1) \\ &= P(m-1,n) + M(m,n-1) - M(m-1,n). \end{aligned}$$

Proof. From Corollary 11 $M(m-1,n) = |W_{m-1,n}|$ and $M(m,n-1) = |W_{m,n-1}|$ and so $P(m,n-1) - M(m,n-1) = |B_{m,n-1}|$ and $P(m-1,n) - M(m-1,n) = |B_{m-1,n}|$. Then the lemma follows from Lemma 6 and Theorem 12. □

Lemma 14. *If either m is odd or n even then*

$$M(m,n) = M(m-1,n) + M(m,n-1).$$

Proof. From Corollary 11. □

Lemma 15. *If either m is odd or n even then*

$$M(m,n) > \frac{1}{2}P(m,n).$$

Proof. From Lemma 9(i) and Corollary 11. □

Corollary 16.

$$M(m,n) \geq \frac{1}{2}\binom{n+m-1}{n}.$$

Now we show two more recursive formulas.

Lemma 17. $M(2k-1,n) = \sum_{i=0}^{n} M(2k-2,i)$.

Proof. The proof is by induction on n. The statement is true for $n=0$. According to Lemma 14

$$M(2k-1,n) = M(2k-2,n) + M(2k-1,n-1).$$

On the other hand, by the induction hypothesis

$$M(2k-1,n-1) = \sum_{i=0}^{n-1} M(2k-2,i)$$

and the statement follows. □

Similarly, we have the following lemma.

Lemma 18. $M(m,2k) = \sum_{i=1}^{2k} M(m-1,i)$.

Proof. The proof is by induction on k and taking into account from Lemma 14 that

$$\begin{aligned} M(m,2k) &= M(m-1,2k) + M(m,2k-1) \\ &= M(m-1,2k) + M(m-1,2k-1) + M(m,2k-2). \end{aligned}$$

□

The values for $M(m,n)$ for small values of m and n are given in Table 1 taken from [3] which may be verified from Theorem 12 and Lemma 14 (we use $M(m,0)=1$ and $M(1,n)=1$). If we compare this table with Table 2 for the numbers $P(m,n)$ we can see that the following Hypothesis 4 of [3] is quite reasonable:

$$\frac{M(m,n)}{P(m,n)} \to 0.5 \text{ as } m \text{ become large.}$$

Now we derive an exact formula for $M(m,n)$ from which the above conjecture follows immediately.

Theorem 19. *If either m is odd or n is even then*

$$M(m,n) = \frac{1}{2}\binom{n+m-1}{n} + \frac{1}{2}\binom{\lfloor n/2\rfloor + \lfloor (m-1)/2\rfloor}{\lfloor n/2\rfloor}.$$

Table 1: $M(m,n)$ [3]: the maximal numbers of non-adjacent fundamental symmetric functions.

	number of values (m)								
n	1	2	3	4	5	6	7	8	9
0		1	1	1	1	1	1	1	1
1	1	1	2	2	3	3	4	4	5
2	1	2	4	6	9	12	16	20	25
3	1	2	6	10	19	28	44	60	85
4	1	3	9	19	38	66	110	170	255
5	1	3	12	28	66	126	236	396	651
6	1	4	16	44	110	236	472	868	1519
7	1	4	20	60	170	396	868	1716	3235
8	1	5	25	85	255	651	1519	3235	6470

Table 2: $P(m,n)$ [3]: the numbers of fundamental symmetric functions.

	number of values (m)								
n	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36	45
3	1	4	10	20	35	56	84	120	165
4	1	5	15	35	70	126	210	330	495
5	1	6	21	56	126	252	462	792	1287
6	1	7	28	84	210	462	924	1716	3003
7	1	8	36	120	330	792	1716	3432	6435
8	1	9	45	165	495	1287	3003	6435	12870

Proof. Consider three cases.

(i) $m = 2k+1$, $n = 2l$. By Lemma 14

$$\begin{aligned} M(2k+1,2l) &= M(2k+1,2l-1) + M(2k,2l) \\ &= M(2k+1,2l-2) + M(2k,2l-1) + M(2k,2l-1) \\ &\quad + M(2k-1,2l) \\ &= 2M(2k,2l-1) + M(2k+1,2l-2) + M(2k-1,2l) \\ &= \ldots. \end{aligned}$$

We continue this process of developing each term $M(m,n)$ if and only if either m is odd or n is even. We stop at the moment when each summand in that development is one of the forms $M(m,0)$, $M(1,n)$, $M(2k,2l-1)$. Note that $M(m,0) = M(1,n) = 1$ for any m and n.

Now, the question is how many times $M(m,0)$ appears in that development. It is easy to see that $M(m,0)$ does not appear at all if m is even. On the other hand,

$M(2r+1,0)$ appears $\binom{k-r+l-1}{l-1}$ times for all $1 \leq r \leq k$. Thus the total number of summands of the form $M(m,0)$ is $\binom{k+l-1}{l}$, since $\binom{k+l-2}{l-1}+\binom{k+l-3}{l-1}+\dots\binom{l-1}{l-1}=\binom{k+l-1}{l}$.

Similarly, we can see that the summand $M(1,n)$ appears if and only if n is even and that $M(1,2t)$, $1 \leq t \leq l$ appears $\binom{k-t+l-1}{k-1}$ times. Hence the total number of summands of the form $M(1,n)$ is $\binom{k+l-2}{k-1}+\binom{k+l-3}{k-1}+\dots+\binom{k-1}{k-1}=\binom{k+l-1}{k}=\binom{k+l-1}{l-1}$.

Denote A the sum of all terms of the form $M(m,n)$ for m even and n odd in the above development, each summand being taken as many times as it appears. Then $M(2k+1,2l)=A+\binom{k+l-1}{l}+\binom{k+l-1}{l-1}$, i.e.

$$M(2k+1,2l)=A+\binom{k+l}{l}. \quad (6)$$

If we apply the same development for the number $P(2k+1,2l)$, according to Theorem 12, it follows that

$$P(2k+1,2l)=2A+\binom{k+l}{l}. \quad (7)$$

From (6) and (7) it follows that $2M(2k+1,2l)-P(2k+1,2l)=\binom{k+l}{l}$. Hence

$$M(2k+1,2l)=\frac{1}{2}\binom{2k+2l}{2l}+\frac{1}{2}\binom{k+l}{l}. \quad (8)$$

(ii) $m=2k+1,\ n=2l+1$. From Lemma 14 it follows that $M(2k+1,2l+1)=M(2k,2l+1)+M(2k+1,2l)$, and now, according to Theorem 12 and (8) we obtain

$$M(2k+1,2l+1) = \frac{1}{2}\binom{2k+2l+1}{2l+1}+\frac{1}{2}\binom{k+l}{k}. \quad (9)$$

(iii) $m=2k$, $n=2l$. Similarly from Lemma 14, Theorem 12 and (8), we obtain

$$M(2k,2l) = \frac{1}{2}\binom{2k+2l-1}{2l}+\frac{1}{2}\binom{k+l-1}{l}. \quad (10)$$

Equations (8), (9) and (10) give the formula given in the theorem. □

Now from Lemmas 1, 6 and Theorem 19 we obtain

Corollary 20.

(a) $$\lim_{m\to\infty}\frac{M(m,n)}{P(m,n)}=\frac{1}{2} \text{ and}$$

(b) $$\lim_{n\to\infty}\frac{M(m,n)}{P(m,n)}=\frac{1}{2}. \quad \square$$

Lastly, by Note 4 we have:

Note 21. The independence number for the adjacent graph of the grid points of the $m-1$-dimensional polytope given by the m equations $x_1=0$, $x_{m-1}=n$, $x_1=x_2,\dots,x_{m-2}=x_{m-1}$ is also given by $M(m,n)$ in Theorem 19.

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