ELG4177 - DIGITAL SIGNAL PROCESSING
Tutorial 8

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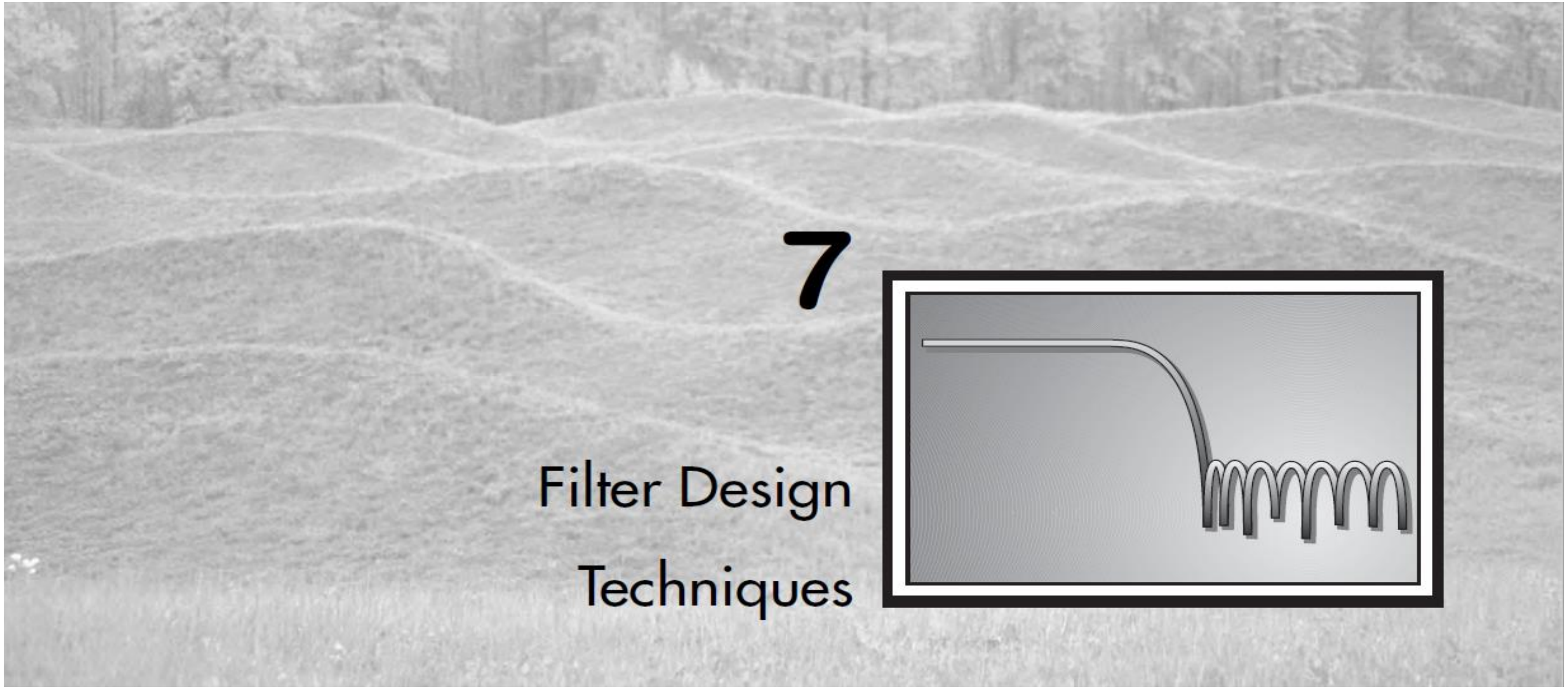


Problems: 7.2, 7.5, 7.6, 7.9, 7.10, 7.11, 7.12, 7.15, 7.16, 7.17, 7.18, 7.22, 7.28, 7.30

7.2. A discrete-time lowpass filter is to be designed by applying the impulse invariance method to a continuous-time Butterworth filter having magnitude-squared function

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}.$$

The specifications for the discrete-time system are those of Example 7.2, i.e.,

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi,$$
$$|H(e^{j\omega})| \le 0.17783, \quad 0.3\pi \le |\omega| \le \pi.$$

Assume, as in that example, that aliasing will not be a problem; i.e., design the continuous-time Butterworth filter to meet passband and stopband specifications as determined by the desired discrete-time filter.

(a) Sketch the tolerance bounds on the magnitude of the frequency response, $|H_c(j\Omega)|$, of the continuous-time Butterworth filter such that after application of the impulse invariance method (i.e., $h[n] = T_d h_c(nT_d)$), the resulting discrete-time filter will satisfy the given design specifications. Do not assume that $T_d = 1$ as in Example 7.2.

(b) Determine the integer order N and the quantity $T_d\Omega_c$ such that the continuous-time Butterworth filter exactly meets the specifications determined in part (a) at the passband edge.

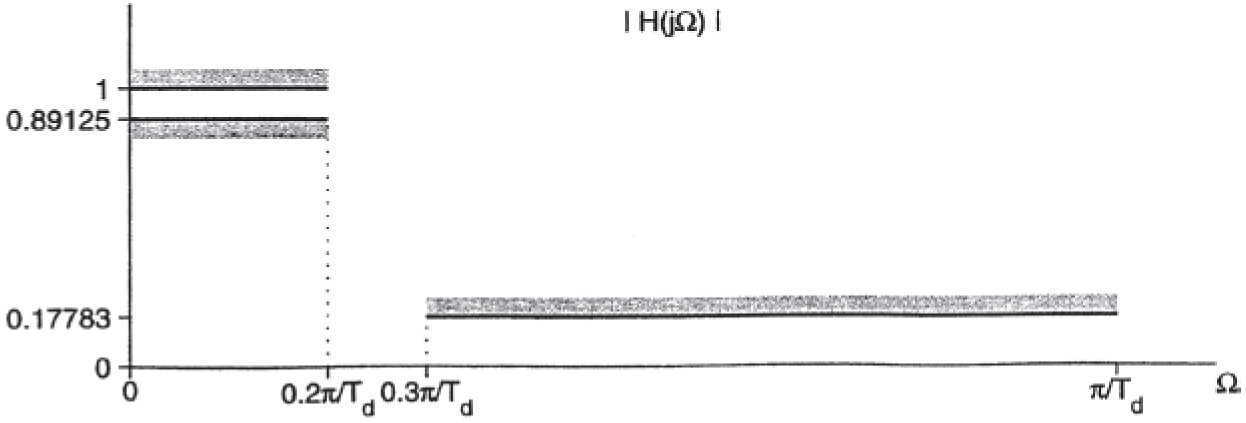
Recall that $\Omega = \omega/T_d$.

(a) Then

$$0.89125 \le |H(j\Omega)| \le 1, \quad 0 \le |\Omega| \le 0.2\pi/T_d$$

$$|H(j\Omega)| \le 0.17783, \quad 0.3\pi/T_d \le |\Omega| \le \pi/T_d$$

The plot of the tolerance scheme is



(b) As in the book's example, since the Butterworth frequency response is monotonic, we can solve

$$|H_c(j0.2\pi/T_d)|^2 = \frac{1}{1+\left(\frac{0.2\pi}{\Omega_c T_d}\right)^{2N}} = (0.89125)^2$$

$$|H_c(j0.3\pi/T_d)|^2 = \frac{1}{1+\left(\frac{0.3\pi}{\Omega_c T_d}\right)^{2N}} = (0.17783)^2$$

to get $\Omega_c T_d = 0.70474$ and $N = 5.8858$. Rounding up to $N = 6$ yields $\Omega_c T_d = 0.7032$ to meet the specifications.

From the magnitude-squared function in eq. (B.1), we observe by substituting $j\Omega = s$ that $H_c(s)H_c(-s)$ must be of the form

$$H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}. \qquad (B.2)$$

The roots of the denominator polynomial are therefore located at values of s satisfying $1 + (s/j\Omega_c)^{2N} = 0$; i.e.,

$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)}, \quad (B.3)$$

where, $k = 0, 1, \ldots, 2N-1$.

Thus, there are $2N$ poles equally spaced in angle on a circle of radius Ω_c in the s-plane. The poles are symmetrically located with respect to the imaginary axis.

Filter design by impulse invariance (II)

We consider the system function of a casual continuous-time filter expressed in terms of a partial fraction expansion, so that

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}. \qquad (7.7)$$

The corresponding impulse response is

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \ge 0, \\ 0, & t < 0. \end{cases} \qquad (7.8)$$

The impulse response of the causal discrete-time filter obtained by sampling $T_d h_c(t)$ is

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] = \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n]. \qquad (7.9)$$

The system function of the causal discrete-time filter is therefore given by

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}. \qquad (7.10)$$

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7.5. We wish to use the Kaiser window method to design a discrete-time filter with generalized linear phase that meets specifications of the following form:

$$\begin{aligned} |H(e^{j\omega})| &\le 0.01, & 0 \le |\omega| \le 0.25\pi, \\ 0.95 \le |H(e^{j\omega})| &\le 1.05, & 0.35\pi \le |\omega| \le 0.6\pi, \\ |H(e^{j\omega})| &\le 0.01, & 0.65\pi \le |\omega| \le \pi. \end{aligned}$$

(a) Determine the minimum length $(M + 1)$ of the impulse response and the value of the Kaiser window parameter β for a filter that meets the preceding specifications.

(b) What is the delay of the filter?

(c) Determine the ideal impulse response $h_d[n]$ to which the Kaiser window should be applied.

Example 7.7 Linear-Phase Lowpass Filter

The desired frequency response is defined as

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \quad (7.69)$$

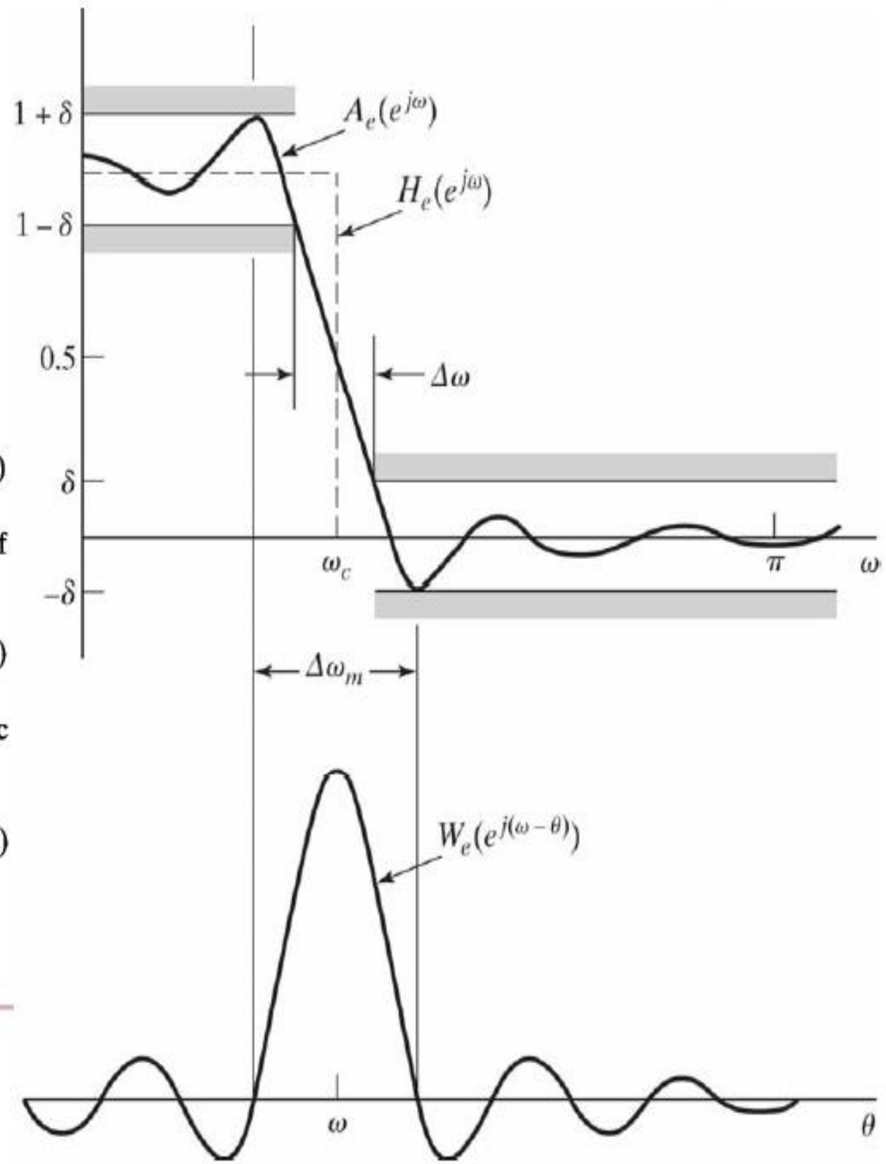
where the generalized linear-phase factor has been incorporated into the definition of the ideal lowpass filter. The corresponding ideal impulse response is

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} e^{j\omega n}\, d\omega = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} \quad (7.70)$$

for $-\infty < n < \infty$. It is easily shown that $h_{lp}[M-n] = h_{lp}[n]$, so if we use a symmetric window in the equation

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n], \quad (7.71)$$

then a linear-phase system will result.



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(a) We must use the minimum specifications!

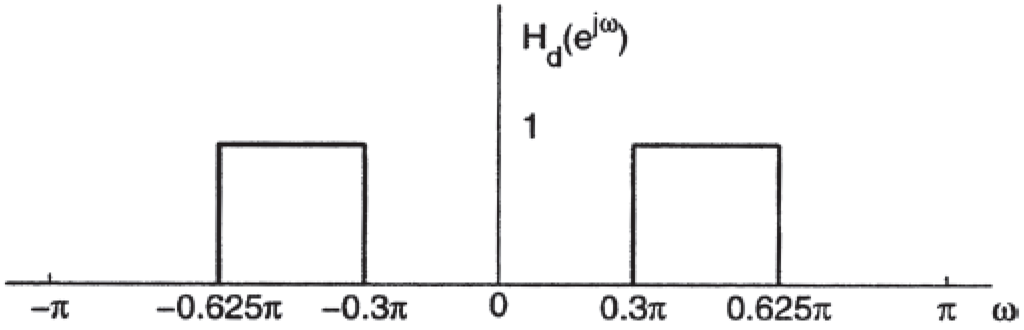
$$\begin{aligned} \delta &= 0.01 \\ \Delta\omega &= 0.05\pi \\ A &= -20\log_{10}\delta = 40 \end{aligned}$$

$$M + 1 = \frac{A - 8}{2.285\Delta\omega} + 1 = 90.2 \rightarrow 91$$

$$\beta = 0.5842(A - 21)^{0.4} + 0.07886(A - 21) = 3.395$$

(b) Since it is a linear phase filter with order 90, it has a delay of 90/2 = 45 samples.

(c)



$$h_d[n] = \frac{\sin(.625\pi(n - 45)) - \sin(.3\pi(n - 45))}{\pi(n - 45)}$$

$$W[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1 - \left[\frac{n-\alpha}{\alpha}\right]^2}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{ow} \end{cases}$$

① $\Delta\omega = \omega_s - \omega_p$

② $A = -20\log_{10}\delta$

③ $\beta = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$

④ $M = \dfrac{A-8}{2.285\Delta\omega}$

7.6. We wish to use the Kaiser window method to design a symmetric real-valued FIR filter with zero phase that meets the following specifications:

$$\begin{aligned} 0.9 < H(e^{j\omega}) &< 1.1, & 0 \le |\omega| \le 0.2\pi, \\ -0.06 < H(e^{j\omega}) &< 0.06, & 0.3\pi \le |\omega| \le 0.475\pi, \\ 1.9 < H(e^{j\omega}) &< 2.1, & 0.525\pi \le |\omega| \le \pi. \end{aligned}$$

This specification is to be met by applying the Kaiser window to the ideal real-valued impulse response associated with the ideal frequency response $H_d(e^{j\omega})$ given by

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 0.25\pi, \\ 0, & 0.25\pi \le |\omega| \le 0.5\pi, \\ 2, & 0.5\pi \le |\omega| \le \pi. \end{cases}$$

(a) What is the maximum value of δ that can be used to meet this specification? What is the corresponding value of β? Clearly explain your reasoning.

(b) What is the maximum value of $\Delta\omega$ that can be used to meet the specification? What is the corresponding value of $M+1$, the length of the impulse response? Clearly explain your reasoning.

(a) The Kaiser formulas say that a discontinuity of height 1 produces a peak error of δ. If a filter has a discontinuity of a different height the peak error should be scaled appropriately. This filter can be thought of as the sum of two filters. This first is a lowpass filter with a discontinuity of 1 and a peak error of δ. The second is a highpass filter with a discontinuity of 2 and a peak error of 2δ. In the region $0.3\pi \le |\omega| \le 0.475\pi$, the two peak errors add but must be less or equal to than 0.06.

$$\begin{aligned} \delta + 2\delta &\le 0.06 \\ \delta_{\max} &= 0.02 \end{aligned}$$

$$A = -20\log(0.02) = 33.9794$$

$$\beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65$$

(b) The transition width can be

$$\begin{aligned} \Delta\omega &= 0.3\pi - 0.2\pi \\ &= 0.1\pi \text{ rad} \end{aligned} \quad or \quad \begin{aligned} \Delta\omega &= 0.525\pi - 0.475\pi \\ &= 0.05\pi \text{ rad} \end{aligned}$$

We must choose the smallest transition width so $\Delta\omega_{\max} = 0.05\pi$ rad. The corresponding value of M is

$$M = \frac{33.9794 - 8}{2.285(0.05\pi)} = 72.38 \rightarrow 73$$

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1 - \left[\frac{n-\alpha}{\alpha}\right]^2}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{ow} \end{cases}$$

① $\Delta\omega = \omega_s - \omega_p$

② $A = -20\log_{10}\delta$

③ $\beta = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$

④ $M = \dfrac{A - 8}{2.285\,\Delta\omega}$

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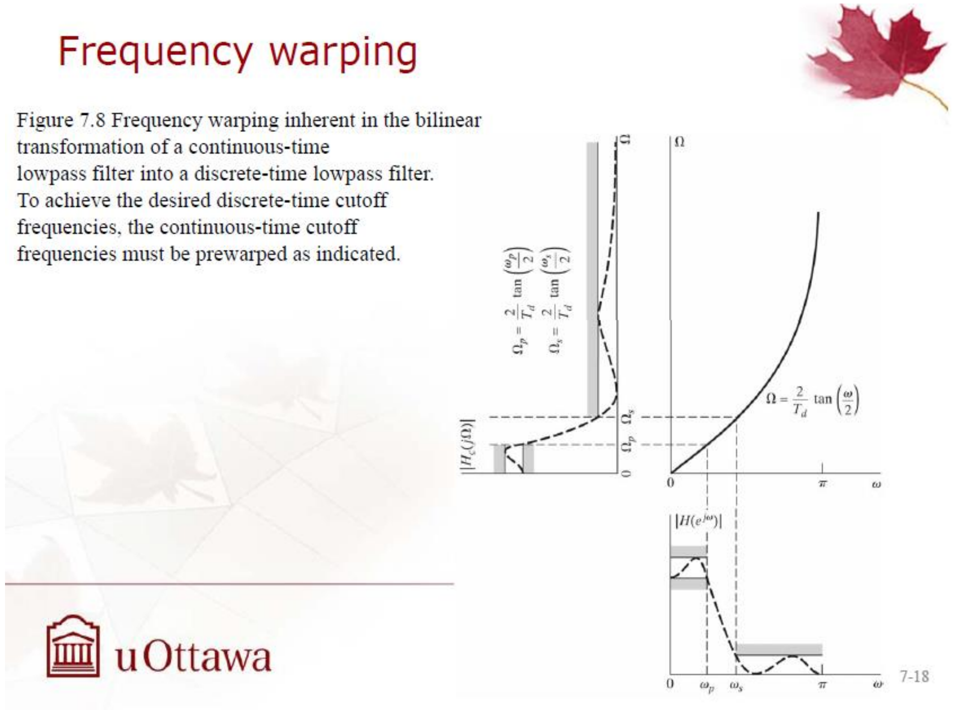
7.9. Suppose we design a discrete-time filter using the impulse invariance technique with an ideal continuous-time lowpass filter as a prototype. The prototype filter has a cutoff frequency of $\Omega_c = 2\pi(1000)$ rad/s, and the impulse invariance transformation uses $T = 0.2$ ms. What is the cutoff frequency ω_c for the resulting discrete-time filter?

Solution

Using the relation $\omega = \Omega T$, the cutoff frequency ω_c for the resulting discrete-time filter is

$$\begin{aligned}\omega_c &= \Omega_c T \\ &= [2\pi(1000)][0.0002] \\ &= 0.4\pi \text{ rad}\end{aligned}$$

7.10. We wish to design a discrete-time lowpass filter using the bilinear transformation on a continuous-time ideal lowpass filter. Assume that the continuous-time prototype filter has cutoff frequency $\Omega_c = 2\pi(2000)$ rad/s, and we choose the bilinear transformation parameter $T = 0.4$ ms. What is the cutoff frequency ω_c for the resulting discrete-time filter?



Solution

Using the bilinear transform frequency mapping equation,

$$
\begin{aligned}
\omega_c &= 2\tan^{-1}\left(\frac{\Omega_c T}{2}\right) \\
&= 2\tan^{-1}\left(\frac{2\pi(2000)(0.4\times 10^{-3})}{2}\right) \\
&= 0.7589\pi \text{ rad}
\end{aligned}
$$

7.11. Suppose that we have an ideal discrete-time lowpass filter with cutoff frequency $\omega_c = \pi/4$. In addition, we are told that this filter resulted from applying impulse invariance to a continuous-time prototype lowpass filter using $T = 0.1$ ms. What was the cutoff frequency Ω_c for the prototype continuous-time filter?

Solution

Using the relation $\omega = \Omega T$,

$$\begin{aligned}\Omega_c &= \frac{\omega_c}{T} \\ &= \frac{\pi/4}{0.0001} \\ &= 2500\pi \\ &= 2\pi(1250)\ \frac{\text{rad}}{\text{s}}\end{aligned}$$

7.12. An ideal discrete-time highpass filter with cutoff frequency $\omega_c = \pi/2$ was designed using the bilinear transformation with $T = 1$ ms. What was the cutoff frequency Ω_c for the prototype continuous-time ideal highpass filter?

Solution

Using the bilinear transform frequency mapping equation,

$$\begin{aligned}\Omega_c &= \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)\\ &= \frac{2}{0.001}\tan\left(\frac{\pi/2}{2}\right)\\ &= 2000\ \frac{\text{rad}}{\text{s}}\\ &= 2\pi(318.3)\ \frac{\text{rad}}{\text{s}}\end{aligned}$$

7.15. We wish to design an FIR lowpass filter satisfying the specifications

$$0.95 < H(e^{j\omega}) < 1.05, \qquad 0 \le |\omega| \le 0.25\pi,$$

$$-0.1 < H(e^{j\omega}) < 0.1, \qquad 0.35\pi \le |\omega| \le \pi,$$

by applying a window $w[n]$ to the impulse response $h_d[n]$ for the ideal discrete-time lowpass filter with cutoff $\omega_c = 0.3\pi$. Which of the windows listed in Section 7.5.1 can be used to meet this specification? For each window that you claim will satisfy this specification, give the minimum length $M + 1$ required for the filter.

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

This filter requires a maximal passband error of $\delta_p = 0.05$, and a maximal stopband error of $\delta_s = 0.1$. Converting these values to dB gives

$$\delta p = -26 \text{ dB}$$
$$\delta s = -20 \text{ dB}$$

This requires a window with a peak approximation error less than -26 dB. Looking in Table 7.1, the Hanning, Hamming, and Blackman windows meet this criterion.

Next, the minimum length L required for each of these filters can be found using the "approximate width of mainlobe" column in the table since the mainlobe width is about equal to the transition width. Note that the actual length of the filter is $L = M + 1$.

Hanning:

$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Hamming:

$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Blackman:

$$\begin{aligned} 0.1\pi &= \frac{12\pi}{M} \\ M &= 120 \end{aligned}$$

7.16. We wish to design an FIR lowpass filter satisfying the specifications

$$0.98 < H(e^{j\omega}) < 1.02, \qquad 0 \le |\omega| \le 0.63\pi,$$

$$-0.15 < H(e^{j\omega}) < 0.15, \qquad 0.65\pi \le |\omega| \le \pi,$$

by applying a Kaiser window to the impulse response $h_d[n]$ for the ideal discrete-time lowpass filter with cutoff $\omega_c = 0.64\pi$. Find the values of β and M required to satisfy this specification.

Solution

Since filters designed by the window method inherently have $\delta_1 = \delta_2$ we must use the smaller value for δ.

$$\begin{aligned} \delta &= 0.02 \\ A &= -20\log_{10}(0.02) = 33.9794 \end{aligned}$$

$$\beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65$$

$$M = \frac{A - 8}{2.285\Delta\omega} = \frac{33.9794 - 8}{2.285(0.65\pi - 0.63\pi)} = 180.95 \rightarrow 181$$

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7.17. Suppose that we wish to design a bandpass filter satisfying the following specification:

$$-0.02 < |H(e^{j\omega})| < 0.02, \qquad 0 \le |\omega| \le 0.2\pi,$$
$$0.95 < |H(e^{j\omega})| < 1.05, \qquad 0.3\pi \le |\omega| \le 0.7\pi,$$
$$-0.001 < |H(e^{j\omega})| < 0.001, \qquad 0.75\pi \le |\omega| \le \pi.$$

The filter will be designed by applying impulse invariance with $T = 5$ ms to a prototype continuous-time filter. State the specifications that should be used to design the prototype continuous-time filter.

Solution

Using the relation $\omega = \Omega T$, the specifications which should be used to design the prototype continuous-time filter are

$$-0.02 < H(j\Omega) < 0.02, \qquad 0 \le |\Omega| \le 2\pi(20)$$
$$0.95 < H(j\Omega) < 1.05, \qquad 2\pi(30) \le |\Omega| \le 2\pi(70)$$
$$-0.001 < H(j\Omega) < 0.001, \qquad 2\pi(75) \le |\Omega| \le 2\pi(100)$$

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7.18. Suppose that we wish to design a highpass filter satisfying the following specification:

$$-0.04 < |H(e^{j\omega})| < 0.04, \qquad 0 \le |\omega| \le 0.2\pi,$$
$$0.995 < |H(e^{j\omega})| < 1.005, \qquad 0.3\pi \le |\omega| \le \pi.$$

The filter will be designed using the bilinear transformation and $T = 2$ ms with a prototype continuous-time filter. State the specifications that should be used to design the prototype continuous-time filter to ensure that the specifications for the discrete-time filter are met.

Solution

Using the bilinear transform frequency mapping equation,

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.2\pi}{2}\right) = 2\pi(51.7126)\ \frac{\text{rad}}{\text{s}}$$
$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.3\pi}{2}\right) = 2\pi(81.0935)\ \frac{\text{rad}}{\text{s}}$$

Thus, the specifications which should be used to design the prototype continuous-time filter are

$$|H_c(j\Omega)| < 0.04, \qquad |\Omega| \le 2\pi(51.7126)$$
$$0.995 < |H_c(j\Omega)| < 1.005, \qquad |\Omega| \ge 2\pi(81.0935)$$

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7.22. In the system shown in Figure P7.22, the discrete-time system is a linear-phase FIR lowpass filter designed by the Parks–McClellan algorithm with $\delta_1 = 0.01$, $\delta_2 = 0.001$, $\omega_p = 0.4\pi$, and $\omega_s = 0.6\pi$. The length of the impulse response is 28 samples. The sampling rate for the ideal C/D and D/C converters is $1/T = 10000$ samples/sec.

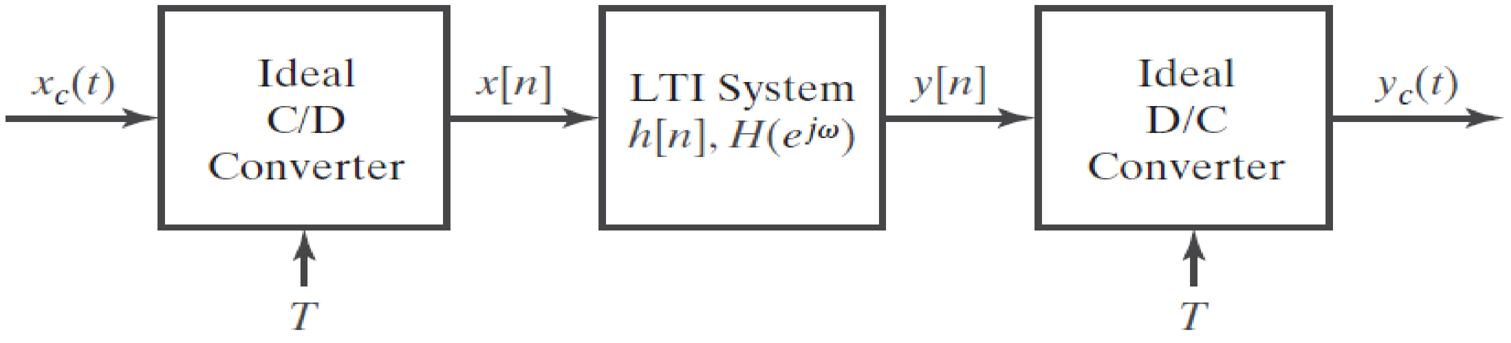

Figure P7.22

(a) What property should the input signal have so that the overall system behaves as an LTI system with $Y_c(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega)$?

(b) For the conditions found in (a), determine the approximation error specifications satisfied by $|H_{eff}(j\Omega)|$. Give your answer as either an equation or a plot as a function of Ω.

(c) What is the overall delay from the continuous-time input to the continuous-time output (in seconds) of the system in Figure P7.22?

A. Strictly speaking, the input $x_c(t)$ must be bandlimited to 5000 Hz to ensure that there is no aliasing when sampled at 10000 samples/sec. As a practical matter, it may be adequate to bandlimit the input to 7000 Hz. Frequency components between 5000 and 7000 Hz will alias to the range $\Omega = 2\pi 3000$ to $2\pi 5000$ rad/s, or $\omega = 0.6\pi$ to π, using $\omega = \Omega T$. Thus the aliased components will fall in the stopband of the discrete-time lowpass filter.

B. For the continuous-time system, the passband edge is $\Omega_p = \omega_p / T = 0.4\pi \times 10000 = 2\pi 2000$ rad/s. The stopband edge is $\Omega_s = \omega_s / T = 0.6\pi \times 10000 = 2\pi 3000$ rad/s. Within the passband the specifications are

$$(1-\delta_1) \le |H_{eff}(j\Omega)| \le (1+\delta_1), \quad |\Omega| \le \Omega_p$$

$$0.99 \le |H_{eff}(j\Omega)| \le 1.02, \quad |\Omega| \le 2\pi 2000.$$

Within the stopband the specifications are

$$|H_{eff}(j\Omega)| \le \delta_2, \quad \Omega_s \le \Omega \le 2\pi 5000$$

$$|H_{eff}(j\Omega)| \le 0.001, \quad 2\pi 3000 \le \Omega \le 2\pi 5000.$$

C. The given filter is a linear phase filter whose impulse response has a length of 28 samples. The group delay of the filter is $\alpha = 27/2 = 13.5$ samples. Since samples are spaced 10^{-4} seconds apart, the delay in seconds is $13.5 \times 10^{-4} = 1.35$ ms.

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7.28. Consider a continuous-time lowpass filter $H_c(s)$ with passband and stopband specifications

$$1-\delta_1 \le |H_c(j\Omega)| \le 1+\delta_1, \qquad |\Omega| \le \Omega_p,$$
$$|H_c(j\Omega)| \le \delta_2, \qquad \Omega_s \le |\Omega|.$$

This filter is transformed to a lowpass discrete-time filter $H_1(z)$ by the transformation

$$H_1(z) = H_c(s)\big|_{s=(1-z^{-1})/(1+z^{-1})},$$

and the same continuous-time filter is transformed to a highpass discrete-time filter by the transformation

$$H_2(z) = H_c(s)\big|_{s=(1+z^{-1})/(1-z^{-1})}.$$

(a) Determine a relationship between the passband cutoff frequency Ω_p of the continuous-time lowpass filter and the passband cutoff frequency ω_{p1} of the discrete-time lowpass filter.

(b) Determine a relationship between the passband cutoff frequency Ω_p of the continuous-time lowpass filter and the passband cutoff frequency ω_{p2} of the discrete-time highpass filter.

(c) Determine a relationship between the passband cutoff frequency ω_{p1} of the discrete-time lowpass filter and the passband cutoff frequency ω_{p2} of the discrete-time highpass filter.

(a) We have

$$\begin{aligned} s &= \frac{1-z^{-1}}{1+z^{-1}} \\ j\Omega &= \frac{1-e^{-j\omega}}{1+e^{-j\omega}} \\ &= \frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} \\ \Omega &= \tan\left(\frac{\omega}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_1}}{2}\right) \longleftrightarrow \omega_{p_1} = 2\tan^{-1}(\Omega_p)$$

(b)

$$\begin{aligned} s &= \frac{1+z^{-1}}{1-z^{-1}} \\ j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} \\ \Omega &= -\cot\left(\frac{\omega}{2}\right) \\ &= \tan\left(\frac{\omega-\pi}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_2}-\pi}{2}\right) \longleftrightarrow \omega_{p_2} = \pi + 2\tan^{-1}(\Omega_p)$$

(c)

$$\tan\left(\frac{\omega_{p_2}-\pi}{2}\right) = \tan\left(\frac{\omega_{p_1}}{2}\right)$$

$$\Rightarrow \omega_{p_2} = \omega_{p_1} + \pi$$

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7.30. Consider designing a discrete-time filter with system function $H(z)$ from a continuous-time filter with rational system function $H_c(s)$ by the transformation

$$H(z) = H_c(s)\big|_{s=\beta[(1-z^{-\alpha})/(1+z^{-\alpha})]},$$

where α is a nonzero integer and β is real.

(a) If $\alpha > 0$, for what values of β does a stable, causal continuous-time filter with rational $H_c(s)$ always lead to a stable, causal discrete-time filter with rational $H(z)$?

(b) If $\alpha < 0$, for what values of β does a stable, causal continuous-time filter with rational $H_c(s)$ always lead to a stable, causal discrete-time filter with rational $H(z)$?

(c) For $\alpha = 2$ and $\beta = 1$, determine to what contour in the z-plane the $j\Omega$-axis of the s-plane maps.

We are given

$$H(z) = H_c(s)\,|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]}$$

where α is a nonzero integer and β is a real number.

(a) *It is true for $\beta > 0$.*
Proof:

$$\begin{aligned}
s &= \beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right] \\
s + sz^{-\alpha} &= \beta - \beta z^{-\alpha} \\
s - \beta &= -\beta z^{-\alpha} - sz^{-\alpha} \\
\beta - s &= z^{-\alpha}(\beta + s) \\
z^{-\alpha} &= \frac{\beta - s}{\beta + s} \\
z^{\alpha} &= \frac{\beta + s}{\beta - s}
\end{aligned}$$

The poles s_k of a stable, causal, continuous-time filter satisfy the condition $\mathcal{R}e\{s\} < 0$. We want these poles to map to the points z_k in the z-plane such that $|z_k| < 1$. With $\alpha > 0$ it is also true that if $|z_k| < 1$ then $|z_k^{\alpha}| < 1$. Letting $s_k = \sigma + j\omega$ we see that

$$\begin{aligned}
|z_k| &< 1 \\
|z_k^{\alpha}| &< 1 \\
|\beta + \sigma + j\Omega| &< |\beta - \sigma - j\Omega| \\
(\beta + \sigma)^2 + \Omega^2 &< (\beta - \sigma)^2 + \Omega^2 \\
2\sigma\beta &< -2\sigma\beta
\end{aligned}$$

But since the continuous-time filter is stable we have $\mathcal{R}e\{s_k\} < 0$ or $\sigma < 0$. That leads to

$$-\beta < \beta$$

This can only be true if $\beta > 0$.

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(b) *It is true for* $\beta < 0$. The proof is similar to the last proof except now we have $|z^\alpha| > 1$.

(c) We have

$$\begin{aligned} z^2 &= \left.\frac{1+s}{1-s}\right|_{s=j\Omega} \\ |z^2| &= 1 \\ |z| &= 1 \end{aligned}$$