ON SPECTRA OF MANY-VALUED LOGIC SYMMETRIC FUNCTIONS

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ABSTRACT

Many-valued logic symmetric functions appearing in various applications are investigated from the standpoint of determining the number of n-ary functions belonging to a considered set (called the spectrum of the set). We give respective spectra of k-valued functions that are ρ-symmetric, self-dual and self-dual ρ-symmetric, where ρ is a partition of $\{1,\ldots,n\}$. We prove that there exist self-dual totally symmetric n-ary k-valued logic functions if and only if the greatest common divisor of k and n is equal to 1. A test for detecting self-dual symmetry property is also described. We also give respective spectra of k-valued symmetric functions that are threshold, multithreshold, monotone and unate (for the monotone and unate functions $k = 3$ only).

1. Introduction

The symmetry of logical functions usually simplifies the synthesis of switching functions. Moreover, symmetric functions have algebraic properties which make it desirable to treat them as a separate class. Simplification, identification and synthesis problems (cf. [7,14,20,24]), circuit or computational complexity [9,10,12,17,32] and construction of decision trees [21,31] for symmetric (mostly switching) functions have been subjects of investigations. Recently, many-valued logic is considered as one of the solutions to overcome various limitations of information density of VLSI (cf. [6]). The many-valuezation of the signals increases information density of a circuit without increasing the size or complexity of the circuit. Many-valued logic symmetric functions and their algebraic properties have been also investigated (cf. [3,4,20,24,30]). In this paper we consider various many-valued symmetric functions appearing in practical applications from algebraic standpoint and present in most cases exact numbers of n-ary functions belonging to them (called *spectra*). Among them we include many-valued functions that are partially symmetric, self-dual and self-dual partially symmetric functions, and multi-valued symmetric functions that are threshold, multithreshold, monotone and unate. A graphical interpretation of three-valued monotone symmetric functions is also presented since it is directly related to the unate and threshold functions.

Self-dual multi-valued functions are considered from functional completeness theory point of view [13,16]. Mullin [22] initiated the study of symmetric self-dual functions by presenting an exact condition for a symmetric Boolean function to be self-dual. The spectra of self-dual totally as well as partially symmetric Boolean functions are shown in [2]. In [29] the spectrum of three-valued self-dual symmetric functions are given. Here we prove that there exist a k-valued n-ary symmetric self-dual function if and only if the greatest common divisor (gcd) of n and k is equal to 1. The spectrum of totally symmetric k-valued logical functions is given in [14]. By using a technique different from one used in [2] we establish formulas for the spectra of ρ-symmetric (i.e. partially symmetric) k-valued functions, self-dual symmetric and ρ-symmetric functions for a given partition ρ of $\{1,\ldots,n\}$.

The 2-valued threshold and unate functions have been extensively studied during the past twenty years for their hardware superiority, their theoretical appeal due to their nature (continuous versus discrete, i.e. linearity versus 0-1), and their relation to 0-1 linear programming. Synthesis of symmetric switching functions using threshold logic elements are discussed (cf. [11]). Enumeration of these functions has been tried in e.g. [23,1].

2. Preliminaries

Let k be a fixed positive integer. Set $E_k := \{0, 1\ldots, k-1\}$. The set of *k-valued logic functions*, i.e. maps

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$$\{f : E_k^n \to E_k \text{ for } n = 1, 2, \ldots\}$$

is denoted by P_k. A function $f(x_1, \ldots, x_n) \in P_k$ is said to be *symmetric* or *totally symmetric* if

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \qquad (1)$$

holds for all $x_1, \ldots, x_n \in E_k$ and every permutation π on $\{1, \ldots, n\}$. A function $f(x_1, \ldots, x_n)$ is said to be *partially symmetric* or *ρ-symmetric* if and only if the set $\{x_1, \ldots, x_n\}$ is divided into subsets (partition ρ) such that the function is totally symmetric in each subset. It is shown that a function is partially symmetric if and only if there exists a transposition σ satisfying (1). This result from [2] for Boolean functions is valid also for k-valued logical functions.

For each assignment $x := (x_1, \ldots, x_n)$ we define an *a-vector* $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{k-1}]$ $(\alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} = n)$ associated with this assignment, where α_i is the number of i's in the assignment. A function is symmetric if and only if the values of the function for all the assignments having the same a-vector are the same [28]. Therefore, if α is a-vector for x then we can define $f(\alpha) := f(x)$. There is a one-to-one correspondence between the set of all a-vectors and the set of all monotone vectors $\{(x_1, \ldots, x_n) | \; x_1 \leq x_2 \ldots x_{n-1} \leq x_n\}$. A symmetric function is determined by its values on all a-vectors, i.e. on the its values on all monotone vectors. The number of the monotone vectors is $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$. Since each of these vectors is mapped into $\{0, 1, \ldots, k-1\}$ the number of n-ary symmetric k-valued functions is given [14] by

$$k^{\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}.$$

Therefore, there are 2^{n+1} and $3^{\begin{pmatrix} n+2 \\ 2 \end{pmatrix}}$ n-ary symmetric functions in the sets of Boolean and 3-valued functions, respectively.

Let $S := \{f \mid f(x_1 + 1, \ldots, x_n + 1) = f(x_1, \ldots, x_n) + 1 \pmod{k}\}$ be the set of *self-dual* functions in k-valued logic. Note that there are some other types of self-dual functions (cf. [16]). It is easy to check that there are $k^{k^{n-1}}$ n-ary k-valued self-dual functions. We will determine the number of self-dual symmetric functions.

A k-valued n-ary function f is *threshold* or *linearly separable* if there exist a real n-vector w and a real $(k-1)$-vector T such that for $i = 0, \ldots, k-1$ and $x \in E_k^n$ we have

$$f(x) = i \Leftrightarrow T_i \leq wx < T_{i+1} \qquad (2)$$

where $T_0 = -\infty$, $T_k = \infty$ and $wx = w_1x_1 + \ldots + w_nx_n$, a real sum (some authors require $T_i < wx < T_{i+1}$ in (2)). Thus f is threshold if the k^n lattice or integer points of the n-dimensional cube $[0, k-1]^n$ are separated by $k-1$ parallel hyperplanes into layers so that f is constant on each layer and its values do not decrease as we move in a direction perpendicular to the hyperplanes. The vectors w and T may be assumed to be integers. Clearly f is a monotone function with respect to the order $0 < 1 \cdots < k-1$ (cf. [26,15]).

Multithreshold Boolean functions are defined as a generalization of Boolean threshold functions. We give a definition of k-valued multithreshold functions from [8]. A k-valued m-multithreshold function f is defined by setting

$$f(x_1, \ldots, x_n) := H(e),$$

where $e = \sum_{j=1}^{n} w_j x_j$ (called *excitation*), w_j is a weight associated with input x_j, $H(e)$ is the output value when the excitation is e and defined in the following way (f, H, $x_j \in \{0, 1, \ldots, k-1\}$). The function H has real $m+1$ threshold $T_0, T_1, \ldots, T_{m-1}, T_m$ $(T_0 = -\infty, T_m = \infty, t_i < t_{i+1}, i = 0, \ldots, m-1)$ which determine the values of H. Clearly $m \leq e_{max} \leq (k-1)\sum_{=1}^{n} |w_j|$. Thus f is multithreshold if the former lattice is separated by $m-1$ parallel hyperplanes into layers so that f is constant on each layer. However, there is no monotonicity conditions.

3. ρ-symmetric functions

We are going to determine the number of partially symmetric functions.

For each partially symmetric function f there exists a partition ρ on $\{1, 2, \ldots, n\}$ such that for a transposition $\sigma = (i, j)$, f is ρ-symmetric if and only if i and j are in the same equivalence class of ρ.

For instance, let $f(x_1, x_2, x_3, x_4, x_5) = (x_1 \vee x_3)x_2x_4 \vee x_5$. Then $(1,3) \in \rho$, $(2,4) \in \rho$ and $\rho = \{\{1,3\}, \{2,4\}, \{5\}\}$. If ρ is the partition which corresponds to a partially symmetric function then f is said to be *ρ-symmetric*. The corresponding a-vectors for ρ-symmetric functions can be composed from a-vectors of their totally symmetric parts in the following way: $\alpha = [\alpha^1, \ldots, \alpha^s]$, where α^i is the a-vector with respect to variables in the i-th class induced by ρ $(1 \leq i \leq s)$.

For example, a-vector for partition ρ defined by $\{\{1,3\}, \{2,4\}, \{5\}\}$ and a 3-valued vector $(0,1,1,1,2)$ is [[1,1,0], [0,2,0] and [0,0,1]].

Theorem 3.1. *The number of ρ-symmetric functions is* $k^{\prod_{i=1}^{s} \binom{r_i+k-1}{k-1}}$, *where $r_1, \ldots, r_s$ is the numbers of elements in s equivalence classes of partition ρ on $\{1, \ldots, n\}$ $(r_1 + \ldots + r_s = n)$.*

Proof. Sum of the members in α^i is r_i. Thus, recalling the one-to-one correspondence between a-vectors and monotone vectors, the number of a-vectors α^i is given by $\binom{r_i+k-1}{k-1}$. Therefore, the number of a-vectors α is $\prod_{i=1}^{n} \binom{r_i+k-1}{k-1}$, which implies the assertion of the theorem. □

For $k = 2$ this result is obtained in [2] using algebraic techniques. Note that for $s = 1$ and $s = n$ the last formula gives the numbers of whole n-ary symmetric functions and whole n-ary functions, respectively.

4. Self-dual symmetric functions

Let us consider self-dual symmetry conditions. From the definition of S follows

$$f(x_1+i, x_2+i, \ldots, x_n+i) = f(x_1, x_2, \ldots, x_n)+i \pmod{k}, \tag{3}$$

i.e. $f([\alpha_0, \ldots, \alpha_{k-1}])$

$$= f([\alpha_i, \alpha_{i+1}, \ldots, \alpha_{k-1}, \alpha_0, \ldots, \alpha_{i-1}]) - i \pmod{k} \tag{4}$$

for each i, $0 \leq i \leq k-1$. Thus we classify all a-vectors so that the set of all a-vectors $\{[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{k-1}, \alpha_0, \ldots, \alpha_{i-1}] | i = 0, \ldots, k-1\}$ forms exactly one class. The value for any a-vector in a class then uniquely determines the values for the remaining $k-1$ a-vectors in this class. Using (3) and (4) we will determine all symmetric self-dual logical functions.

Theorem 4.1. *If greatest common divisor* $\gcd(n,k)$ *is greater then 1 then there is no n-ary k-valued symmetric self-dual function.*

Proof. Let $n = dn_1$, $k = dk_1$ and $d > 1$. Consider an a-vector $\alpha = [(0^{k/d-1}, n/d)^d]$, where x^m denotes $x \cdots x$ (m times). The a-vector α contains k members sum of which is n. This is, in fact, the vector $a = ((k/d-1)^{n/d}, (2k/d-1)^{n/d}, \ldots, (k-1)^{n/d})$. Let $b := a + k/d$ (a componentwise addition), i.e. $b = ((2k/d-1)^{n/d}, (3k/d-1)^{n/d}, \ldots, (k-1)^{n/d}, (k/d-1)^{n/d})$. From (3) we have $f(x_1 + k/d, \ldots, x_n + k/d) = f(x_1, \ldots, x_n) + k/d$. Substituting a for x, we have $f(b) = f(a) + k/d \pmod{k}$. Therefore $f(a) \neq f(b)$. However a and b correspond to the same a-vector α. This contradicts the symmetry of the function. □

For example, let $k = 4$ and $n = 6$. Then $f([0,3,0,3]) = f(1,1,1,3,3,3) = f(0,0,0,2,2,2)+1 = f(3,3,3,1,1,1)+2 = f([0,3,0,3]) + 2 \pmod{4}$ and we have a contradiction.

Theorem 4.2. *If* $\gcd(n,k) = 1$ *then the number of n-ary symmetric self-dual functions in P_k is* $n^{\binom{n+k-1}{k-1}/k}$.

Proof. It is sufficient to prove that if $\gcd(n,k) = 1$, then

$$[\alpha_0, \ldots, \alpha_{k-1}] = [\alpha_i, \alpha_{i+1}, \ldots, \alpha_{k-1}, \alpha_0, \ldots, \alpha_{i-1}] \tag{5}$$

is satisfied for none of i, $1 \leq i \leq k-1$. Suppose that the equation (5) is valid for some $1 \leq i \leq k-1$. Then (5) is valid for each number ri $(r \geq 1)$ (recall that ri denotes $ri \pmod{k}$). These numbers form a subgroup of $(E_k, +)$ and from group theory it follows that for the minimal number d in this subgroup $d \mid k$ is satisfied. Thus, from (5) and the definition of a-vector, we have $n = (\alpha_0 + \ldots + \alpha_{d-1})k/d$ and therefore $\gcd(n,k) \geq d$. Thus d=1 and hence $\alpha_0 = \alpha_1 = \ldots = \alpha_{k-1}$. We conclude $n = k\alpha_0$ which contradicts to $\gcd(n,k) = 1$. □

Corollary 4.1. *If p is a prime number then there are $p^{\binom{n+p-1}{p-1}/p}$ n-ary self-dual symmetric functions for $n \not\equiv 0 \pmod{p}$ and 0 otherwise.*

Let $r_1, \ldots, r_s$ be the numbers of the elements in the equivalence classes of a partition ρ $(r_1 + \ldots + r_s = n)$. Then we can state the following theorem.

Theorem 4.3. *The number of self-dual ρ-symmetric functions is*

$$k^{1/k \prod_{i=1}^{s}} \binom{r_i + k - 1}{k - 1}$$

if $\gcd(r_1, \ldots, r_s, k) = 1$ *and 0 otherwise.*

Proof. A similar proof analogous to the last two theorems can be given by using the following a-vector: $\alpha := [[(0^{k/d-1}, r_1/d)^d], \ldots, [(0^{k/d-1}, r_s/d)^d]]$ for $\gcd(r_1, \ldots, r_s, k) > 1$. In the other case it is sufficient to show that (5) is not satisfied for any a-vector α^i with respect to symmetry group containing r_i variables $(1 \le i \le s)$ and thus for any a-vector α of a given ρ-symmetric function. We omit the complete proof to avoid repeating the same considerations. □

By using presented results a procedure for identification of self-dual symmetric functions can be described. A test for detection of self-dual and symmetric properties is one which firstly examines symmetries of given function using a well-known technique [7,14,20] and then checks $\gcd(r_1, \ldots, r_s, k)$. If the greatest common divisor is 1 then (4) can be applied for totally or partially symmetric functions in order to certify self-dual property. Note that it is sufficient to verify (4) only for a representative (on the right-hand side of (4)) in each class of a-vectors. As a representative we can choose, for instance, the minimal one in the lexicographic order.

5. Symmetric monotone and unate functions

A function f is *non-decreasing in x_i* if from $x_i' \le x_i''$ follows

$$f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \le f(x_1, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_n)$$

where variables take the values from $\{0, 1, \ldots, k-1\}$. Similarly we can define *non-increasing in x_i* function by changing latter "$\le$" into "$\ge$". A function which is either non-decreasing or non-increasing in x_i is said to be *unate in* x_i. Clearly if a function is both non-decreasing and non-increasing in x_i then x_i is a fictitious variable in f.

A function which is either non-decreasing, non-increasing or unate in all variables is said to be a *non-decreasing, non-increasing* or *unate* function, respectively. A function is *monotone* if it is either non-decreasing or non-increasing.

Symmetric unate or threshold functions will be proved to be monotone. It is well-known that there are $n+2$ symmetric n-ary non-decreasing Boolean functions (cf. [2]). We will present ternary non-decreasing symmetric functions of n variables because they are related to the symmetric unate and threshold functions.

Theorem 5.1. *The number of n-ary 3-valued non-decreasing symmetric functions is* $\binom{2n+3}{n+1}$.

Proof. A graphical representation of non-decreasing symmetric 3-valued functions can be given as follows. For an a-vector $[\alpha_0, \alpha_1, \alpha_2]$ let $(x, y) := (\alpha_0, \alpha_2)$; $\alpha_1 = n - x - y$. Then all a-vectors of f can be represented as a set L_n of non-negative integer points in R^2 space. The set L_n consists of integer-points $\{(x, y) \mid x \ge 0, y \ge 0, x + y \le n \ (x, y \text{ integers})\}$. **Figure 1** gives these points for $n = 4$.

We define a relation $\le$ in the set of points (x, y) in the following way:

$$(x', y') \le (x'', y'') \Leftrightarrow x' \ge x'' \ \wedge \ y' \le y''.$$

For each symmetric function f we define a function F on L_n by setting $F(x, y) := f([x, n - x - y, y])$. It is easy to check that f is non-decreasing function if and only if

$$(x', y') \le (x'', y'') \ \Rightarrow F(x', y') \le F(x'', y'') \qquad (6)$$

is satisfied for each $(x', y'), (x'', y'') \in L_n$. Thus the number of n-ary 3-valued non-decreasing symmetric functions is equal to the number of non-decreasing functions F on L_n (F is non-decreasing function if it satisfies (6)).

Let A_i be the set of points $(x, y) \in L_n$ satisfying $F(x, y) = i$ $(i = 0, 1, 2)$. These sets completely determine the function F.

If F is a non-decreasing function, then the sets A_i, $i = 0, 1, 2$ can be separated at most by the following two paths each of which is composed of $n + 1$-length segment (Figure.1): Starting from the point (-1/2,-1/2), each segment

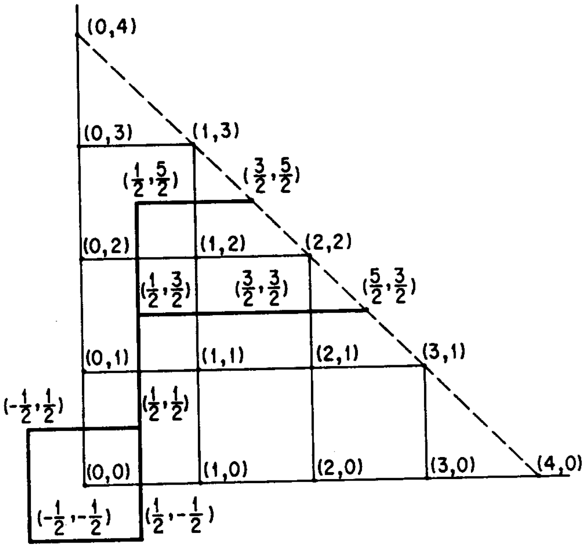

Figure 1: Representation of 3-valued monotone symmetric functions.

of the both paths connects the last point (u,v) of the predecessor segment with point either $(u,v+1)$ or $(u+1,v)$. We denote a segment connecting the points (u,v) and $(u,v+1)$ by 1 and segment connecting the points (u,v) and $(u+1,v)$ by 0.

For instance the two paths on Figure 1 can be denoted by the sequences 10110 and 01100. Sequences $a_1a_2\ldots a_{n+1}$ and $b_1b_2\ldots b_{n+1}$ $(a_i, b_i \in \{0,1\},\ 1 \le i \le n+1)$ separating the sets A_0, A_1, A_2 define a non-decreasing function if and only if $a_1+a_2+\ldots+a_j \le b_1+b_2+\ldots+b_j$ is satisfied for each j, $1 \le j \le n+1$. Let $c_i := 2b_i + a_i$. Thus $c_i \in \{0,1,2,3\}$. We call this sequence *characteristic sequence* if and only if the number of 2's in the sequence $c_1\ldots c_j$ is greater than or equal to the number of 1's in the same sequence for each j, $1 \le j \le n+1$. Then F is a non-decreasing function if and only if $c_1\ldots c_{n+1}$ is a characteristic sequence. Thus the number of n-ary 3-valued non-decreasing symmetric functions is equal to the number $\gamma(n+1)$ of characteristic sequences $c_1\ldots c_{n+1}$.

Let *m-number* $m(c,n)$ for a characteristic sequence $c = c_1\ldots c_n$ be the difference between the number of 2's and the number of 1's in the sequence and let $\gamma_i(n)$ be the number of characteristic sequences c of the length n for which $m(c,n) = i$ is satisfied. Then $\gamma_{-1}(n+1) = \gamma_{n+1}(n) = \gamma_{n+2}(2) = 0$.

We have

$$\gamma_i(n+1) = \gamma_{i-1}(n) + 2\gamma_i(n) + \gamma_{i+1}(n)$$

for $0 \le i \le n+1$. Because $m(c,n+1) = i$ if and only if one of the following conditions is satisfied:

$m(c,n) = i-1$ and $c_{n+1} = 2$ (m-number increases),
$m(c,n) = i$ and $c_{n+1} = 0$ or 3 (m-number is the same),
$m(c,n) = i+1$ and $c_{n+1} = 1$ (m-number decreases).

By induction on n we can prove that $\gamma_i(n) = \binom{2n}{n+i} - \binom{2n}{n+i+2}$ for $0 \le i \le n$. Proof is omitted since it is standard one. Then $\gamma(n) = \sum_{i=0}^{n} \gamma_i(n) = \binom{2n+1}{n}$. Finally $\gamma(n+1) = \binom{2n+3}{n+1}$ is the number of n-ary 3-valued non-decreasing symmetric functions. □

Note that the characteristic sequences can be interpreted as paths in the square plane tessellation starting from (0,0) so that a point $(-1,y)$ will never be reached. Therefore there are one-to-one correspondence between n-ary 3-valued non-decreasing symmetric functions and such paths with $n+1$ segments.

The set of threshold functions has been shown to be a proper subset of the set of unate functions. The problem of unate function enumeration is similar to the problem of monotone function enumeration. Consider symmetric unate functions.

Theorem 5.2. *A symmetric function is unate function if and only if it is a monotone function.*

Proof. Clearly monotone functions are unate functions. Suppose f is non-decreasing in x_i. We prove that f is non-decreasing in x_j for each j, $1 \le j \le n$. Without loss of generality we can assume $i = 1, j = 2$ and $x_2' \le x_2''$. Then $f(x_1, x_2', x_3, \ldots, x_n) = f(x_2', x_1, x_3, \ldots, x_n) \le f(x_2'', x_1, x_3, \ldots, x_n) = f(x_1, x_2'', x_3, \ldots, x_n)$ which verifies our assertion. The same result holds for non-increasing f. Thus a unate function is either non-decreasing or non-increasing for all variables. □

Thus there are $2n+2$ Boolean unate symmetric functions.

Corollary 5.1. *The number of 3-valued unate symmetric functions is* $2\binom{2n+3}{n+1}-3$.

Proof. The number of n-ary symmetric non-decreasing 3-valued logical functions is $\binom{2n+3}{n+1}$. Only the constant functions 0,1,2 are non-increasing and non-decreasing at the same time among them. □

Enumeration of k-valued unate functions is an open combinatorial problem.

6. Symmetric threshold functions

We are interested in the k-valued threshold functions realizable with integer weights and integer thresholds only (thus we assume some scale factor on the unit of E_k to guarantee this). We will determine all symmetric threshold functions in P_k.

Theorem 6.1. *The number of n-ary k-valued symmetric threshold functions is*

$$2\binom{kn-n+k}{k-1}-k.$$

Proof. Suppose f is a symmetric threshold function. Then from (2) we conclude

$$\begin{array}{lcccl} T_i & \le & w_1x_1+\cdots+w_nx_n & < & T_{i+1}, \\ T_i & \le & w_1x_2+w_2x_3+\cdots+w_{n-1}x_n+w_nx_1 & < & T_{i+1}, \\ T_i & \le & w_1x_3+w_2x_4+\cdots+w_{n-1}x_1+w_nx_2 & < & T_{i+1}, \\ & & \vdots & & \\ T_i & \le & w_1x_n+w_2x_1+w_3x_2+\cdots+w_nx_{n-1} & < & T_{i+1}, \end{array}$$

because $f(x_j,x_{j+1},\ldots,x_n,x_1,x_2,\ldots,x_{j-1})=f(x_1,\ldots,x_n)=i$, $1\le j\le n$ (definition of symmetric functions). Sum of the above inequalities gives

$$nT_i \le (\sum_{j=1}^{n} w_j)(\sum_{j=1}^{n} x_j) < nT_{i+1}, \text{ i.e.}$$

$$T'_i \le \quad x_1+\cdots+x_n \quad < T'_{i+1} \text{ or} \qquad (7)$$

$$T'_i \le -(x_1+\cdots+x_n) < T'_{i+1} \qquad (8)$$

where $T'_i = nT_i/|(\sum_{j=1}^{n} w_j)|$.

Therefore, the value of symmetric threshold function depends only on the sum $x_1+\cdots+x_n$. It follows from $0\le x_1+\cdots+x_n\le(k-1)n$ that there are $(k-1)n+1$ possible values for $x_1+\cdots+x_n$ and we have only to settle $k-1$ values $T'_1\le T'_2\cdots T'_{k-2}\le T'_{k-1}$ among them (recall that we assume all T'_i integers). It is easy to see that this threshold configuration can be done in $\binom{kn-n+k}{k-1}$ different ways. Bearing in mind positive case (7) and negative case (8) we conclude that the number of symmetric threshold functions is $2\binom{kn-n+k}{k-1}-k$ because only the constant functions can be expressed in both (7) and (8) ways. □

In [15] a similar construction is used in order to prove that for a Boolean partially symmetric function there are separating hyperplanes with equal coefficients w_j for variables in the same symmetry group.

Corollary 6.1. *All monotone non-decreasing or non-increasing Boolean symmetric functions are threshold.*

Proof. For $k=2$ there are $2(n+2)-2=2n+2$ n-ary symmetric Boolean threshold functions. This coincides with the number of Boolean monotone non-decreasing and non-increasing symmetric functions. □

For $k=3$ there are $2(2n+3)(n+1)-3$ 3-valued symmetric n-ary threshold functions. But the number of n-ary symmetric monotone nondecreasing 3-valued functions is $\binom{2n+3}{n+1}$ and we conclude that there are monotone symmetric nonthreshold functions. The same conclusion can be stated for any $k>2$.

Theorem 6.2. *There are $k^{(k-1)n+1}$ n-ary k-valued symmetric multithreshold functions.*

Proof. Similarly as for threshold symmetric functions we can prove that a symmetric function is multithreshold if and only if from $x_1+\cdots+x_n=y_1+\cdots+y_n$ follows $f(x_1,\ldots,x_n)=f(y_1,\ldots,y_n)$. Since $0\le x_1+\cdots+x_n\le(k-1)n$ we conclude that there are at most $(k-1)n+1$ different values for $x_1+\cdots+x_n$ and hence there are $k^{(k-1)n+1}$ symmetric n-ary multithreshold functions in k-valued logic. □

All 2^{n+1} symmetric Boolean functions are multithreshold because $x_1 + \cdots + x_n$ is the number of 1's (i.e. a-number). For $k > 2$ there exist symmetric functions that are not multithreshold. An example is a function satisfying $f(0,1,2,x_3,\ldots,x_n) \neq f(1,1,1,x_3,\ldots,x_n)$.

A similarity between symmetric and multithreshold functions are discussed in [5]. In [25] an algorithm based on a possibility of assigning equal weights (coordinates w_i) to variables in a symmetry group of a given Boolean function (for totally or partially symmetric Boolean functions) for multithreshold element synthesis is presented. But, no proof for this possibility is given.

7. Concluding remarks

Symmetric functions are of special practical importance, because any function can be constructed from symmetric "gates" (e.g. $\mathrm{NAND}(x,y)$) and only symmetric gates are usually used in practice (nonsymmetry complicates the situation, e.g. delays and so on). We have determined the spectra (the numbers of n-ary functions)) for several kinds of symmetric functions which arise in applications. They include ρ-symmetric functions, self-dual functions, self-dual ρ-symmetric functions, symmetric monotone or unate 3-valued functions, symmetric threshold functions and symmetric multithreshold functions.

Some other studies of symmetric functions include the following. The size of PLA's (programmable logic arrays) with two-bit decoders which are sufficient to realize a symmetric function is far less than that for an arbitrary function [27]. Symmetric property of the minimal covering problem is used to reduce the search space in branch-and-bound method [33]. As an application of this technique to the design of minimal PLA's, the minimal sum derivation of symmetric switching functions is considered there. Realization of symmetric Boolean monotone non-decreasing functions (called voting functions) by networks have been studied in literature (cf. [15]).

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