Tutorial 2 : chapter 3

Q 3.1

a) $\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$

$= \sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}$ ROC: $|z| > \frac{1}{2}$

b) $\mathcal{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\} = -\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} u[-n-1]$

$= -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{n=1}^{\infty}(2z)^n$

$= \frac{-2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}}$

ROC: The values of z for which the sequence is absolutely summable

$\left|\sum_{n=1}^{\infty}(2z)^n\right| < \infty$

$|2z| < 1$

$\therefore$ ROC is $|z| < \frac{1}{2}$

c) $\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[-n]\right\} = \sum_{n=-\infty}^{0}(2z)^n = \frac{1}{1-2z}$

ROC: $|2z| < 1 \rightarrow |z| < \frac{1}{2}$

d) $\mathcal{Z}\{\delta[n]\} = z^0 = 1$ ROC: all z

e) $\mathcal{Z}\{\delta[n-1]\} = z^{-1}$ ROC: $|z| > 0$

f) $\mathcal{Z}\{\delta[n+1]\} = z$ (2)

ROC: $0 \le |z| < \infty$

g) $Z\{(\frac{1}{2})^n (u[n] - u[n-10])\} = \sum_{n=0}^{9} (\frac{1}{2})^n z^{-n}$

$= \sum_{n=0}^{9} \left(\frac{1}{2z}\right)^n = \frac{1-(2z)^{-10}}{1-(2z)^{-1}}$

ROC: $|z| > 0$

<u>Note</u>: $\sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}}$

<u>Note</u>: There are finite number of terms in the summation, hence the sum will be finite for $|a| < \infty$ and $|z| \neq 0$ only. Therefore, the

ROC is $|z| > 0$

Q 3.2

$$x[n] = \begin{cases} n & 0 \le n \le N-1 \\ N & n \ge N \end{cases}$$

$$= n\, u[n] - (n-N)\, u[n-N]$$

from the z-Transform Table:

$n\, x[n] \longrightarrow -z \frac{d}{dz} X(z) = -z \frac{d}{dz} \cdot \frac{1}{1-z^{-1}}$

ROC1: $|z| > 1$

$\therefore \quad n\,u[n] \longrightarrow \dfrac{z^{-1}}{(1-z^{-1})^2} \qquad ROC1: |z| > 1$

$x[n-n_0] \longrightarrow X(z) \cdot z^{-n_0}$

$\therefore \quad (n-N)\,u[n-N] \longrightarrow \dfrac{z^{-N-1}}{(1-z^{-1})^2}$

$\&$ $ROC\,2: |z| > 1$

$$\therefore \quad X(z) = \frac{z^{-1} - z^{-N-1}}{(1-z^{-1})^2}$$

$ROC = ROC\,1 \cap ROC\,2 : |z| > 1$

Q 3.3

a) $x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$

$$X_a(z) = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

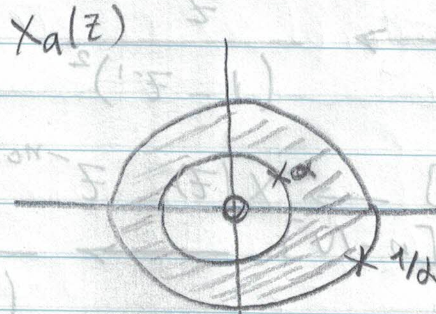
$$= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}}$$

$$\sum_{n=0}^{\infty} (\alpha z)^n - 1 = \frac{1}{1-\alpha z} - 1 = \frac{\alpha z}{1-\alpha z}$$

$$X_a(z) = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}$$

$ROC: |\alpha| < |z| < \dfrac{1}{|\alpha|}$

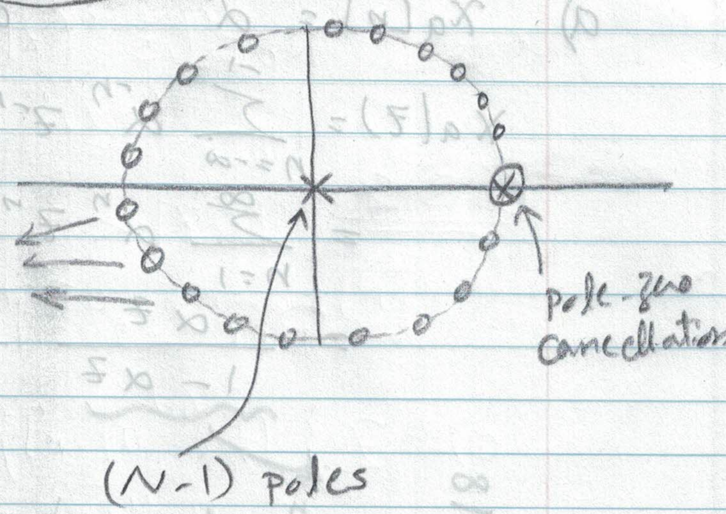


b) $x_b[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & n \ge N \\ 0 & n < 0 \end{cases}$

$$X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

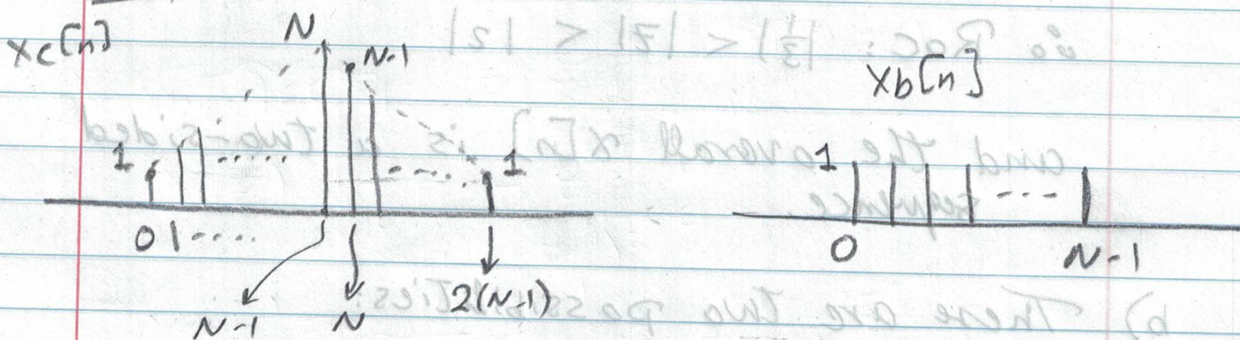
$$= \frac{\overbrace{z^N - 1}^{N \text{ zeros at } 1}}{\underbrace{z^{N-1}}_{(N-1) \text{ poles at zero}} \underbrace{(z-1)}_{\text{pole at } 1}}$$

ROC: $|z| \neq 0$



c) $x_c[n] = \begin{cases} n+1 & 0 \le n \le N-1 \\ 2N-1-n & N \le n \le 2(N-1) \\ 0 & \text{o.w.} \end{cases}$

Note



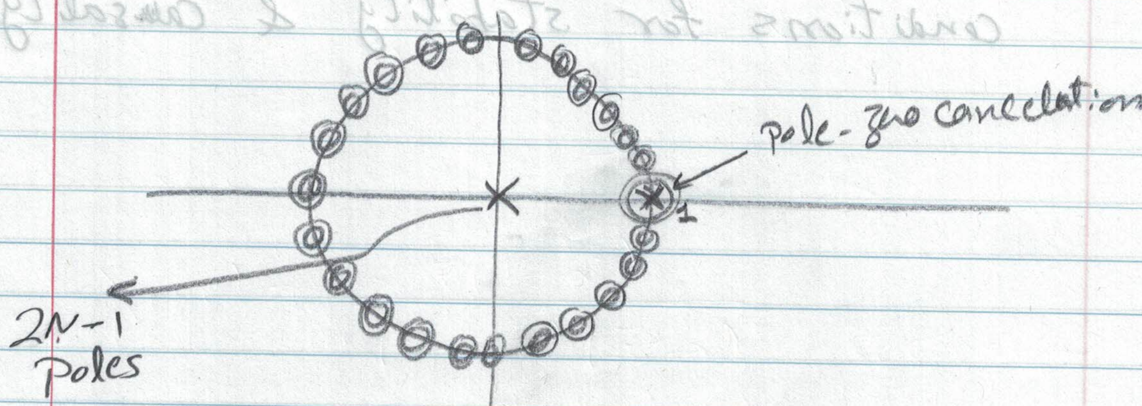
It can be observed that:

$$x_c[n] = x_b[n] * x_b[n-1]$$

$$\therefore X_c(z) = z^{-1} X_b(z) X_b(z)$$

$$= z^{-1}\left(\frac{z^N - 1}{z^{N-1}(z-1)}\right)^2 = \frac{(z^N - 1)^2}{z^{2N-1}(z-1)^2}$$

ROC: $z \neq 0, 1$



Q 3.4

a) For the FT of x[n] to exist, the ROC should include the unit circle

$\therefore$ ROC: $|\frac{1}{3}| < |z| < |2|$

and the overall x[n] is a two-sided sequence.

b) There are two possibilities:

$|\frac{1}{3}| < |z| < |2|$

and $|2| < |z| < |3|$

c) For stability, ROC must include the unit circle

For causality, ROC must extend from the outside pole.

The ROC have to be a connected region, hence we cannot combine the previous conditions for stability & causality.

Q 3.7

a) $x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$

$$X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

RoC: $\frac{1}{2} < |z| < 1$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}}$$

$$= \frac{1-z^{-1}}{1+z^{-1}}$$

$H(z)$ is causal $\Rightarrow$ RoC: $|z| > 1$

b) Since on of the poles of $X(z)$ is cancelled by the zero of $H(z)$, the RoC of $Y(z)$

is the Region satisfying the other two constraints:

$|z| > \frac{1}{2}$, $|z| > 1 \Rightarrow |z| > \frac{1}{2} \cap |z| > 1$

$\therefore$ RoC of $Y(z)$: $|z| > 1$

c) $$Y(z) = \frac{-1/3}{1-\frac{1}{2}z^{-1}} + \frac{1/3}{1+z^{-1}}$$

$$y[n] = \left(\frac{-1}{3}\right)\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

Q 3.11

a) if $x[n]$ is causal, then

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

→ no positive powers of z is included in this sum.

$\therefore$ $X(z)$ must converge at $z = \infty$

$\therefore$ $z = \infty$ must be in the ROC of $X(z)$

or, $\lim_{z \to \infty} X(z) \neq \infty$

a) $\lim_{z \to \infty} \frac{(1 - z^{-1})^2}{(1 - \frac{1}{2} z^{-1})} = 1 \rightarrow$ Could be causal

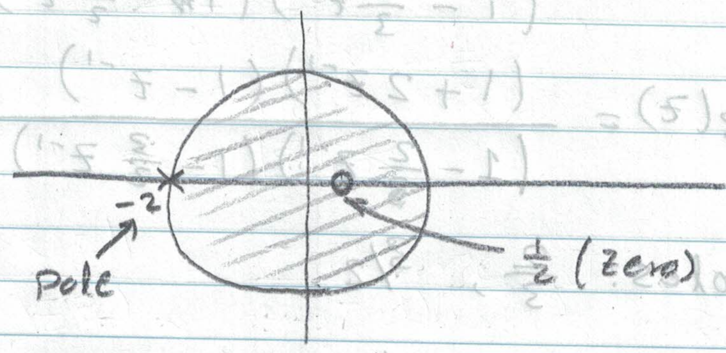
b) $\lim_{z \to \infty} \frac{(z - 1)^2}{(z - \frac{1}{2})} = \infty \Rightarrow$ Not causal

c) $\lim_{z \to \infty} \frac{(z - 1/4)^5}{(z - \frac{1}{2})^6} = 0 \Rightarrow$ Causal

d) $\lim_{z \to \infty} \frac{(z - 1/4)^6}{(z - \frac{1}{2})^5} = \infty \Rightarrow$ Not causal.

Q 3.12

a) $X_1(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}}$ RoC: $|z| < 2$

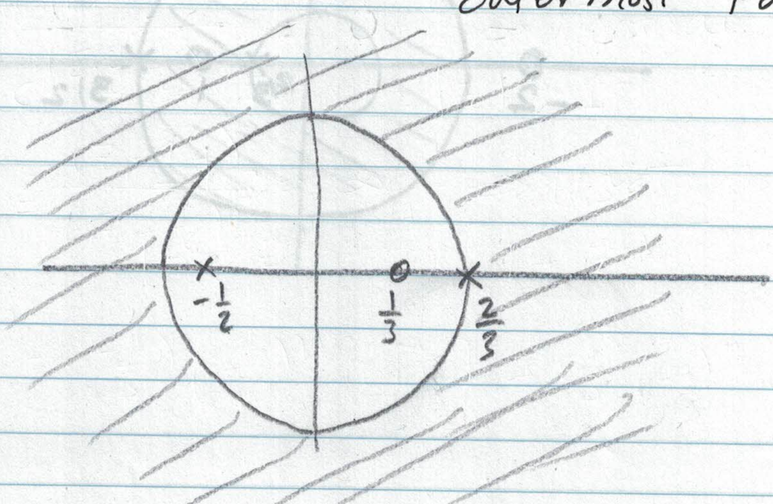


b) $X_2(z) = \dfrac{1-(1/3)z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$

Poles : $-\frac{1}{2}$, $\frac{2}{3}$
Zero : $1/3$

$x_2[n]$ is causal, then:

Roc : $|z| > 2/3$ (extends from the outermost pole)



c) $$X_3(z) = \frac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6} z^{-1} + z^{-2}}$$

$$= \frac{(1 + z^{-1} - 2z^{-2})}{(1 - \frac{2}{3} z^{-1})(1 - \frac{3}{2} z^{-1})}$$

$$X_3(z) = \frac{(1 + 2z^{-1})(1 - z^{-1})}{(1 - \frac{2}{3} z^{-1})(1 - \frac{3}{2} z^{-1})}$$

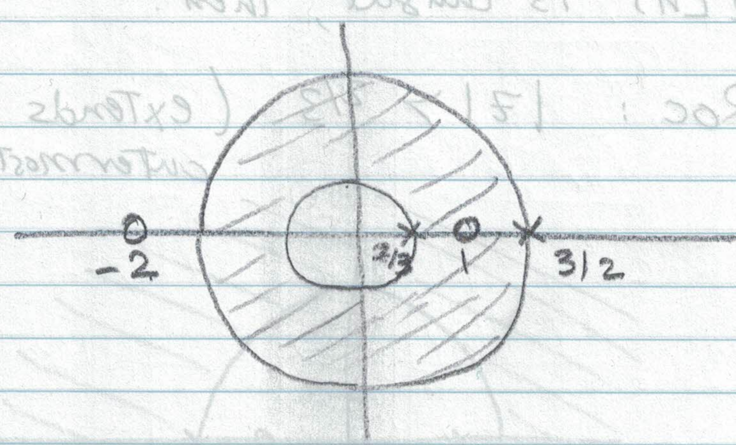
Poles: $\frac{2}{3}$, $3/2$

Zeros: -2, 1

$x_3[n]$ is absolutely summable.

$\therefore$ The ROC must include the unit circle

ROC: $2/3 < |z| < 3/2$



Q 3.17

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1]$$

$$Y(z)\left[1 - \frac{5}{2}z^{-1} + z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

$$= \frac{1 - z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{2/3}{1 - 2z^{-1}} + \frac{1/3}{1 - \frac{1}{2}z^{-1}}$$

The different values of $h[0]$ comes from different ROC choices

If ROC is

a) $|z| < \frac{1}{2}$

$$h[n] = -\frac{2}{3}(2^n)u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

$$h[0] = 0$$

b) $\frac{1}{2} < |z| < 2$

$$h[n] = -\frac{2}{3}(2^n)u[-n-1] + \frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$$

$$h[0] = 1/3$$

c) $|z| > 2$

$$h[n] = \frac{2}{3}(2^n)u[n] + \frac{1}{3}(\tfrac{1}{2}^n)u[n-1]$$

$$h[0] = 1$$

d) $|z| > 2$ or $|z| < 1/2$

$$h[n] = \left(\frac{2}{3}\right)(2^n)u[n] - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^n u[n-1]$$

$$h[0] = 2/3$$

Q 3.21

a) Poles: 0.25, −0.5

Since the system is causal, the ROC must extend from the outermost pole

ROC: $|z| > 0.5$

b) ROC contains the unit circle, hence the system is stable.

c) $$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 0.25z^{-1} - 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})}$$

$$Y(z)\left[1 + 0.5z^{-1} - 0.25z^{-1} - 0.125z^{-2}\right] =$$

$$X(z)\left[4 + 0.25z^{-1} - 0.5z^{-2}\right]$$

∴ Difference eqn:

$$y[n] + 0.25\,y[n-1] - 0.125\,y[n-2] =$$

$$4\,x[n] + 0.25\,x[n-1] - 0.5\,x[n-2]$$

d) $$H(z) = \frac{4 + (1/4)z^{-1} - (1/2)z^{-2}}{1 + (1/4)z^{-1} - (1/8)z^{-2}}$$

$$= \frac{4(1 + (1/16)z^{-1} - \frac{1}{8}z^{-2})}{1 + (1/4)z^{-1} - (1/8)z^{-2}}$$

$$= 4 - \frac{(3/4)z^{-1}}{1 + (1/4)z^{-1} - (1/8)z^{-2}}$$

$$= 4 - \frac{(3/4)z^{-1}}{(1 - (1/4)z^{-1})(1 + (\frac{1}{2})z^{-1})}$$

$$= 4 - \frac{1}{(1 - (1/4)z^{-1})} + \frac{1}{(1 + (1/2)z^{-1})}$$

$$h[n] = 4\,\delta[n] - (\tfrac{1}{4})^n u[n] + (-\tfrac{1}{2})^n u[n]$$

e)

$x[n] = u[-n-1] \longrightarrow$ left-sided sequence

$$X(z) = \frac{-1}{1 - z^{-1}}$$ ROC: $|z| < 1$

$Y(z) = H(z)\,X(z)$; $ROC_Y : ROC_H \cap ROC_X$

$$Y(z) = \frac{\left(-4 - (1/4)\, z^{-1} + (1/2)\, z^{-2}\right)(-1)}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 + \frac{1}{2} z^{-1}\right)\left(1 - z^{-1}\right)}$$

ROC : $\frac{1}{2} < |z| < 1$

~~Tutorial 3~~

Q 3.33

a) $X(z) = \dfrac{3z^{-3}}{(1-\frac{1}{4}z^{-1})^2}$ $x[n]$ is left_sided

from Z-Table :

$$n\,x[n] \longrightarrow -z\frac{d}{dz}X(z)$$

$$X(z) = 3z^{-3}\cdot\frac{(-\frac{1}{4}z^{-2})}{(1-\frac{1}{4}z^{-1})^2}\cdot\left(\frac{1}{-\frac{1}{4}z^{-2}}\right)$$

$$= -12\,z^{-1}\underbrace{\left[\frac{(-1/4)z^{-2}}{(1-(\frac{1}{4})z^{-1})^2}\right]}_{X_1(z)}$$

$$n\,x_1[n] \longrightarrow -z\frac{d}{dz}X_1(z)$$

$$X(z) = -12\,z^{-1}\left[-z\frac{d}{dz}\left(\frac{1}{(1-\frac{1}{4}z^{-1})}\right) * \frac{-1}{z}\right]$$

$$= 12\,z^{-2}\underbrace{\left[-z\frac{d}{dz}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\right]}_{n\,x_1[n]}$$

$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1]$ $\longrightarrow$ Note That $x[n]$ is left-sided.

$$\therefore x[n] = 12\,x_1[n-2]$$

$$= -12\left(\frac{1}{4}\right)^{n-2} u[-(n-2)-1]$$

$$= -12\left(\frac{1}{4}\right)^{n-2} u[-n+1]$$

b) $X(z) = \sin(z)$ ROC includes $|z| = 1$

Maclaurin expansion:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\therefore X(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

$$\therefore x[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \delta[n+2k+1]$$

c) $X(z) = \dfrac{z^7 - 2}{1 - z^{-7}}$ $|z| > 1$

$$= \frac{z^7 - 1 - 1}{1 - z^{-7}} = \frac{z^7 - 1}{1 - z^{-7}} - \frac{1}{1 - z^{-7}}$$

$$= z^7 - \frac{1}{1 - z^{-7}}$$

we have $\displaystyle\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}$

$$\therefore \sum_{k=0}^{\infty} z^{-7k} = \frac{1}{1 - z^{-7}}$$

$$X(z) = z^7 - \sum_{k=0}^{\infty} z^{-7k}$$

$$\therefore x[n] = \delta[n+7] - \sum_{n=0}^{\infty} \delta[n - 7k]$$