PROPERTIES OF THE ROC FOR THE z-TRANSFORM

PROPERTY 1: The ROC will either be of the form $0 \le r_R < |z|$, or $|z| < r_L \le \infty$, or, in general the annulus, i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Q 3.1

1. Determine the z-transform, including the ROC, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n] - u[n-10])$.

Q 3.2

2. Determine the z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases}$$

Q 3.3

3. Determine the z-transform of each of the following sequences. Include with your answer the ROC in the z-plane and a sketch of the pole–zero plot. Express all sums in closed form; α can be complex.

(a) $x_a[n] = \alpha^{|n|}, \qquad 0 < |\alpha| < 1.$

(b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2(N-1), \\ 0, & \text{otherwise.} \end{cases}$

Hint: Note that $x_b[n]$ is a rectangular sequence and $x_c[n]$ is a triangular sequence. First, express $x_c[n]$ in terms of $x_b[n]$.

Q 3.4

4. Consider the z-transform $X(z)$ whose pole–zero plot is as shown in Figure P4.

(a) Determine the ROC of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) How many possible two-sided sequences have the pole–zero plot shown in Figure P4?

(c) Is it possible for the pole–zero plot in Figure P4 to be associated with a sequence that is both stable and causal? If so, give the appropriate ROC.

The z-Transform

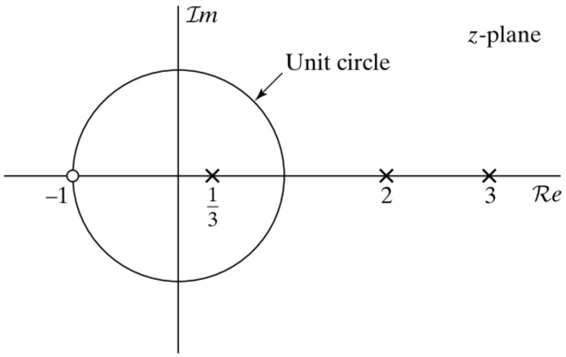

Figure P4

Q 3.7

7. The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}.$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

(b) What is the ROC for $Y(z)$?

(c) Determine $y[n]$.

Q 3.11

11. Following are four z-transforms. Determine which ones *could* be the z-transform of a *causal* sequence. Do not evaluate the inverse transform. You should be able to give the answer by inspection. Clearly state your reasons in each case.

(a) $\dfrac{(1-z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)}$

(b) $\dfrac{(z-1)^2}{\left(z-\frac{1}{2}\right)}$

(c) $\dfrac{\left(z-\frac{1}{4}\right)^5}{\left(z-\frac{1}{2}\right)^6}$

(d) $\dfrac{\left(z-\frac{1}{4}\right)^6}{\left(z-\frac{1}{2}\right)^5}$

Q 3.12

12. Sketch the pole–zero plot for each of the following z-transforms and shade the ROC:

(a) $X_1(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$, ROC: $|z| < 2$

(b) $X_2(z) = \dfrac{1 - \frac{1}{3}z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)}$, $x_2[n]$ causal

(c) $X_3(z) = \dfrac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$, $x_3[n]$ absolutely summable.

Q 3.17

17. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

Determine all possible values for the system's impulse response $h[n]$ at $n = 0$.

Q 3.21

21. A causal LTI system has the following system function:

$$H(z) = \frac{4 + 0.25z^{-1} - 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})}$$

(a) What is the ROC for $H(z)$?

(b) Determine if the system is stable or not.
(c) Determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$.
(d) Use a partial fraction expansion to determine the impulse response $h[n]$.
(e) Find $Y(z)$, the z-transform of the output, when the input is $x[n] = u[-n-1]$. Be sure to specify the ROC for $Y(z)$.
(f) Find the output sequence $y[n]$ when the input is $x[n] = u[-n-1]$.

Q 3.33

33. Determine the inverse z-transform of each of the following. You should find the z-transform properties in Section 4 helpful.

(a) $X(z) = \dfrac{3z^{-3}}{\left(1 - \frac{1}{4}z^{-1}\right)^2}$, $\quad x[n]$ left sided

(b) $X(z) = \sin(z)$, $\quad$ ROC includes $|z| = 1$

(c) $X(z) = \dfrac{z^7 - 2}{1 - z^{-7}}$, $\quad |z| > 1$

Q 3.8

8. The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output $y[n]$.
(c) Is the system stable? That is, is $h[n]$ absolutely summable?