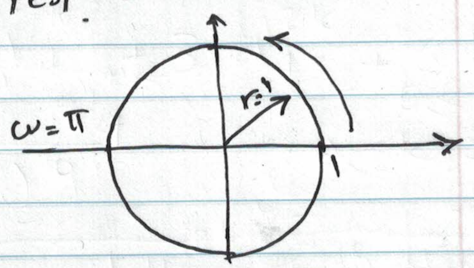
# Tutorial 6

## Recall the following:

**DTFT**: $\underbrace{x[n]}_{\downarrow} \xrightarrow{\text{DTFT}} \underbrace{X(e^{j\omega})}_{\downarrow} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

Discrete-time signal

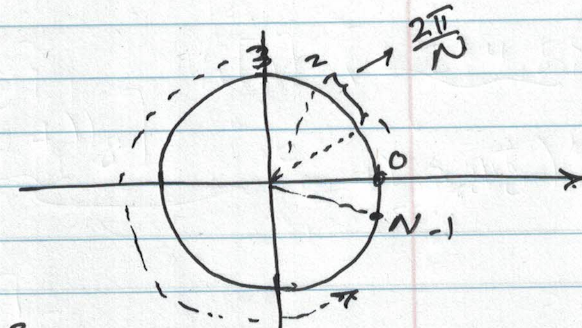
Frequency response $\Rightarrow$ $\omega$ is continuous $[0, 2\pi]$




**DFT**: $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$

where $\omega = \frac{2\pi}{N} k$

$\therefore$ DFT is a discretized version of DTFT




$\therefore$ $\omega$ in DFT became discrete.

$$\Rightarrow X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

**Q 8.4**

$$x[n] = \alpha^n u[n] \quad , \quad |\alpha| < 1$$

Periodic sequence $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$ , $|\alpha|<1$

(a)
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \frac{1}{1-\alpha e^{-j\omega}} \quad , \quad |\alpha| < 1$$

(b) DFS:
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

where $W_N^{kn} = (e^{-j2\pi/N})^{kn}$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x[n+rN] W_N^{kn}$$

$$= \sum_{n} \sum_{r} \alpha^{n+rN} u[n+rN] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \alpha^{n+rN} W_N^{kn}$$

$$= \sum_{r=0}^{\infty} \alpha^{rN} \sum_{n=0}^{N-1} \alpha^n W_N^{kn}$$

$$= \sum_{r=0}^{\infty} \alpha^{rN} \left( \frac{1-\alpha^N W_N^{kN}}{1-\alpha W_N^{k}} \right)$$

$$= \left( \frac{1}{1-\alpha^N} \right) \left( \frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}} \right)$$

$$\tilde{X}[k] = \frac{1}{1 - \alpha e^{-j\frac{2\pi k}{N}}} \quad ; \quad |\alpha| < 1$$

ⓒ $$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$\therefore \tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

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## Q 8.6

$$x[n] = \begin{cases} e^{j\omega_0 n} & 0 \le n \le N-1 \\ 0 & \text{o.w.} \end{cases}$$

ⓐ $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} e^{j\omega_0 n}\, e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n}$$

$$= \frac{1 - e^{-j(\omega-\omega_0)N}}{1 - e^{-j(\omega-\omega_0)}}$$

$$= \frac{e^{-j(\omega-\omega_0)\frac{N}{2}}}{e^{-j(\omega-\omega_0)/2}} \cdot \frac{\sin\left[(\omega-\omega_0)\frac{N}{2}\right]}{\sin\left[\frac{\omega-\omega_0}{2}\right]}$$

ⓑ N-point DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \quad ; \quad 0 \le k \le N-1$$

$$= \sum_{n=0}^{N-1} e^{j\omega_0 n}\, W_N^{kn} = \sum_{n=0}^{N-1} e^{j\omega_0 n}\, e^{-j\frac{2\pi}{N}kn}$$

$$X[k] = \frac{1 - e^{-j\left(\frac{2\pi k}{N} - w_0\right)N}}{1 - e^{-j\left(\frac{2\pi}{N}k - w_0\right)}}$$

$$= e^{-j\left[\frac{2\pi k}{N} - w_0\right]\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left[\left(\frac{2\pi k}{N} - w_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N} - w_0\right)/2\right]}$$

Note: $X[k] = X(e^{jw})\Big|_{w = \frac{2\pi k}{N}}$

(e) if $w_0 = \frac{2\pi \cdot k_0}{N}$

$$X[k] = \frac{1 - e^{-j(k-k_0)2\pi}}{1 - e^{-j(k-k_0)2\pi/N}}$$

$$= e^{-j\frac{2\pi}{N}(k-k_0)\left(\frac{N-1}{2}\right)} \cdot \frac{\sin[\pi(k-k_0)]}{\sin[\pi(k-k_0)/N]}$$

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Q 8.8 $\quad x[n] = (0.5)^n u[n]$

$$Y[k] = X(e^{jw})\Big|_{w = \frac{2\pi k}{10}}$$

$$X(e^{jw}) = \sum x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n u[n] e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-jwn}$$

$$X(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$$Y[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{10}k} = \frac{1}{1 - \frac{1}{2} e^{-j\left(\frac{2\pi}{10}\right)k}}$$

$$Y[k] = \sum_{n=0}^{9} y[n]\, W_N^{kn} \quad ; \text{ where } N = 10$$

**Note:**

$$\left(\tfrac{1}{2}\right)^n \xrightarrow{N\text{-pt DFT}} \frac{1 - \left(\frac{1}{2}\right)^N}{1 - \frac{1}{2} e^{-j\frac{2\pi k}{N}}}$$

by analogy:

$$\therefore\ y[n] = \frac{(1/2)^n}{1 - \left(\frac{1}{2}\right)^{10}}$$

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$\boxed{Q\,8.10}$

from the figure, $x_2[n]$ is a circular shift of $x_1[n]$

$$x_2[n] = x_1[((n-4))_8] \longrightarrow ((n-4) \bmod 8)$$

from property 5:

$$x_1[((n-4))_8] \xrightarrow{DFT} W_8^{4k}\, X_1[k]$$

$$x[((n-m))_N] \xrightarrow{DFT} W_N^{km}\, X[k]$$

$$\therefore\ X_2[k] = W_8^{4k}\, X_1[k] = e^{-j\pi k}\, X_1[k]$$

$$\boxed{X_2[k] = (-1)^k\, X_1[k]}$$

## Q 8.12

$x[n] = \cos\left(\frac{\pi n}{2}\right) = \{1, 0, -1, 0\}$

$h[n] = 2^n = \{1, 2, 4, 8\}$

(a) $X[k] = \sum_{n=0}^{3} \cos\left(\frac{\pi n}{2}\right) W_4^{kn}$ $\quad 0 \leq k \leq 3$

$= W_4^0 + (-1) W_4^{2k} = 1 - e^{-j\frac{2\pi}{4}(2k)}$

$X[k] = 1 - e^{-j\pi k}$

(b) $H[k] = \sum_{n=0}^{3} 2^n W_4^{nk}$ $\quad 0 \leq k \leq 3$

$H[k] = 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k}$

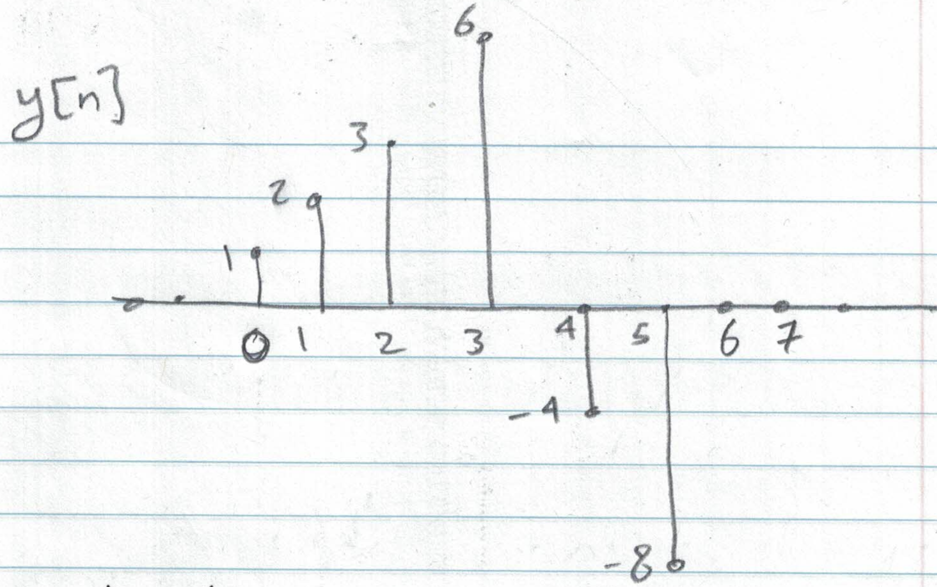
(c) To perform circular convolution.

$x[n] = \underbrace{\{1, 0, -1, 0\}}_{\text{4-point seq}}$ $\quad h[n] = \underbrace{\{1, 2, 4, 8\}}_{\text{4-point seq}}$

$\underbrace{y[n]}_{\text{4-point output sequence}} = x[n] \; (4) \; h[n]$

<u>Method 1</u>

Linear convolution $y[n] = x[n] * h[n]$




Since $N = 4$
then we expect to get aliasing

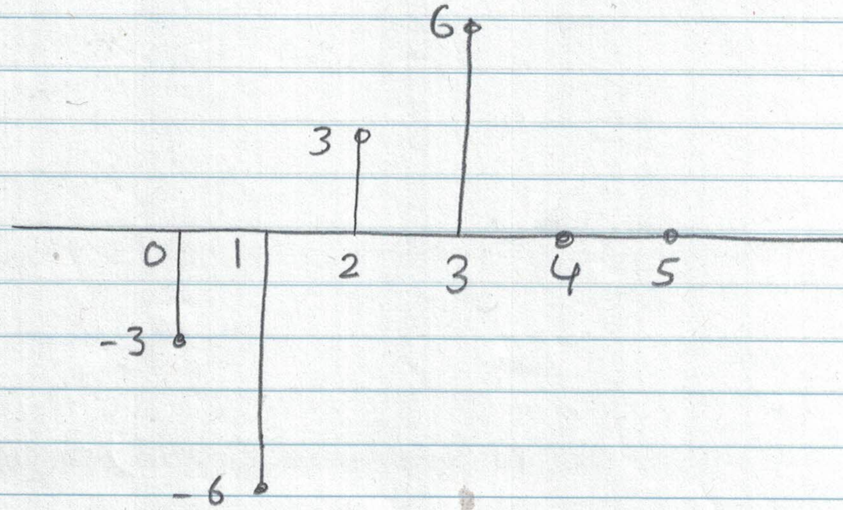
the linear convolution o/p length is $= 4+4-1 = 7$

while circular convolution gives an o/p of

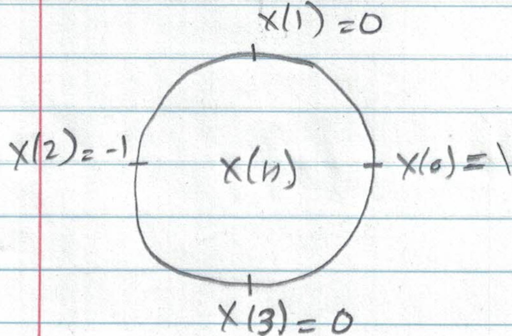
length 4 $\Rightarrow$ Therefore, an aliasing occurs

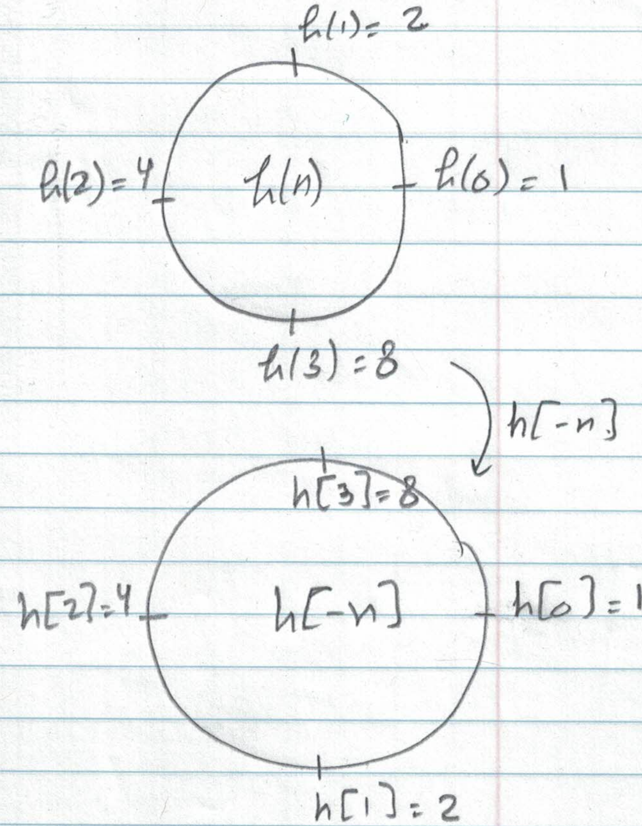
due to samples (4, 5, 6) ; And the

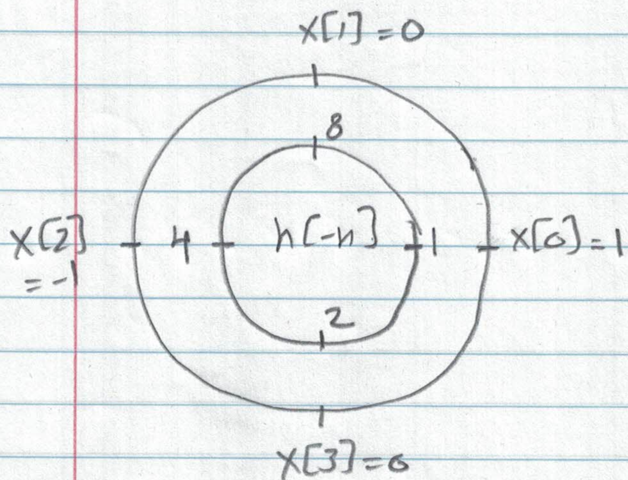
output of circular convolution becomes:

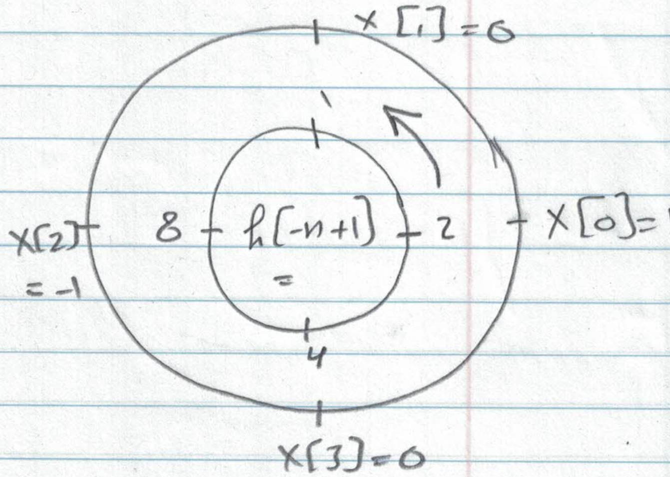


## Method 2: Circular Convolution



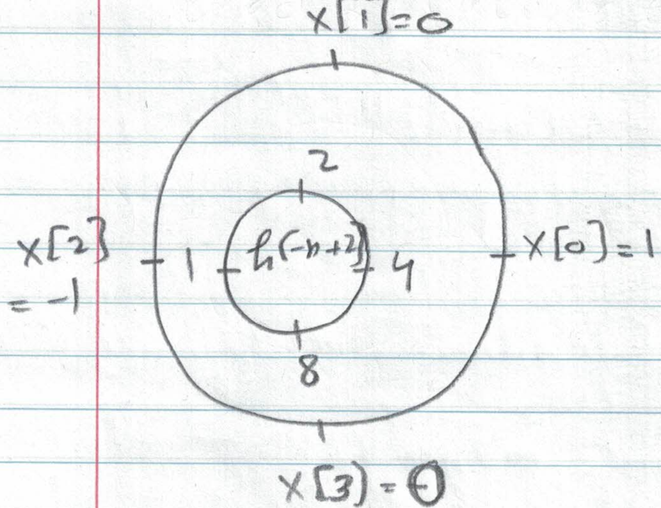




$$y[0] = 1 + (0)(8) + (-1)(4) + (2)(0)$$

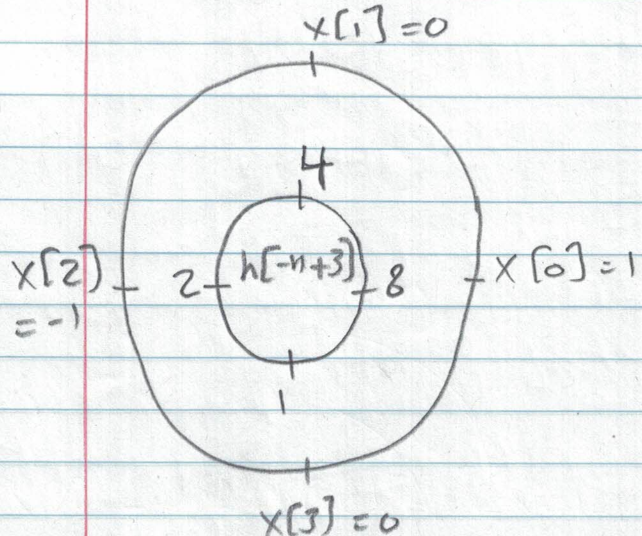
$$= -3$$

$$y[1] = (1)(2) + (0)(1) + 8(-1) + (4)(0)$$

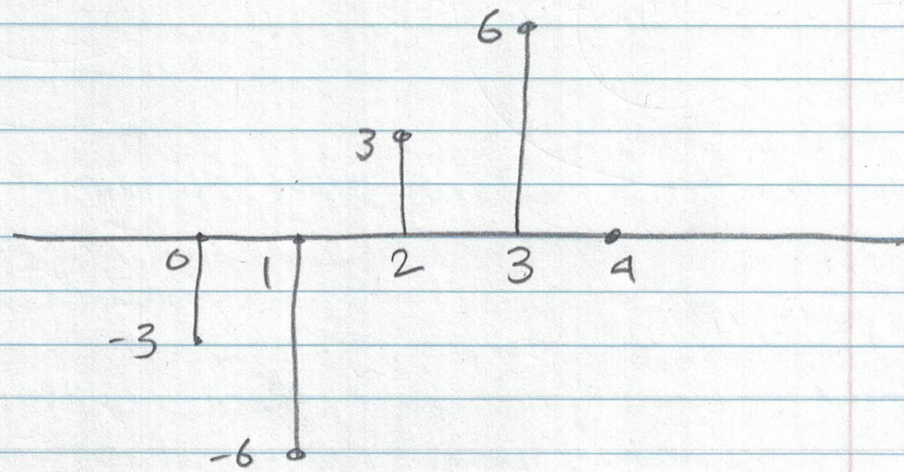
$$= -6$$




$$y[2] = (1)(4)+(0)(2)+(1)(-1)+8(0)$$
$$= 3$$




$$y[3] = (1)(8)+(4)(0)+(2)(-1)+(1)(0)$$
$$= 6$$



(d) using DFT values from parts (a) & (b)

$$Y[k] = X[k]\, H[k]$$

$$= 1 + 2W_4^{k} + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k}$$

$$- 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k}$$

$$W_4^{4k} = W_4^{0} \quad \text{and} \quad W_4^{5k} = W_4^{k}$$

$$Y[k] = -3 - 6W_4^{k} + 3W_4^{2k} + 6W_4^{3k}$$

$$0 \le k \le 3$$

Take IDFT

$$y[n] = -3\,\delta[n] - 6\,\delta[n-1] + 3\,\delta[n-2]$$

$$+ 6\,\delta[n-3]$$

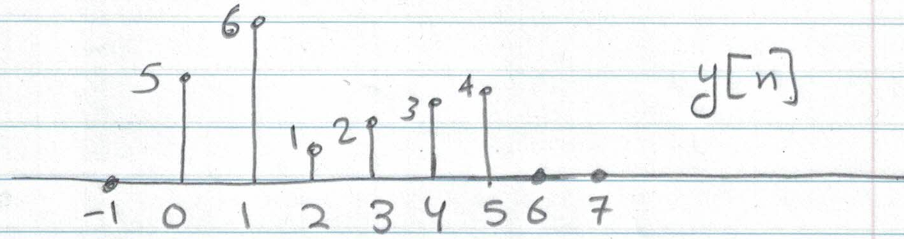
# Tutorial 7

## Q 8.11

$$y[n] = x_1[n] \; ⑥ \; x_2[n]$$

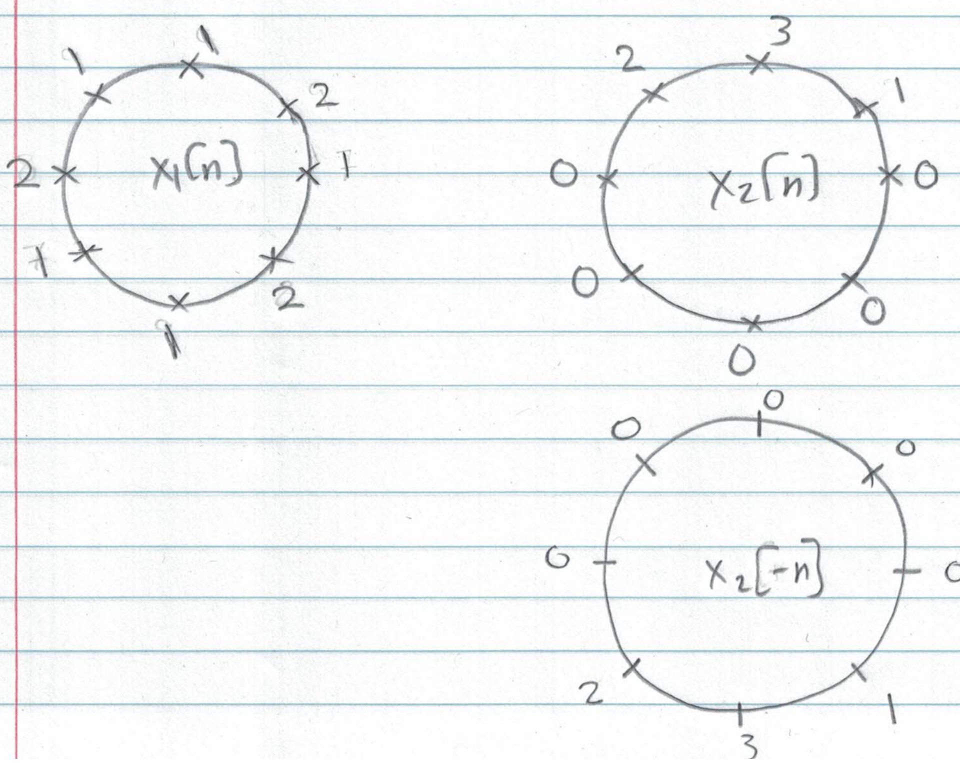
$$= x_1[n] \; ⑥ \; \delta[n-2]$$

$$= x_1[((n-2))_6]$$




## Q 8.14

$$x_3[n] = x_1[n] \; ⑧ \; x_2[n]$$

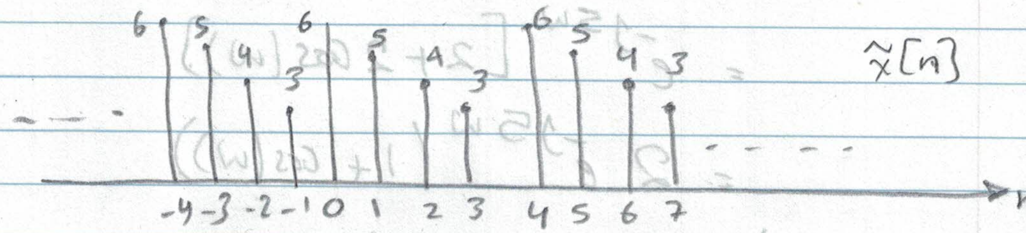


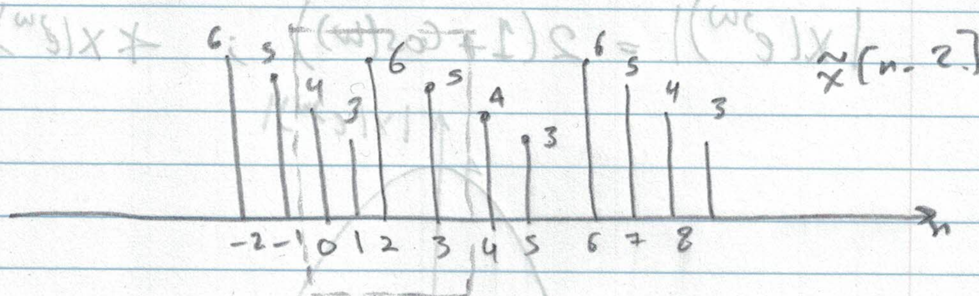
# Tutorial 10

**Q 8.24**

$x_1[n] = x[((n-2))_4] \qquad 0 \le n \le 3$

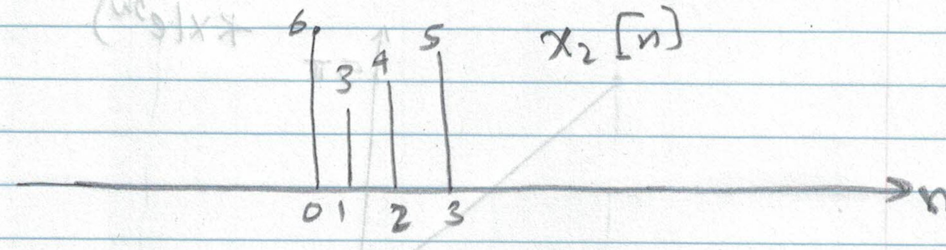
let $\tilde{x}[n]$ be the periodic signal of $x[n]$




$\rightarrow x_1[n] = x[((n-2))_4]$

$x_2[n] = x[((-n))_4]$

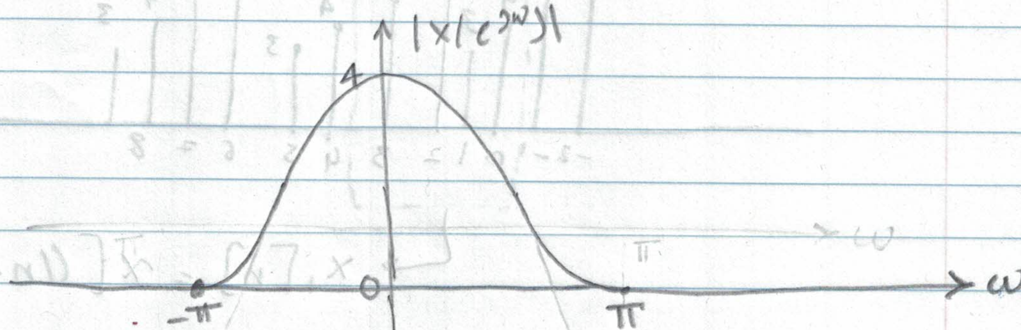


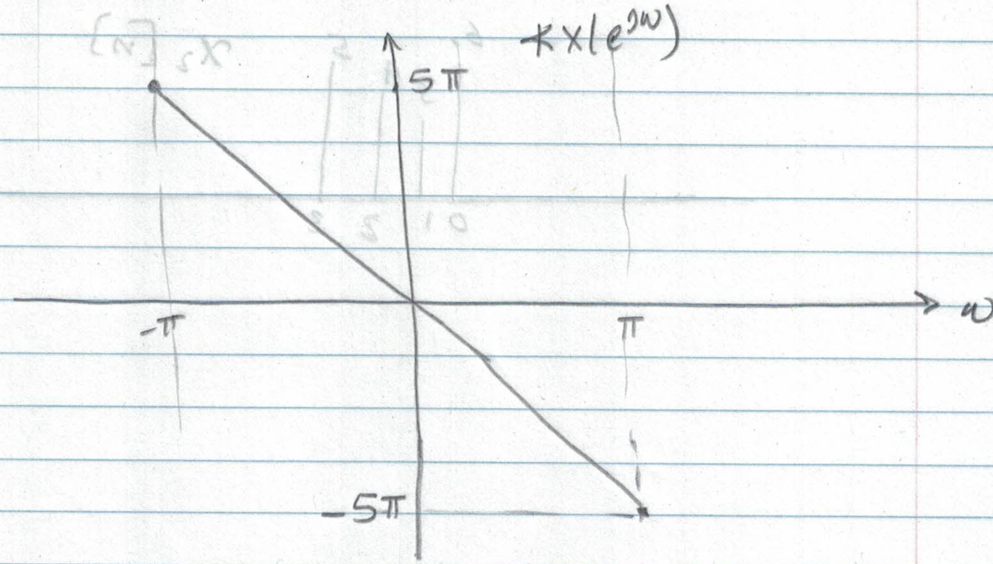
Q8.25

ⓐ $x[n] = \delta[n-4] + 2\delta[n-5] + \delta[n-6]$

$$X(e^{jw}) = e^{-j4w} + 2e^{-j5w} + e^{-j6w}$$
$$= e^{-j5w}\left[e^{jw} + 2 + e^{-jw}\right]$$
$$= e^{-j5w}\left[2 + 2\cos(w)\right]$$
$$= 2e^{-j5w}\left(1 + \cos(w)\right)$$

$$|X(e^{jw})| = 2(1+\cos(w)) \quad ; \quad \angle X(e^{jw}) = -5w$$



(b) $N$-DFT $\Rightarrow$ $\omega = \frac{2\pi K}{N}$ ; $K = 0, 1, \ldots, N-1$

* The set of frequencies for which $X(e^{j\omega})$ is pure real is $\omega = \frac{2\pi K}{10}$

$\therefore K = 0, 1, \ldots, 9$ $\Rightarrow$ $\boxed{\measuredangle X(e^{j\omega}) = 5\omega = \pi M}$

$\therefore$ The DFT will be real when

$N = 1, 2, 5, 10$

(c) The 3-point aliased version of $x[n]$ is $\delta[n] + \delta[n-1] + 2\delta[n-2] = x_3[n]$

from part (a) $\Rightarrow$ $|X(e^{j\omega})| = 2(1+\cos(\omega))|$

$$|X(K)| = 2\left[1 + \cos\left(\frac{2\pi K}{N}\right)\right]$$

$|X_3[K]| \longrightarrow |X_3[0]| = |X[0]| = 4$

$|X_3[1]| = 2\left(1 + \cos\left(\frac{2\pi}{3}\right)\right) = 1$

$|X_3[2]| = 2\left(1 + \cos\left(\frac{2\pi}{3} \cdot 2\right)\right) = 1$

$$|X_3[K]| = 4\delta[K] + \delta[K-1] + \delta[K-2]$$

$\downarrow$

$$x_3[n] = 2\delta[n] + \delta[n-1] + \delta[n-2] = x_1[n]$$

**Q 8.33**

$$H[k] = \frac{1}{5}\,\delta[k-1] + \frac{1}{3}\,\delta[k-7]$$

Compute IDFT:

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, e^{j\frac{2\pi k n}{N}}$$

$$N = 10, \quad n = 0, 1, 2, \ldots, 9$$

$$h[n] = \frac{1}{10}\left[\frac{1}{5}\, e^{j\left(\frac{2\pi}{16}\right)n} + \frac{1}{3}\, e^{j\left(\frac{2\pi}{16}\right)7n}\right]$$

Then,

$$H(e^{j\omega}) = \sum_{n=0}^{9} h[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{9} \frac{1}{10}\left[\frac{1}{5}\, e^{j\left(\frac{2\pi}{10}\right)n} + \frac{1}{3}\, e^{j\left(\frac{2\pi}{10}\right)7n}\right] e^{-j\omega n}$$

$$= \frac{1}{10}\cdot\frac{1}{5}\sum_{n=0}^{9} e^{j\left(\frac{2\pi}{10}-\omega\right)n} + \frac{1}{10}\cdot\frac{1}{3}\sum_{n=0}^{9} e^{j\left(\frac{2\pi}{16}\cdot 7-\omega\right)n}$$

$$= \frac{1}{10}\left[\frac{1}{5}\cdot\frac{1-\left(e^{j\left(\frac{2\pi}{10}-\omega\right)}\right)^{10}}{1-e^{j\left(\frac{2\pi}{10}-\omega\right)}} + \frac{1}{3}\cdot\frac{1-\left(e^{j\left(\frac{2\pi}{10}\cdot 7-\omega\right)}\right)^{10}}{1-e^{j\left(\frac{2\pi}{10}\cdot 7-\omega\right)}}\right]$$

$$= \frac{1}{10}\left[\frac{1}{5}\cdot\frac{1-e^{-j10\omega}}{1-e^{j(2\pi/10-\omega)}} + \frac{1}{3}\cdot\frac{1-e^{-j10\omega}}{1-e^{j(2\pi\cdot 7/10-\omega)}}\right]$$

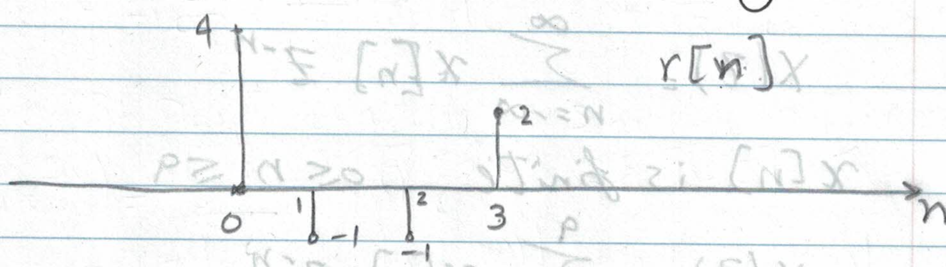
## Q 8.46

$$x[n] = 3\delta[n] - \delta[n-1] + 2\delta[n-3] + \delta[n-4] - \delta[n-6]$$

(a) $R[k] = X(e^{j\omega})\big|_{\omega = 2\pi k/4} \qquad 0 \le k \le 3$

$R[k]$ is a 4-point DFT of $x[n]$

* Inverting $R[k]$ creates aliasing.




(b) $h[n] = \delta[n] - \delta[n-4]$

$Y[k] = X[k]\,H[k] \qquad 0 \le k \le 7$

$\therefore\; y[n] = x[n] \; N \; h[n]$

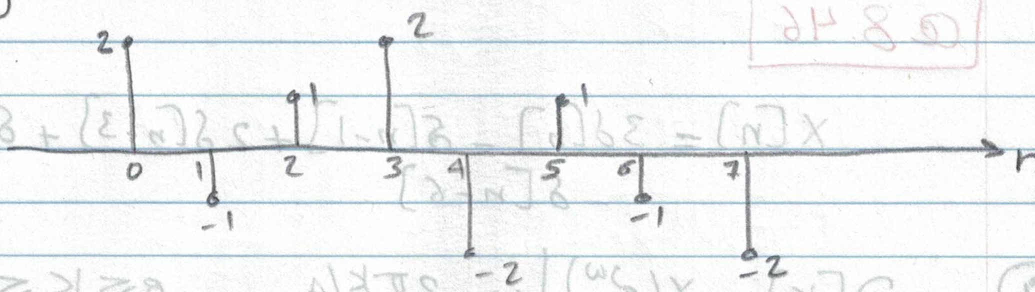
To get $y[n]$, we perform linear convolution and then apply aliasing

$$x[n] * h[n] = 3\delta[n] - \delta[n-1] + 2\delta[n-3] + \delta[n-4] - \delta[n-6] - \big[3\delta[n-4] - \delta[n-5] + 2\delta[n-7] + \delta[n-8] - \delta[n-10]\big]$$

$$= 3\delta[n] - \delta[n-1] + 2\delta[n-3] - 2\delta[n-4] + \delta[n-5] - \delta[n-6] - 2\delta[n-7] - \delta[n-8] + \delta[n-10]$$

6

$y[n]$



Q 8.50

$$X_1[k] = X(z)\Big|_{z=\frac{1}{2}e^{j\left[\frac{2\pi}{10}k+\frac{\pi}{10}\right]}}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$x[n]$ is finite $\quad 0 \le n \le 9$

$$X(z) = \sum_{n=0}^{9} x[n]\, z^{-n}$$

$$z = \frac{1}{2} e^{[2\pi k/10 + \pi/10)}$$

$$X(z) = \sum_{n=0}^{9} x[n]\left(\frac{1}{2} e^{\left(\frac{2\pi k}{10}+\frac{\pi}{10}\right)}\right)^{-n}$$

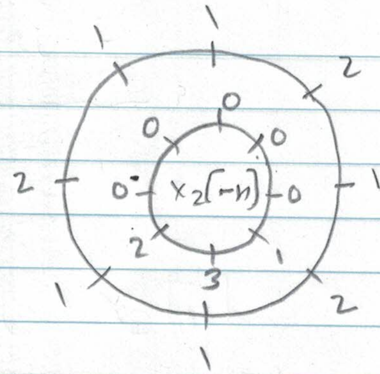
Now,
$$X_1[k] = \sum_{n=0}^{9} x_1[n]\, W_{10}^{kn}$$

and $W_{10}^{kn} = e^{-j\frac{2\pi}{10}kn}$

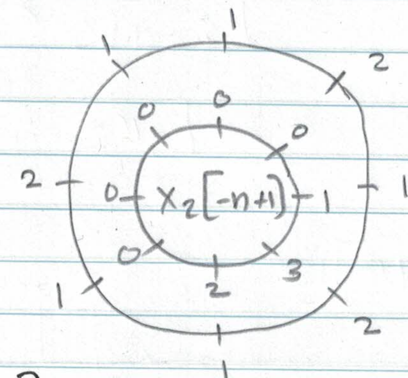
$$\boxed{\therefore\ x_1[n] = x[n]\left(\frac{1}{2}e^{j(\pi/10)}\right)^{-n}}$$

$\boxed{2}$

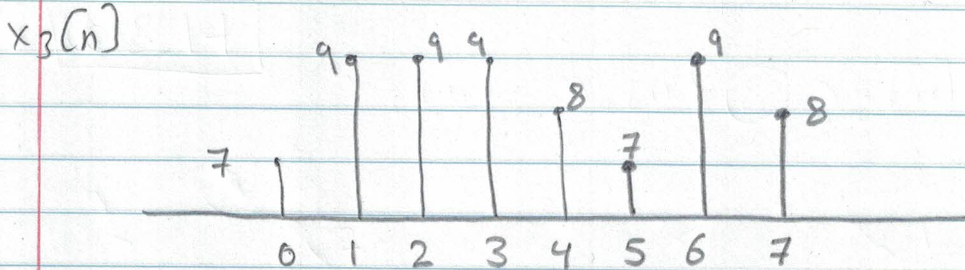
$$x_3[0] = 2(1) + 3(1) + 2(1) = 7$$




$$x_3[1] = (1)(1) + 2(1) + 2(3) = 9$$




repeat this until $x_3[8]$




$$x_3[2] = 9$$

Q 8.18

$$X_1[k] = X[k]\, e^{j\frac{2\pi\, 3k}{5}}$$

$$x_1[n] = x[((n+3))_5]$$

$$\therefore\ x_1[0] = x[3]$$

$$2 = c$$

$$\therefore\ \boxed{c = 2}$$

Q 8.38

$$X_1(z) = \sum_{n=0}^{N-1} x_1[n]\, z^{-n}$$

at $z = \frac{1}{2} e^{-j2\pi k/N}$

$$X_1(z)\Big|_{z=\frac{1}{2}e^{-j\frac{2\pi}{N}k}} = \sum_{n=0}^{N-1} x_1[n] \left(\frac{1}{2} e^{-j\frac{2\pi k}{N}}\right)^{-n}$$

$$\boxed{X_1(z) = \sum_{n=0}^{N-1} x_1[n] \left(\frac{1}{2}\right)^{-n} e^{+j\frac{2\pi kn}{N}}}$$ (A)

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$X_2^*[k] = \sum x_2^*[((-n))_N]\, e^{-j\frac{2\pi kn}{N}}$$

Note $\quad x^*[((-n))_N] \xrightarrow{DFT} X^*[k]$

$$\therefore \boxed{X_2[k] = \sum_{n=0}^{N-1} x_2[((-n))_N] \, e^{\frac{j2\pi kn}{N}}} \quad (B)$$

Comparing A & B

$$\therefore \quad x_1[n]\left(\tfrac{1}{2}\right)^{-n} = x_2[((-n))_N]$$

$$n = 0, \ldots, N-1$$