Q 8.4

8.4. Consider the sequence $x[n]$ given by $x[n] = \alpha^n u[n]$. Assume $|\alpha| < 1$. A periodic sequence $\tilde{x}[n]$ is constructed from $x[n]$ in the following way:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN].$$

(a) Determine the Fourier transform $X(e^{j\omega})$ of $x[n]$.
(b) Determine the DFS coefficients $\tilde{X}[k]$ for the sequence $\tilde{x}[n]$.
(c) How is $\tilde{X}[k]$ related to $X(e^{j\omega})$?
8.6. Consider the complex sequence

\[ x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise.} \end{cases} \]

(a) Find the Fourier transform \( X(e^{j\omega}) \) of \( x[n] \).
(b) Find the \( N \)-point DFT \( X[k] \) of the finite-length sequence \( x[n] \).
(c) Find the DFT of \( x[n] \) for the case \( \omega_0 = 2\pi k_0/N \), where \( k_0 \) is an integer.
8.8. Let \( X(e^{j\omega}) \) denote the Fourier transform of the sequence \( x[n] = (0.5)^n u[n] \). Let \( y[n] \) denote a finite-duration sequence of length 10; i.e., \( y[n] = 0, n < 0 \), and \( y[n] = 0, n \geq 10 \). The 10-point DFT of \( y[n] \), denoted by \( Y[k] \), corresponds to 10 equally spaced samples of \( X(e^{j\omega}) \); i.e., \( Y[k] = X(e^{j2\pi k/10}) \). Determine \( y[n] \).
8.10. The two eight-point sequences $x_1[n]$ and $x_2[n]$ shown in Figure P8.10 have DFTs $X_1[k]$ and $X_2[k]$, respectively. Determine the relationship between $X_1[k]$ and $X_2[k]$.
8.12. Suppose we have two four-point sequences \( x[n] \) and \( h[n] \) as follows:

\[
x[n] = \cos \left( \frac{\pi n}{2} \right), \quad n = 0, 1, 2, 3, \\
h[n] = 2^n, \quad n = 0, 1, 2, 3.
\]

(a) Calculate the four-point DFT \( X[k] \).

(b) Calculate the four-point DFT \( H[k] \).

(c) Calculate \( y[n] = x[n] \oplus h[n] \) by doing the circular convolution directly.

(d) Calculate \( y[n] \) of part (c) by multiplying the DFTs of \( x[n] \) and \( h[n] \) and performing an inverse DFT.
8.11. Figure P8.11 shows two finite-length sequences $x_1[n]$ and $x_2[n]$. Sketch their six-point circular convolution.
8.14. Two finite-length signals, \( x_1[n] \) and \( x_2[n] \), are sketched in Figure P8.14. Assume that \( x_1[n] \) and \( x_2[n] \) are zero outside of the region shown in the figure. Let \( x_3[n] \) be the eight-point circular convolution of \( x_1[n] \) with \( x_2[n] \); i.e., \( x_3[n] = x_1[n] \circledast_8 x_2[n] \). Determine \( x_3[2] \).
8.18. Figure P8.18-1 shows a sequence $x[n]$ for which the value of $x[3]$ is an unknown constant $c$. The sample with amplitude $c$ is not necessarily drawn to scale. Let

$$X_1[k] = X[k]e^{j2\pi 3k/5},$$

where $X[k]$ is the five-point DFT of $x[n]$. The sequence $x_1[n]$ plotted in Figure P8.18-2 is the inverse DFT of $X_1[k]$. What is the value of $c$?
8.34. Suppose that $x_1[n]$ and $x_2[n]$ are two finite-length sequences of length $N$, i.e., $x_1[n] = x_2[n] = 0$ outside $0 \leq n \leq N - 1$. Denote the $z$-transform of $x_1[n]$ by $X_1(z)$, and denote the $N$-point DFT of $x_2[n]$ by $X_2[k]$. The two transforms $X_1(z)$ and $X_2[k]$ are related by:

$$X_2[k] = X_1(z)
|_{z = \frac{1}{2} e^{-j\frac{2\pi k}{N}}}, \quad k = 0, 1, \ldots, N - 1$$

Determine the relationship between $x_1[n]$ and $x_2[n]$. 
8.24. Figure P8.24 shows a finite-length sequence $x[n]$. Sketch the sequences

$$x_1[n] = x[((n - 2))_4], \quad 0 \leq n \leq 3,$$

and

$$x_2[n] = x[((-n))_4], \quad 0 \leq n \leq 3.$$
8.25. Consider the signal \( x[n] = \delta[n - 4] + 2\delta[n - 5] + \delta[n - 6] \).

(a) Find \( X(e^{j\omega}) \) the discrete-time Fourier transform of \( x[n] \). Write expressions for the magnitude and phase of \( X(e^{j\omega}) \), and sketch these functions.

(b) Find all values of \( N \) for which the \( N \)-point DFT is a set of real numbers.

(c) Can you find a three-point causal signal \( x_1[n] \) (i.e., \( x_1[n] = 0 \) for \( n < 0 \) and \( n > 2 \)) for which the three-point DFT of \( x_1[n] \) is:

\[
X_1[k] = |X[k]| \quad k = 0, 1, 2
\]

where \( X[k] \) is the three-point DFT of \( x[n] \)?
8.33. An FIR filter has a 10-point impulse response, i.e.,

\[ h[n] = 0 \quad \text{for} \ n < 0 \text{ and for } n > 9. \]

Given that the 10-point DFT of \( h[n] \) is given by

\[ H[k] = \frac{1}{5} \delta[k - 1] + \frac{1}{3} \delta[k - 7], \]

find \( H(e^{j\omega}) \), the DTFT of \( h[n] \).
8.46. Each part of this problem may be solved independently. All parts use the signal \( x[n] \) given by
\[
x[n] = 3\delta[n] - \delta[n - 1] + 2\delta[n - 3] + \delta[n - 4] - \delta[n - 6].
\]

(a) Let \( X(e^{j\omega}) \) be the DTFT of \( x[n] \). Define
\[
R[k] = X\left(e^{j\omega}\right)\bigg|_{\omega = \frac{2\pi k}{4}}, \quad 0 \leq k \leq 3
\]
Plot the signal \( r[n] \) which is the four-point inverse DFT of \( R[k] \).

(b) Let \( X[k] \) be the eight-point DFT of \( x[n] \), and let \( H[k] \) be the eight-point DFT of the impulse response \( h[n] \) given by
\[
h[n] = \delta[n] - \delta[n - 4].
\]
Define \( Y[k] = X[k]H[k] \) for \( 0 \leq k \leq 7 \). Plot \( y[n] \), the eight-point DFT of \( Y[k] \).
8.50. The DFT of a finite-duration sequence corresponds to samples of its $z$-transform on the unit circle. For example, the DFT of a 10-point sequence $x[n]$ corresponds to samples of $X(z)$ at the 10 equally spaced points indicated in Figure P8.50-1. We wish to find the equally spaced samples of $X(z)$ on the contour shown in Figure P8.50-2; i.e., we wish to obtain

$$X(z)ig|_{z=0.5e^{j(2\pi k/10+(\pi/10))}}.$$

Show how to modify $x[n]$ to obtain a sequence $x_1[n]$ such that the DFT of $x_1[n]$ corresponds to the desired samples of $X(z)$.

\[ \text{Figure P8.50-1} \]

\[ \text{Figure P8.50-2} \]