CLASSIFICATIONS AND BASIS ENUMERATIONS IN MANY-VALUED LOGICS - A SURVEY -

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ABSTRACT

The functions in a closed subset C of P_k may be classified by their membership in the maximal subsets of C, leading to a natural classification of bases of C into equivalence classes (called aggregates). This paper presents clasifications of functions and enumerations of aggregates for the sets P_2, P_3, their maximal sets, several modifications of algebra of logic and some closed subsets of P_k. Sheffer functions for these sets are also considered. Finally this paper states some problems and provides an extensive bibliography.

1. Functional completeness problem and classification in P_k

Let k be a fixed positive integer and let $E_k = \{0,1...,k-1\}$. The set of *k-valued (logical) functions* (i.e. maps $f : E_k^n \to E_k$ for $n = 1,2,\ldots$) is denoted by P_k. A subset F of P_k is said to be *closed* if it contains all *compositions* (or *superpositions*) of its members (cf. [21,67]). More precisely, for $f \in P_k$ m-ary and $g \in P_k$ n-ary put $r := m+n-1$ and define the r-ary $h = f * g$ by setting

$$h(x_1,\ldots,x_r) := f(g(x_1,\ldots,x_n),x_{n+1},\ldots,x_r) \qquad (1)$$

for all $x_1,\ldots,x_r \in E_k$. Further, for $m > 1$ put

$$\begin{aligned}(\varsigma f)(x_1,\ldots,x_m) &:= f(x_2,\ldots,x_m,x_1),\\ (\tau f)(x_1,\ldots,x_m) &:= f(x_2,x_1,x_3,\ldots,x_m),\\ (\triangle f)(x_1,\ldots,x_{m-1}) &:= f(x_1,x_1,\ldots,x_{m-1})\end{aligned}$$

for all $x_1,\ldots,x_m \in E_k$ and $\varsigma f = \tau f = \triangle f = f$ for f unary ($m = 1$). Now F is closed if $f * g$, ςf, τf and $\triangle f \in F$ whenever $f,g \in F$.

For closed sets F and H such that $F \subset H$ (proper inclusion), F is *H-maximal* set if there is no closed set G such that $F \subset G \subset H$ (i.e. H covers F in the set of closed sets ordered by $\subseteq$). A subset X of H is *complete in H* if H is the least closed set containing X. In the sequel we always assume that H has the following property: Each proper closed subset of H extends to an H-maximal set. (This property need not hold, in fact there is an example of such P_8-maximal set [45, 97]). It is known that then there are finitely many H-maximal sets, say $H_1,\ldots,H_m$. It is known and easy to see that a subset of functions in H is complete in H if and only if it is contained in no H-maximal set (completeness condition)(cf. [67]).

Investigations of completeness and related topics, usually called *functional completeness problems*, are mathematically importnat and have a wide range of applications including their direct relationship to logical circuit design. A complete set X in H is called a *base of H* if no proper subset of X is complete in H. The *rank* of a base is the number of its elements. A function f is *Sheffer* for H if $\{f\}$ is a base (of rank 1) of H. In other words, a function f is Sheffer for H if and only if every $g \in H$ is a composition of a finite number of copies of f. Clearly f is Sheffer for H if and only if it belongs to no H-maximal sets. Typical examples of binary functions (i.e. Boolean two-variable functions) that are Sheffer for P_2 are the Sheffer (or better Nicode's) strokes NAND and NOR of the algebra of logic. A Shefer stroke describes the "operation" of a two-input one-output gate (or element) G such that every Boolean function $f(x_1,\ldots,x_n)$ may be represented by the output of a combinatorial (i.e. feedback-free) circuit with inputs $x_1,\ldots,x_n$ and built solely from copies of G (however, the number of the gates needed for the representation may be

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large). A comprehensive survey on Sheffer functions can be found in [67]. A variation of the definition of completeness is the concept of "*complete with constants*", abbreviated *c-complete*, which assumes that for composition besides f one can freely utilize constant-valued functions. More precisely, let Q denote the set of unary constant functions from H. A subset X of H is c-complete in H if $X \cup Q$ is complete in H. This makes sense in real combinatorial circuits, since the constant-valued functions (i.e. constant signals) are usually obtained with no extra cost. In particular, f is c-Sheffer for H means $\{f\}$ is c-complete in H.

The functions from H may be classified by their membership in the H-maximal sets. Let $H_1, \ldots, H_m$ be the H-maximal sets. As mentioned above, a subset X of H is complete in H if and only if for each $1 \leq i \leq m$ there is $f_i \in X \cap (H \setminus H_i)$ (the f_i's need not be distinct). This leads to the following: Define the map $\varphi : H \rightarrow \{0,1\}^m$ by setting $\varphi(f) := a_1 \ldots a_m$ where $a_i = 0$ if $f \in H_i$ and $a_i = 1$ if $f \notin H_i$ (here $a_1 \ldots a_m$ stands for the more customary $(a_1 \ldots a_m)$ or $< a_1 \ldots a_m >$). We call $\varphi(f)$ the *characteristic vector* of f. We put $f \equiv g$ if $f, g \in H$ have the same characteristic vector, i.e. if $\varphi(f) = \varphi(g)$. Clearly $\equiv$ is an equivalence relation on H (it is the standard kernel of φ) and so it partitions H into pairwise disjoint nonempty sets called *(equivalence) classes.* Note that for $f \equiv g$ we have either $f, g \in H_i$ or $f, g \notin H_i$ for all $i = 1, \ldots, m$. We write AB for $A \cap B$, A^1 for A and A^0 for $H \setminus A$ (A, B subsets of H). Clearly each class is of the form $H_1^{a_1} \ldots H_m^{a_m}$ where $(1-a_1) \ldots (1-a_m)$ is a characteristic vector (i.e. it is a non-empty set of the form $H_1^{a_1} \ldots H_m^{a_m}$ with $a_1 \ldots a_m \in \{0,1\}^m$). If $f \in X \subseteq H$ and $f \equiv g$ then clearly X is complete in H if and only if $(X \setminus \{f\}) \cup \{g\}$ is complete in H. In other words, it suffices to study the completeness in H up to the equivalence $\equiv$. It is easy to see that $X \subseteq H$ is complete in H if and only if $s = \sum_{f \in X} \varphi(f)$ (the usual componentwise sum of real m-vectors) is positive (i.e. has all coordinates ≥ 1). Once we know all the characteristic vectors, we can find all complete sets in H and all bases by a direct combinatorial check (which may be done by a simple computer program). If to $a_1 \ldots a_m \in \{0,1\}^m$ we associate $A = \{i : a_i = 1\}$ and if $A_1, \ldots, A_l$ are the subsets of $\{1, \ldots, m\}$ corresponding to the characteristic vectors, the completeness problem is reduced to the listing of $\{A_1, \ldots, A_l\}$ covering $\{1, \ldots, m\}$ and the basis problem to the listing of such coverings which are irredundant (no proper subset covers $\{1, \ldots, m\}$). The study of classes also provides information on the closed sets which are the intersections of families of H-maximal sets, which is of independent interest (e.g. for $H = P_3$ with one exception the least nontrivial intersections are all minimal clones). The characteristic vectors can also be applied to seek the set of classes of functions which makes a given incomplete set complete.

For the description of closed sets containing all projections (i.e. functions e_i^n defined by setting $e_i^n(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in E_k$, $1 \leq i \leq n$) called *clones,* we need the following essential concept of "functions preserving a relation" [cf. 67].

Let $h \geq 1$. An h-ary *relation* ρ on E_k is a subset of E_k^h (i.e. a set of h-tuples over E_k) whose elements are written as columns. Given row n-vectors $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ ($i = 1, \ldots, h$) we write $(\mathbf{a}_1, \ldots, \mathbf{a}_h)^T \in \rho^n$ to indicate that $(a_{1j}, \ldots, a_{hj})^T \in \rho$ for all $j = 1, \ldots, n$, where T denotes the transpose (this means that the $h \times n$ matrix with rows $\mathbf{a}_1, \ldots, \mathbf{a}_h$ has all columns in ρ). We say that an n-ary $f \in P_k$ *preserves* ρ if

$$(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_h))^T \in \rho \text{ whenever } (\mathbf{a}_1, \ldots, \mathbf{a}_h)^T \in \rho^n.$$

Then the set of functions *preserving* ρ is denoted by Pol ρ:

$$Pol\ \rho = \{f \mid (\mathbf{a}_1, \ldots, \mathbf{a}_h)^T \in \rho^n \Rightarrow (f(\mathbf{a}_1), \ldots, f(\mathbf{a}_h))^T \in \rho\}.$$

It is known that each $Pol\ \rho$ is a clone and conversely to each clone C there are relations $\rho_1, \rho_2, \ldots$ such that $Pol\ \rho_1 \supseteq Pol\ \rho_2 \supseteq \ldots \supseteq C$ and $C = \bigcap_{i=1}^{\infty} Pol\ \rho_i$. In particular, if H is a clone, then all H-maximal sets are of the form $Pol\ \rho$ for some relation ρ.

Throughout this paper by $x + y$ and xy we denote $x + y \pmod{k}$ and $xy \pmod{k}$, respectively. Intersection of sets $X_1, \ldots, X_r$ will be denoted by $X_1 \ldots X_r$. Finally, let x^r denote $x \ldots x$ (r times) whenever x is a component of a vector. A relation ρ is often written as matrix whose columns list all the elements of ρ (in some order).

Let $f \in P_k$ be n-ary. The *dual* f^+ of f is defined by setting

$$f^+(x_1,\dots,x_n) := f(x_1+1,\dots,x_n+1)+1$$

for all $x_1,\dots,x_n \in E_2$. We say that f is *self-dual* if $f = f^+$. Again, we say that f is *linear* if there are $a_0,\dots,a_n \in E_k$ so that $f(x_1,\dots,x_n) = a_0 + a_1x_1 + \cdots + a_nx_n$ holds for all $x_1,\dots,x_n \in E_k$. Finally, f is *monotone* (isotone, order-preserving) if $f(x_1,\dots,x_n) \le f(y_1,\dots,y_n)$ whenever $x_1 \le y_1,\dots,x_n \le y_n$ (here $\le$ denotes the natural order $0 < 1 < \dots < k-1$).

Theorem 1.1. [62] *P_2 has exactly the following 5 maximal sets:*

$$\begin{aligned}
T_0 &= Pol(0) = \{f \mid f(0,\dots,0) = 0\}\\
&\quad \textit{(the set of functions preserving 0)},\\
T_1 &= Pol(1) = \{f \mid f(1,\dots,1) = 1\}\\
&\quad \textit{(the set of functions preserving 1)},\\
S &= Pol\begin{pmatrix} 01\\ 10\end{pmatrix} \ \textit{(the set of selfdual functions)},\\
L &= Pol(\{(a,b,c,d)^T \in E_2^4 \mid a+b = c+d\})\\
&\quad \textit{(the set of linear functions)},\\
M &= Pol\begin{pmatrix} 010\\ 011\end{pmatrix} \ \textit{(the set of monotone functions)}.
\end{aligned}$$

The P_k-maximal sets for $k > 2$ are known [21,64] and some of them are discussed in Sections 4–6.

2. Classifications of functions and bases in P_2

There are 15 classes of functions of P_2 [20,16,27]. We present them by their characteristic vectors. The components of the characteristic vectors are given with respect to the order T_0, T_1, S, L, M of the P_2-maximal sets.

1. 11111 2. 11011 3. 01111 4. 10111 5. 11001
6. 10101 7. 01101 8. 00111 9. 10100 10. 01100
11. 00110 12. 00011 13. 00010 14. 00001 15. 00000

For instance, the class 6 represents the set $\overline{T_0}T_1\overline{S}L\overline{M}$, where $\overline{X}$ denotes $P_2 \setminus X$. The class 9 (10) consists only of the constant function 1 (0).

There are two conditions for $X \subseteq P_2$ to be a base: completeness and irredundancy. As mentioned above, X is complete iff $\sum_{f\in X}\varphi(f)$ is a positive integer vector and X is irredundant iff for each $g \in X$ the sum $\sum_{f\in X\setminus\{g\}}\varphi(f)$ has at least one 0 coordinate. An *aggregate* is the set of all bases having the same set of characteristic vectors.

There are 42 aggregates for P_2 [16,27]:

1 aggregate of rank 1: (1);
17 aggregates of rank 2:
(2,3),(2,4),(2,6),(2,7),(2,8),(2,9),
(2,10),(2,11),(3,4),(3,5),(3,6),(3,9),
(4,5),(4,7),(4,10),(5,8),(5,11);
22 aggregates of rank 3:
(5,6,12),(5,6,13),(5,7,12),(5,7,13),(5,9,12),
(5,9,13),(5,10,12),(5,10,13),(6,7,8),(6,7,11),
(6,7,12),(6,7,13),(6,8,10),(6,10,11),(6,10,12),
(6,10,13),(7,8,9),(7,9,11),(7,9,12),(7,9,13),
(8,9,10),(9,10,12);
2 aggregates of rank 4: (9,10,11,14),(9,10,13,14).

Note that there are only three aggregates containing constant functions: (9,10,12), (9,10,11,13) and (9,10,13,14).

The number of bases of P_2 consisting of n-ary functions (functions with at most n variables) in an aggregate can be calculated as a product of the numbers of n-ary functions in the classes determined by the characteristic vectors. Summing these numbers for all aggregates for a rank we obtain corresponding data for the bases of the rank and finally the number of all bases containing n-ary functions.

Each function of the class 1 (i.e.$\overline{T_0}\,\overline{T_1}\,\overline{SLM}$) is a base of P_2 i.e. a Sheffer function. There is a well-known condition for a function to be Sheffer function in P_2 (cf. [67,13]). A function f is Sheffer if and only if $f \notin T_0 \cup T_1 \cup S$. The number of Sheffer functions of n variables is $2^{2^n-2} - 2^{2^{n-1}-1}$ (cf. [67,13]). Thus for large n almost 25% of n-ary operations are Sheffer.

The c-completeness in P_2 is sometimes called completeness in *ES-algebra*, where ES stands for extended superposition. Clearly X is c-complete if and only if $X \nsubseteq M$ and $X \nsubseteq L$. There are four classes of functions and two aggregates in ES-algebra [71]. A function f is c-Sheffer for P_2 if and only if $f \notin M \cup L$ [20,15]. Thus the number of Boolean c-Sheffer functions is $2^{2^n} - 2^{n+1} + n + 2 - \Psi(n)$ [15], where $\Psi(n)$ denotes the number of monotone functions of n variables and called the Dedekind number. When n is large, almost all Boolean functions are c-Sheffer [15].

The set M contains four maximal sets [93,13]: $M^0 = MT_0, M^1 = MT_1, D$ and C, where $D = [\{0,1,\vee\}]$ (the set of constants and disjunctions) and $C = [\{0,1,\wedge\}]$ (the set of constants and conjunctions). There are 6 classes of functions and two aggregates in M [71]. There is no Sheffer function for the set M.

The set T_0 contain 4 maximal sets [93,13]: LT_0, M^0, T_0T_1 and N_0, where

$$N_0 = Pol\begin{pmatrix} 010 \\ 100 \end{pmatrix}.$$

There are 10 classes of functions and 14 aggregates [71].

The map $f \to f^+$ (introduced in Section 1) respects composition (i.e. is compatible with $*, \varsigma, \tau$ and $\triangle$). In particular, it induces an (order) automorphism of closed sets (i.e. $A \subseteq B \Leftrightarrow A^+ \subseteq B^+$ where $A^+ := \{f^+ : f \in A\}$) and carries complete sets and bases onto complete sets and bases. As $T_0^+ = T_1$, all statements concerning T_0 translate to T_1. For example, the T_1-maximal sets are LT_1, M^1, T_0T_1 and $N_1 = Pol\begin{pmatrix} 101 \\ 011 \end{pmatrix}$ (for the latter: given an h-ary relation ρ put $\rho^+ := \{(a_1+1,\ldots,a_n+1)|(a_1,\ldots,a_n) \in \rho\}$, then $(Pol\ \rho)^+ = Pol\ \rho^+$).

The sets L and S will be considered in Section 5 as subsets of P_k.

3. Classifications for various modifications of algebra of logic

In Section 1 we have defined composition and closed sets in a way suitable for propositional logic and universal algebra. There are several variations of composition depending on the method of constructing a network from gates or restrictions imposed by real circuit requirements. We list them mostly by names and references. The corresponding classifications are given in [78,71,52,17,18] and Sheffer (c-Sheffer) functions in [71,56].

The operation $*$ defined by (1) in Section 1 was replaced by a special composition in the *Algebra* Φ^0 [7,8]. The notions of composition under ***r-line coding*** and ***up-to-coding completeness*** were introduced in [12]. A similar completeness was studied in [17,25]. This is also related to the so-called *SP algebra* described in [13].

So far we have considered gates with the output immediately reacting to changes on the inputs. Real gates react with a certain delay. The completeness for the gates having unit-delay or having positive-integer-delays was studied in [30] and its modifications for the unit-delay in [17,18]. Still another construction method for unit-delay called ***sequential circuit completeness*** was introduced in [60]. The notion of ***GS-algebra*** described in [13] is also related to the above concept. The situation is complex and cannot be adequately explained here (a short survey can be found in [56]). We refer the reader to [30,88,59,14] for more details.

4. Classifications of P_3 and its maximal sets

Determination of maximal sets for the set P_k and its closed sets has been the subject of investigation in a large number of papers [62,93,20,21,64,65,39,1,10,31,33-36].

Theorem 4.1. [21] *P_3 has exactly 18 maximal sets.*

The first attempt to derive classes of functions of P_3 was done in [46]. But, it counted several characteristic vectors twice as different classes, consequently the number of classes of functions reported there was incorrect; this was corrected in [69]. The numbers of classes of functions and aggregates for P_3 and for all P_3-maximal sets are reported in [53] (the numbers of irredundant incomplete sets are also given there). Two algorithms for the enumeration of aggregates are given in [47,51] and [69,51,75,77]. They are compared in [51,75].

The set P_3 has exactly 6,239,721 aggregates [69]. The number of aggregates containing the constant functions {0,1,2} is exactly 1,391 [47]: There are 2 such aggregates of rank 4, 633 of rank 5 and 756 of rank 6. There are two proofs that the maximal rank of a base of P_3 is 6: computational [47] and theoretical [79]. Note that there is no base of rank 7 while there exist irredundant incomplete sets with 7 elements. Also note that there is no Sheffer function in some maximal sets of P_3 (cf. [87,94]).

Sheffer functions in P_3 have been studied (cf. [67,82]). The number of n-ary ($n \geq 2$) Sheffer functions in P_3 is [82]:

$$8 \cdot 3^{3^n-3} - 2 \cdot 3^{3^{n-1}-1} - 2^{2^n-1}3^{3^n-2^n} - 6\prod_{i=1}^{n-1}(2^{2^i}+1)^{\begin{pmatrix} n \\ i \end{pmatrix}}$$
$$+ 6\prod_{i=1}^{n-1}(2^{2^{i-1}}+1)^{\begin{pmatrix} n \\ i \end{pmatrix}}$$

Thus the number of Sheffer functions with 2 and 3 variables are 3,774 and 2,110,663,244,298, respectively [42,82]. The corresponding formula for the number of symmetric Sheffer functions is given in [95].

5. Classifications of the functions for some closed subsets of P_k

There are several classification results for closed sets in P_k [74,71,54,55]. A function is *linear* if there are $a_0, \dots, a_n \in E_k$ so that

$$f(x_1, \dots, x_n) = a_0 + a_1x_1 + \dots + a_nx_n$$

holds for all $x_1, \dots, x_n \in E_k$. The set of linear functions has been investigated (cf. [1,2,36]). It is P_k-maximal if and only if k is a prime number [21]. Let L be the set of linear functions of P_k and $T_m = \{f \mid f(m, \dots, m) = m\}$ the set of functions preserving m $(0 \leq m \leq k-1)$.

Theorem 5.1. [1,2] *There are exactly $p+2$ maximal sets of L in prime-valued logic P_p:*

$$\begin{array}{ll} L_m = & LT_m,\ 0 \leq m \leq p-1, \\ L_S = & LS = \{a_0 + a_1x_1 + \dots + a_nx_n \mid a_1 + \dots + a_n = 1\} \\ & \quad \textit{(the set of linear selfdual functions)}, \\ L^{(1)} = & \{a_0 + a_1x_i \mid a_0, a_1 \in \{0, 1, \dots, p-1\},\ i > 0\} \\ & \quad \textit{(the set of essentially unary linear functions).} \end{array}$$

There are exactly $2p+4$ classes of functions of the set L [74]. Their characteristic vectors listed with respect to the above order of maximal sets are:

$$\begin{array}{ll} 1: & 0^{p+2} \text{ (i.e. } p+2 \text{ zeros)} \\ 2: & 0^{p+1}1 \\ 3 \leq r \leq p+3: & 1^{r-3}01^{p+3-r}0 \\ p+4 \leq r \leq 2p+4: & 1^{r-p-4}01^{2p+5-r}. \end{array}$$

Let $f(x_1, \dots, x_n) = a_0 + a_1x_1 + \dots + a_nx_n$ be a linear function in P_p. The function x is in the class 1, and the function $a_1x_1 + \dots + a_nx_n$ is in the class 2 for $n \geq 2$ and $a_1 + \dots + a_n = 1$. The functions $a_0 + x$ are in the class $p+3$ for $a_0 \neq 0$, and the functions $a_0 + a_1x_1 + \dots + a_nx_n$ for $a_0 \neq 0$ and $a_1 + \dots + a_n = 1$, $n \geq 2$ are in the class $2p+4$. The constant function $f = i$ belongs to the class $i+3$ $(0 \leq i \leq p-1)$. Let $a_1 + \dots + a_n \neq 1$ and let a be the number determined uniquely by $a(1 - a_1 - \dots - a_n) = a_0$, i.e. $a_0 + a_1a + \dots + a_na = a$ $(a \in E_p)$. Then the function $f(x_1, \dots, x_n) = a_0 + a_1x_1 + \dots + a_nx_n$ belongs to the class $p+4+a$, because it preserves a.

No Sheffer function for L exists. However, each $f \in L \setminus L^{(1)}$ is c-Sheffer as $0 \notin T_m$ $(m \geq 1)$, $1 \notin T_0$, $0 \notin S$. The number of such n-ary functions is $p^{n+1} - np(p-1) - p$ $(n \geq 2)$. As $n \to \infty$ the proportion of c-Sheffer n-ary linear functions (among n-ary linear functions) goes rapidly to 1.

Bases of rank 2 are composed of any two functions of classes i and j, where i and j satisfy the condition

a) $p+4 \leq i < j \leq 2p+4$, or
b) $3 \leq i \leq p+3 < j \leq 2p+4$ and $j \neq i+p+1$.

Bases of rank 3 contain a function of the class 2 and two functions, one each from the classes i and j, where $3 \leq i < j \leq p+3$. Thus L contains exactly $4\binom{p+1}{2}$ aggregates; $3\binom{p+1}{2}$ of rank 2 and $\binom{p+1}{2}$ of rank 3. The maximal rank of a base of L is 3.

The H-maximal sets for the above $p+2$ L-maximal sets H (p prime) are determined in [1] and their classification is in [74].

Let $S = Pol\begin{pmatrix} 0 & 1 & \dots & k-2 & k-1 \\ 1 & 2 & \dots & k-1 & 0 \end{pmatrix}$. It is easy to verify that S is a set of selfdual functions in k-valued logic (i.e. f such that $f(x_1+1, \dots, x_n+1) = f(x_1, \dots, x_n) + 1$). Note that there are another types of selfdual functions (cf. [65]).

Theorem 5.2. [96] *There are exactly two S-maximal sets in prime-valued logic P_p:*

$$S_L = SL \text{ and } S_0 = ST_0.$$

A linear function $a_0 + a_1x_1 + \dots + a_nx_n$ is selfdual if $a_1 + \dots + a_n = 1$. If this holds, the function $a_1x_1 + \dots + a_nx_n$ belongs to the set S_LS_0 (class 00) and the functions $a_0 + a_1x_1 + \dots + a_nx_n$ for $a_0 \neq 0$ belong to the set $S_L\overline{S}_0$ (class 01).

The number of n-ary Sheffer functions in S is $(p-1)p^{n-1}(p^{p^{n-1}-1} - 1)$. Note that c-Sheffer is the same as Sheffer because no constant function belongs to S. There are exactly two aggregates for S; one each for ranks 1 and 2.

Let $P_{k,2}$ be the set of functions with domain E_k and range E_2. Let f and g be n-ary functions such that $f \in P_{k,2}$

and $g \in P_2$. Then $pr\ f = g$ if and only if $f(\mathbf{a}) = g(\mathbf{a})$ for all $\mathbf{a} \in E_2^n$. For $X \subseteq P_2$, the inverse image of X is the subset

$$X' := pr^{-1}(X) = \{f \in P_{k,2} \mid pr\ f \in X\}.$$

Theorem 5.3. [5,31,36] *$P_{k,2}$ has $5 + (k-2)(k+1)/2$ maximal sets:*

$$T_0', T_1', S', L', M' \text{ and}$$
$$Z_{i,t} := P_{k,2} Pol \begin{pmatrix} 01i \\ 01t \end{pmatrix}, \ 0 \le t < i \le k-1, \ i \ge 2,$$

where T_0, T_1, S, L and M are the P_2-maximal sets.

For each function $f \in P_{k,2}$ define a binary relation Q_f on the set E_k :

$$(i,t) \in Q_f \Leftrightarrow \left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \in \begin{pmatrix} 01i \\ 01t \end{pmatrix} \Rightarrow f(\mathbf{a}) = f(\mathbf{b}) \right)$$

for $0 \le t < i \le k-1$, $i \ge 2$ ($\mathbf{a}, \mathbf{b} \in E_k^n$). Then we have [54]:

$$f \in Z_{i,t} \Leftrightarrow (i,t) \in Q_f \text{ and}$$
$$f \text{ is a constant on the set } E_2^n \Leftrightarrow (0,1) \in Q_f.$$

In fact, if $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \in \begin{pmatrix} 01i \\ 01t \end{pmatrix}$ then $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfy $b_j = a_j$ for $a_j \in \{0,1\}$ and $b_j = t$ for $a_j = i$ ($1 \le j \le n$). From $f \in Z_{i,t}$ follows $\begin{pmatrix} f(\mathbf{a}) \\ f(\mathbf{b}) \end{pmatrix} \in \begin{pmatrix} 01i \\ 01t \end{pmatrix}$, i.e. $f(\mathbf{a}) = f(\mathbf{b}) = 0, f(\mathbf{a}) = f(\mathbf{b}) = 1$ or $f(\mathbf{a}) = i, f(\mathbf{b}) = t$. However, from $\{i,t\} \not\subseteq E_2$ the last case is not possible. Therefore $f(\mathbf{a}) = f(\mathbf{b})$ and $(i,t) \in Q_f$. Q_f is an equivalence relation. This equivalence relation describes in a way the structure of $P_{k,2}$, because classes of functions of $P_{k,2}$ directly correspond to equivalence relations. More precisely, for each function of $P_{k,2}$ we associate an equivalence relation Q according to its characteristic vector. Then the relation is determined uniquely by the class of functions. Conversely, for each Q there corresponds exactly one characteristic vector of $P_{k,2}$ up to the maximal sets $\{Z_{i,t}\}$. The remaining part ("P_2"-part) of it is determined by the 15 classes of functions of P_2 described in Section 1. Recall that among 15 classes of functions of P_2 two classes contain constant functions only (the case $(0,1) \in Q$). The remaining 13 classes of functions correspond to the case $(0,1) \notin Q$. Thus, we have 13 or 2 classes of functions of $P_{k,2}$ for each Q, depending on $(0,1) \in Q$. Hence the number of classes of functions of $P_{k,2}$ is $13A_k - 11A_{k-1}$, where A_k denotes the number of equivalence relations on the set of k elements. Although the number of maximal sets of $P_{k,2}$ is $O(k^2)$ and the number of classes of functions is $O(k!)$, the maximal rank of a base of $P_{k,2}$ is $k+2$ [54]. Similar investigations for all the 6 families of maximal sets of $P_{k,2}$ were done except M' (the monotone set). The maximal rank of a base for each of the above maximal sets of $P_{k,2}$ is also $O(k)$ [55]. The enumeration of Sheffer functions in $P_{k,2}$ is an open problem as well as the classification of M'.

6. An overview and some open problems

The number of P_k-maximal sets was approximated in [83,84] and the exact formula for it was determined in [66]:

k	2	3	4	5	6	7
maximal sets	5	18	82	643	15,182	7,848,984

Classification of P_k is barely possible for $k = 4$. We give some subsets of P_k whose maximal sets are known. Perhaps the most interesting P_k-maximal set is the set L of linear functions. Let $k = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$, $\alpha_1, \ldots, \alpha_m \ge 1, p_1, \ldots, p_m$: prime numbers. All the maximal sets of L are described as follows [36]:

1) $2^m - 1$ maximal sets
   $T_d := L_d \cup \bigcup_{n \ge 1} \{f \in L \mid \exists b, a_0, \ldots, a_n :$
   $b|d \ \wedge \ b \ne 1 \ \wedge f(\mathbf{x}) = a_0 + b \sum_{i=1}^n a_i x_i\}$,
   where $\mathbf{x} = (x_1, \ldots, x_n)$ and
   $L_d := \bigcup_{n \ge 1} \{f \in L \mid \exists a_0, \ldots, a_n, j \ :$
   $f(\mathbf{x}) = a_0 + a_j x_j + d \sum_{i=1, i \ne j}^n a_i x_i\}, d = p_{i_1} \ldots p_{i_t}$,
   $\{p_{i_1}, \ldots, p_{i_t}\} \subseteq \{p_1, \ldots, p_m\}, \ 1 \le t \le m.$
2) m maximal sets of type
   $L_{*,p_i} := \bigcup_{n \ge 1} \{f \in L \mid \exists a_0, \ldots, a_n \in E_k :$
   $f(\mathbf{x}) = a_0 + \sum_{i=1}^n a_i x_i \wedge a_1 + \ldots + a_n = 1 \pmod{p_i}\}$,
   $1 \le i \le m.$
3) $p_1 + \ldots + p_m$ maximal sets
   $L \cap Pol(j, p_i + j, 2p_i + j, \ldots, k - p_i + j)$ for all j
   satisfying $0 \le j \le p_i - 1, \ 1 \le i \le m.$

The special case of $k = p^m$ or $k = 2 \cdot p$ ($m > 1, \ p > 2, p$: prime) is also investigated in [36].

Another interesting maximal set is the set of special self-dual functions S (cf. Section 4) for k not a prime number

[35]. All $2\prod_{i=1}^{m}(\alpha_i+1)-3$ maximal sets of S are described as

$$\{S\cap Pol\,\gamma_r,\ S\cap Pol\,\rho_t\ \mid\ r\in T\setminus\{1\},\ t\in T\setminus\{1,k\}\},$$

where $T := \{x \mid k\equiv 0 \pmod{x}\}$, $\gamma_r := \{x\in E_k \mid x\equiv 0 \pmod{r}\}$ and $\rho_t := \{(x,y)\in E_k^2 \mid y-x\equiv 0 \pmod{t}\}$. Some cases of selfdual functions are also described in [40].

Compositions of partial k-valued functions are investigated in [11,37,63]. Define $P_{k,l} := \bigcup_{n\geq 1}\{f(x_1,\ldots,x_n) \mid f : \{0,1,\ldots,k-1\}^n \to \{0,\ldots,k,\ldots,l-1\}\}$, $l>k$, with the operation of composition defined by:

$$f\circ g = \begin{cases} f*g & \text{if } W(g)\subseteq\{0,\ldots,k-1\},\\ f & \text{otherwise,}\end{cases}$$

where $W(g)$ denotes the range of g. $P_{k,l}$ is a generalization of the partial k-valued logic. $P_{2,l}$ has exactly the following 8 maximal sets [89]:

$$\{f\in P_{2,l} \mid |W(f)|\leq l-1\},\ Pol^*(0),\ Pol^*(1),$$

$$Pol^*\begin{pmatrix}0\\1\end{pmatrix},\ Pol^*\begin{pmatrix}01\\10\end{pmatrix},\ Pol^*\begin{pmatrix}001\\011\end{pmatrix},$$

$$Pol^*\begin{pmatrix}000111\\001101\\010011\\011001\end{pmatrix},\ Pol^*\begin{pmatrix}00011011\\00110101\\01001101\\01100011\end{pmatrix},$$

where $Pol^*\rho := \{f\in P_{2,l} \mid (\mathrm{a}_1,\ldots,\mathrm{a}_h)^T\in\rho^n \Rightarrow (f(\mathrm{a}_1),\ldots, f(\mathrm{a}_h))^T\in\rho\cup(\{0,\ldots,l-1\}^h\setminus\{0,1\}^h)^T\}$. $P_{2,l}$ is being classified. Besides, it is known that $P_{3,l}$, $l>3$ has exactly 58 maximal sets ([89], a slightly different number of the maximal sets is reported in [63]).

Define $P_3(2) :\doteq \bigcup_{n\geq 1}\{f(x_1,\ldots,x_n)\in P_3 \mid |W(f)|\leq 2\}$. $P_3(2)$ has exactly the following 13 maximal sets [89]: $\bigcup_{n\geq 1}\{f \mid \exists f_0,\ldots,f_n\in P_3^1(2)$(the set of unary functions) : $f(x_1,\ldots,x_n) = f_0(f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n) \bmod 2)\}$ and the classes $P_3(2)\cap\rho$ where $\rho\in$

$$\{\begin{pmatrix}012001\\012122\end{pmatrix},\begin{pmatrix}012112\\012200\end{pmatrix},\begin{pmatrix}012220\\012011\end{pmatrix},\begin{pmatrix}01201\\01210\end{pmatrix},$$

$$\begin{pmatrix}01202\\01220\end{pmatrix},\begin{pmatrix}01212\\01221\end{pmatrix},(0\,1),\ (0\,2),\ (1\,2),\begin{pmatrix}0120102\\0121020\end{pmatrix},$$

$$\begin{pmatrix}0120121\\0121012\end{pmatrix},\begin{pmatrix}0120212\\0122021\end{pmatrix}\}.$$

Define $P_{\{0,1\},\{a,3\}} := \bigcup_{n,m\geq 1}\{f^{n,m} \mid f^{n,m} : \{0,1\}^n\times\{a,3\}^m \to \{0,1\}\}$, $a\in\{0,2\}$, and with a similar generalization of superposition. $P_{\{0,1\},\{a,3\}}$ has exactly 10 maximal sets for $a=0$ and 21 maximal ones for $a=2$ ([85] also cf. [32]). The maximal sets for Pol (0) are also known [33].

Finding maximal sets for other subsets of P_k and under various modifications of composition are open problems. Among them we find part of automata theory [9,30], where some maximal sets are given. Uniform delay composition with unit-delay for P_3 was solved in [58], and with positive-integer-delays for P_3 in [14] (30 and 49 maximal sets). Composition with delay was also treated in the general case in [41,68].

The enumeration of Sheffer functions as well as c-Sheffer may be considered in many of the above cases (cf. [67]). For example, the number of n-ary 3-valued c-Sheffer functions is known only for n=2 [57].

Maximal rank of a base is an open problem in many cases. The problem is mentioned early in [21,6] especially for P_k. It is known that some closed subsets of P_k, $k\geq 3$ have an infinite base or no base [24]. Also it is known that for $k\geq 8$ some P_k-maximal sets have no finite basis [45].

The classification and basis enumeration can be used to calculate the number of n-ary bases [75,81,29,61,3,4]. In many cases, this has not yet been done. The corresponding classifications and basis enumerations for the symmetric functions are surveyed in [76].

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