ELG4177 - DIGITAL SIGNAL PROCESSING
Tutorial 7

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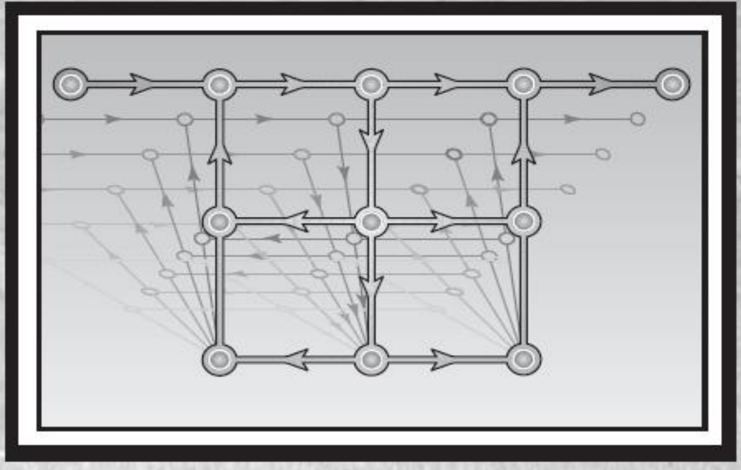
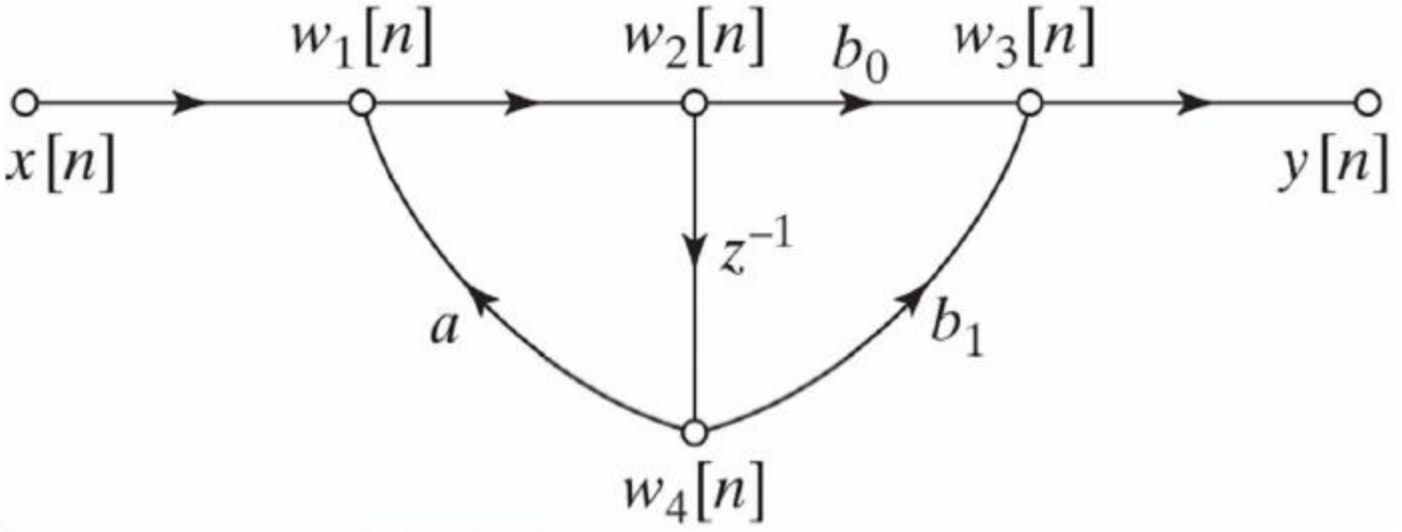
6

Structures for Discrete-Time Systems



Signal Flow Graph

Signal flow graph with the delay branch indicated by z^{-1}



$$w_1[n] = aw_4[n] + x[n], \tag{6.18a}$$

$$w_2[n] = w_1[n], \tag{6.18b}$$

$$w_3[n] = b_0 w_2[n] + b_1 w_4[n], \tag{6.18c}$$

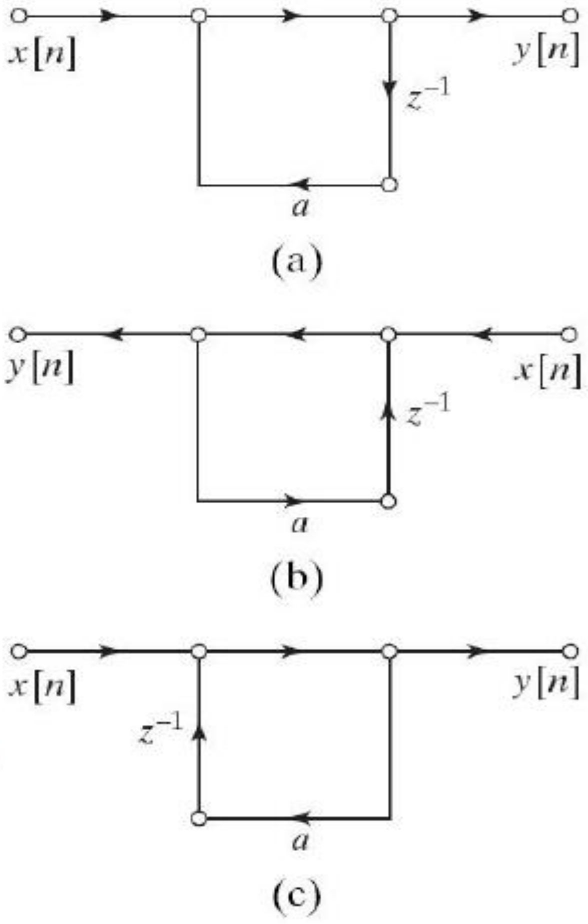
$$w_4[n] = w_2[n-1], \tag{6.18d}$$

$$y[n] = w_3[n]. \tag{6.18e}$$

Transposed Form

Example 6.7 Transposed form

(a) Flow graph of simple 1st-order system.
(b) Transposed form of (a).
(c) Structure of (b) redrawn with input on left.



IIR Filter Structure

6.3 Basic structures for IIR systems

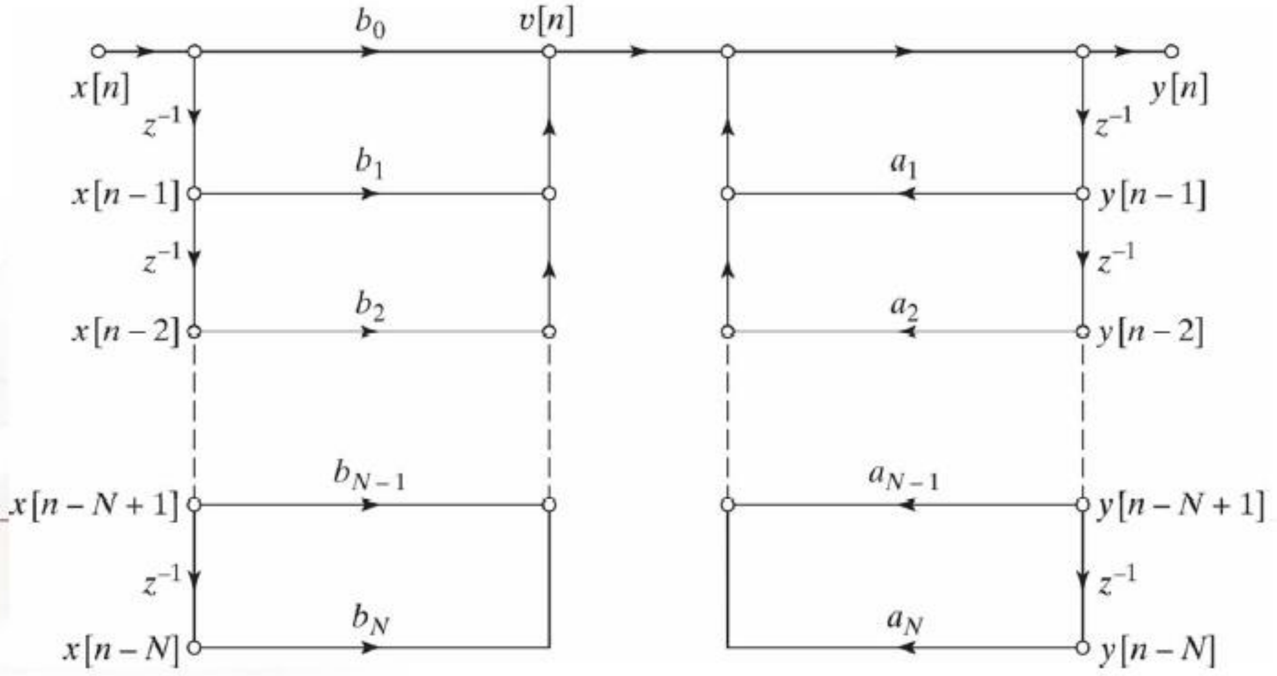
- Direct forms
- Cascade forms
- Parallel forms

Direct Form I Structure

Signal flow graph of direct form I structure for an Nth-order system

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad (6.26)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}. \quad (6.27)$$



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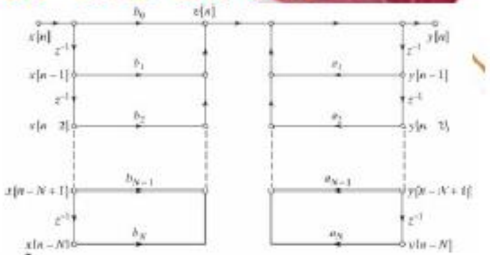
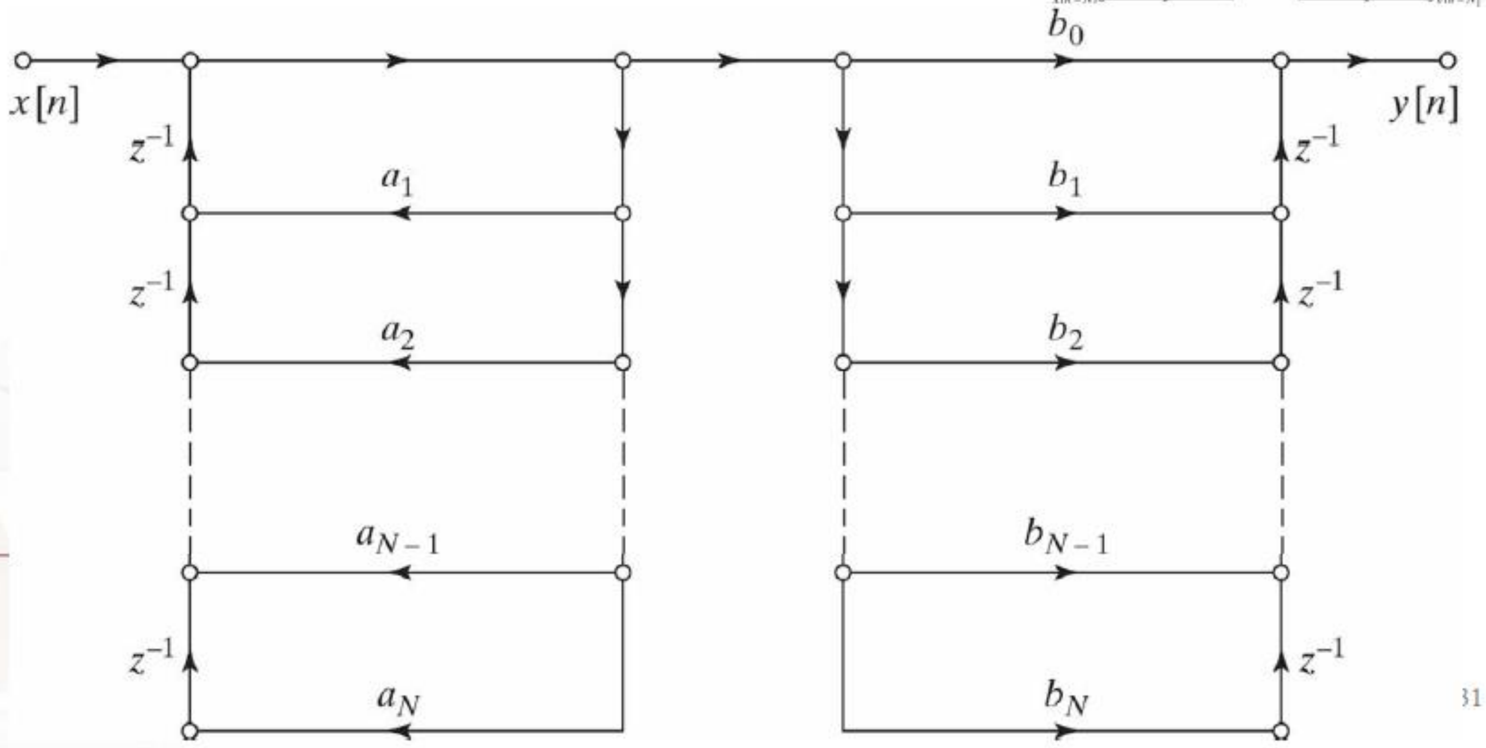
Transposed Direct Form I Structure

Figure 6.27 General flow graph resulting from applying the transposition theorem to the direct form I structure of Figure 6.14



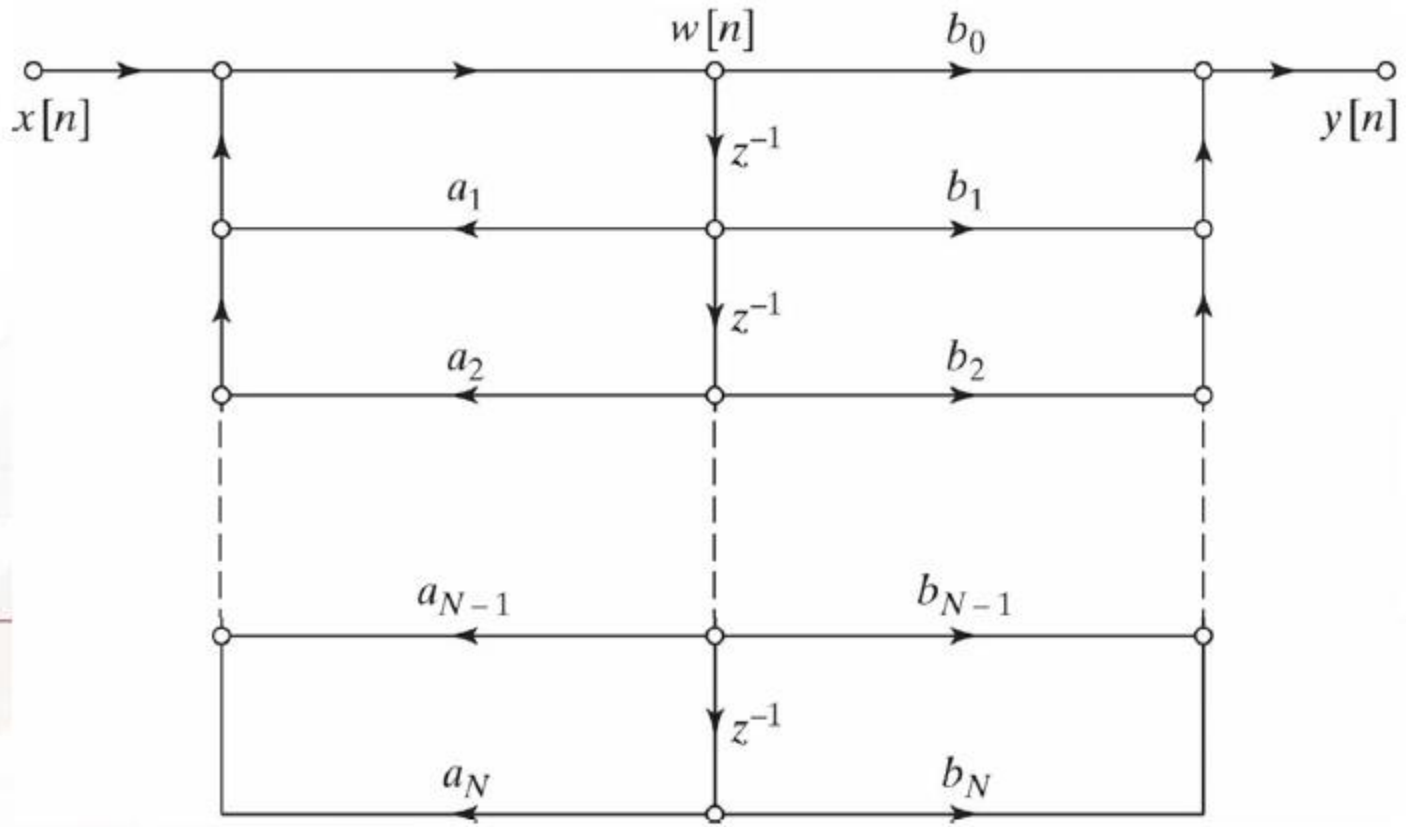


Direct Form II Structure

Signal flow graph of direct form II structure for an Nth-order system

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad (6.26)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}. \quad (6.27)$$



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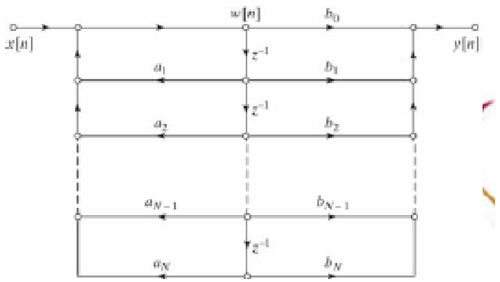
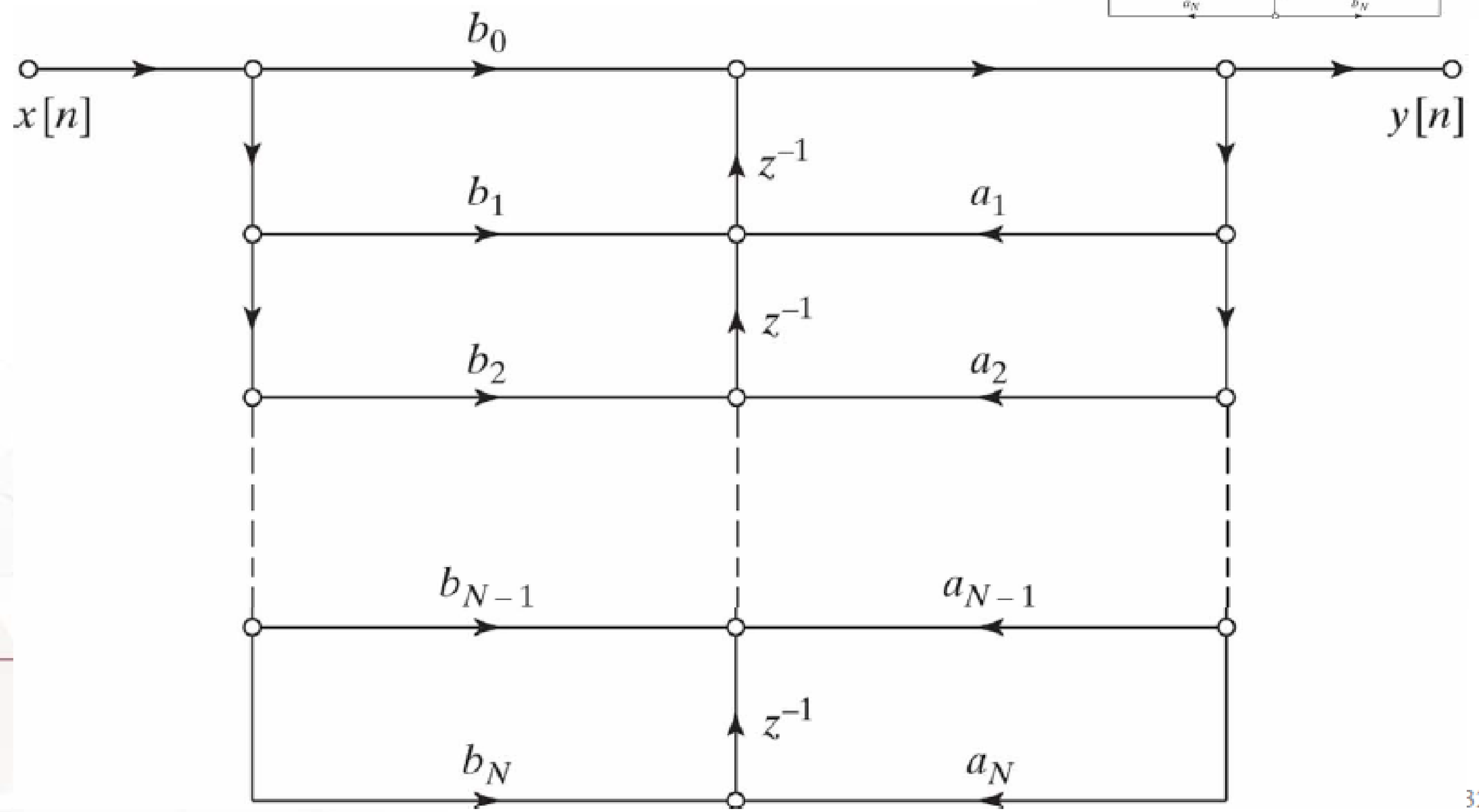
Transposed Direct Form II Structure

Figure 6.28 General flow graph resulting from applying the transposition theorem to the direct form II structure of Figure 6.15





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Cascade Form

Cascade form

$$H(z) = A\frac{\prod_{k=1}^{M_1}(1-f_k z^{-1})\prod_{k=1}^{M_2}(1-g_k z^{-1})(1-g_k^* z^{-1})}{\prod_{k=1}^{N_1}(1-c_k z^{-1})\prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}, \quad (6.29)$$

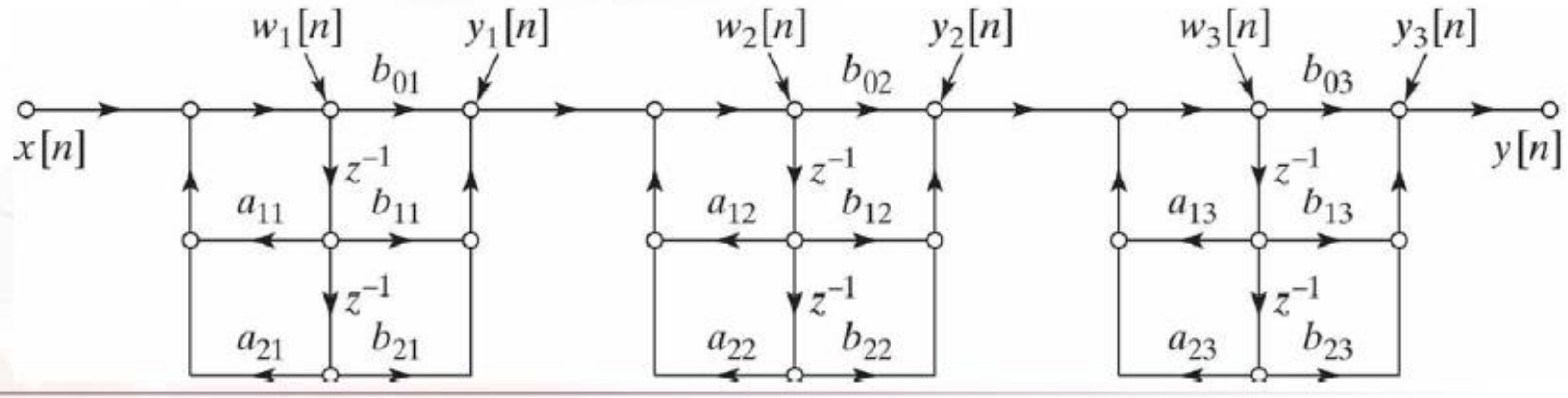
$$H(z) = \prod_{k=1}^{N_s}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}, \quad (6.30)$$

$$y_0[n] = x[n], \quad (6.31a)$$

$$w_k[n] = a_{1k}w_k[n-1] + a_{2k}w_k[n-2] + y_{k-1}[n], \quad k = 1, 2, \ldots, N_s, \quad (6.31b)$$

$$y_k[n] = b_{0k}w_k[n] + b_{1k}w_k[n-1] + b_{2k}w_k[n-2], \quad k = 1, 2, \ldots, N_s, \quad (6.31c)$$

$$y[n] = y_{N_s}[n]. \quad (6.31d)$$



Parallel form structure

Parallel form

We can express a rational system function as a partial fraction expansion in the form

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_1} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k(1 - e_k z^{-1})}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})}, \qquad (6.34)$$

where $N = N_1 + 2N_2$. If $M \geq N$, then $N_p = M - N$; otherwise, the first summation in eq. (6.34) is not included. If the coefficients a_k and b_k are real in eq. (6.27), then the quantities A_k, B_k, C_k, c_k, and e_k are all real. In this form, the system function can be interpreted as representing a parallel combination of 1^{st}- and 2^{nd}-order IIR systems, with possibly N_p simple scaled delay paths. Alternatively, we may group the real poles in pairs, so that

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}, \qquad (6.35)$$

where, $N_s = \lfloor (N+1)/2 \rfloor$ is the largest integer contained in $(N+1)/2$, and if $N_p = M - N$ is negative, the first sum is not present.

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Parallel form structure

Parallel form structure for 6th-order system

(M = N = 6) with the real and complex poles grouped in pairs

The general difference equations for the parallel form with 2^{nd}-order direct form II sections are

$$w_k[n] = a_{1k}w_k[n-1]+a_{2k}w_k[n-2]+x[n],$$

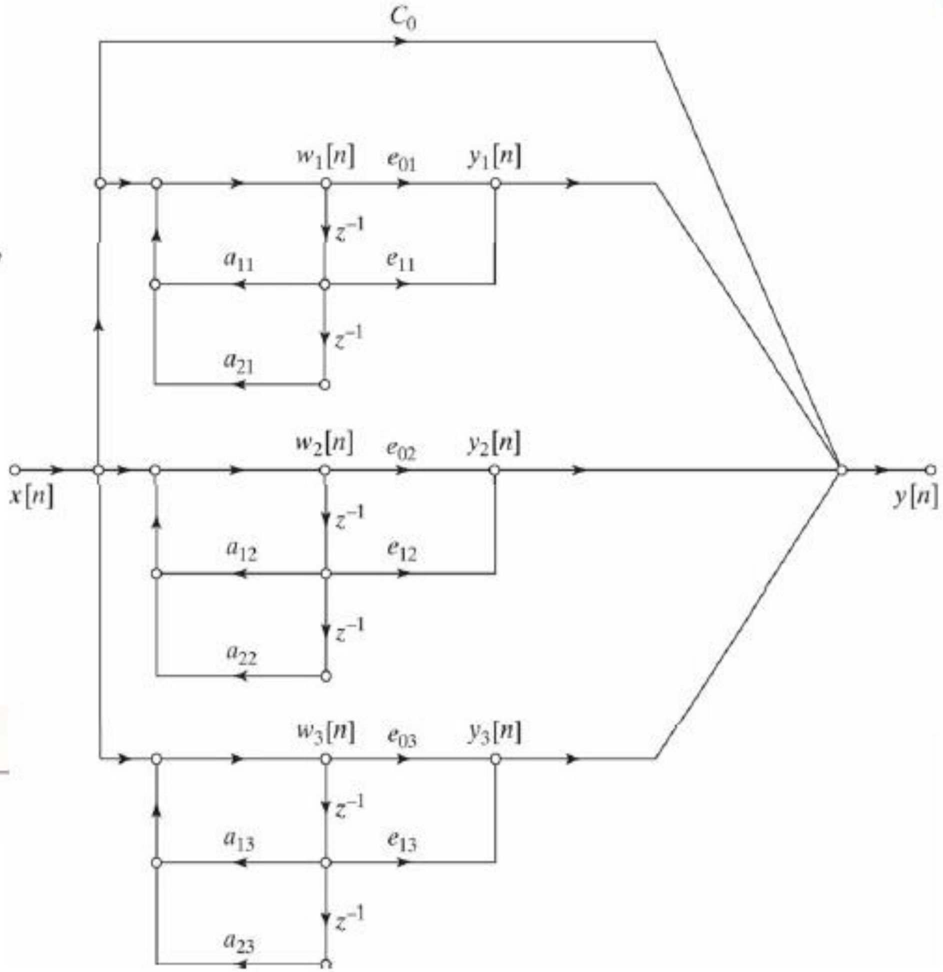
where $k = 1, 2, \ldots, N_s$ (6.36a).

$$y_k[n] = e_{0k}w_k[n] + e_{1k}w_k[n-1],$$

where $k = 1, 2, \ldots, N_s$ (6.36b).

$$y[n] = \sum_{k=0}^{N_p} C_k x[n-k] + \sum_{k=1}^{N_s} y_k[n]. (6.36c)$$

If $M < N$, then the first summation in eq. (6.36c) is not included.



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FIR Filter Structure

Direct form realization of an FIR system

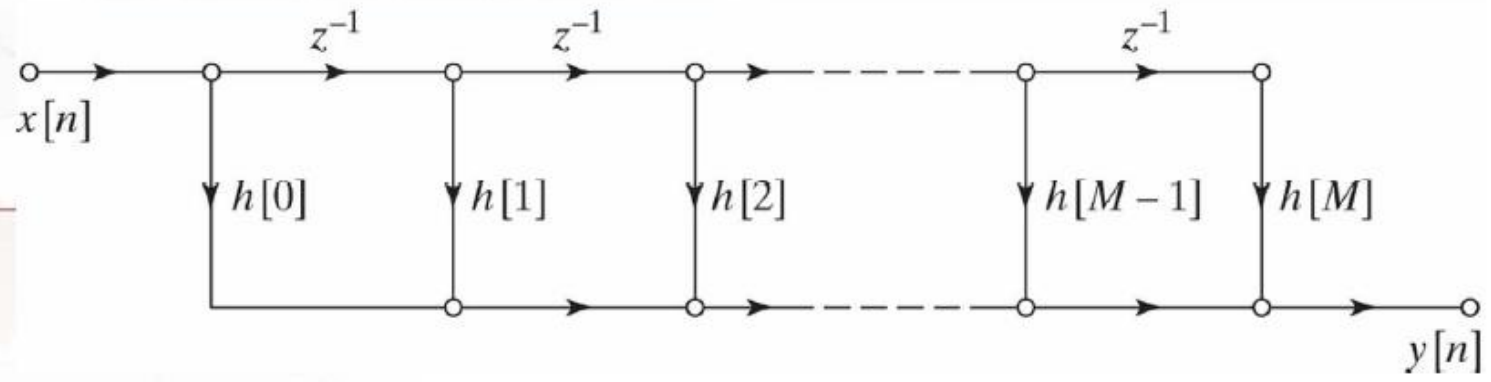
For causal FIR systems, the system function has only zeros (except for poles at $z = 0$), and since the coefficients a_k are all zero, the difference equation of Eq. (6.9) reduces to

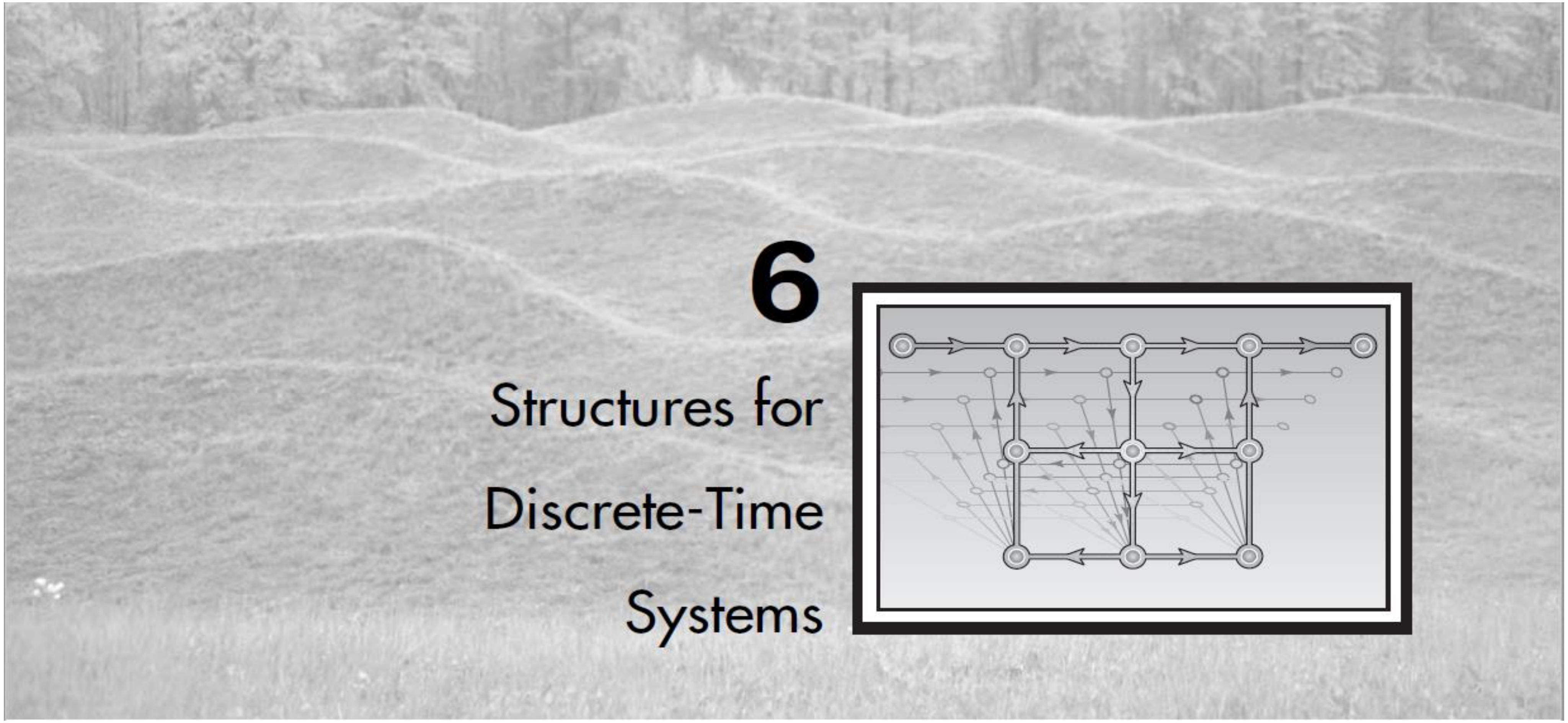
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]. \qquad (6.46)$$

This can be recognized as the discrete convolution of $x[n]$ with the impulse response

$$h[n] = \begin{cases} b_n & n = 0, 1, \ldots, M, \\ 0 & \text{otherwise.} \end{cases} \qquad (6.47)$$

This structure is also referred to as a *tapped delay line* structure or a *transversal filter* structure.



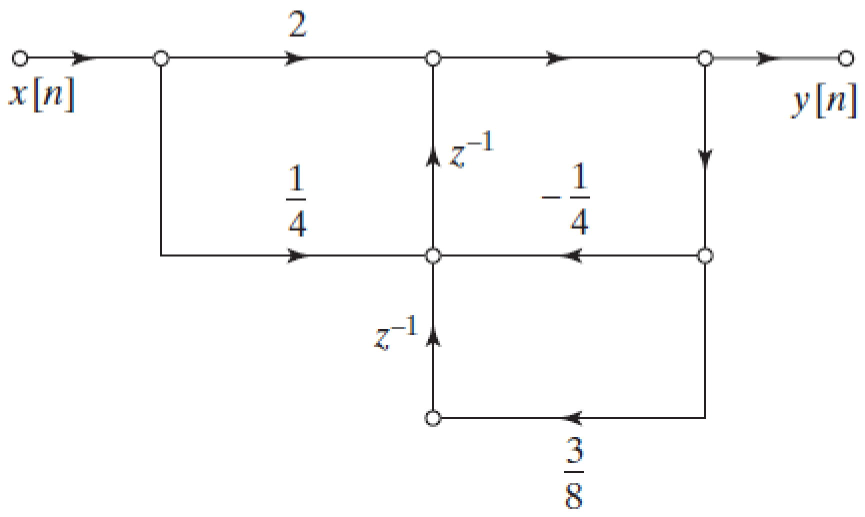


Problems: 6.4, 6.5, 6.6, 6.9, 6.10, 6.11, 6.13, 6.18, 6.19, 6.27, 6.37, 6.38

6.4. Consider the system in Figure P6.3(d).

(a) Determine the system function relating the z-transforms of the input and output.

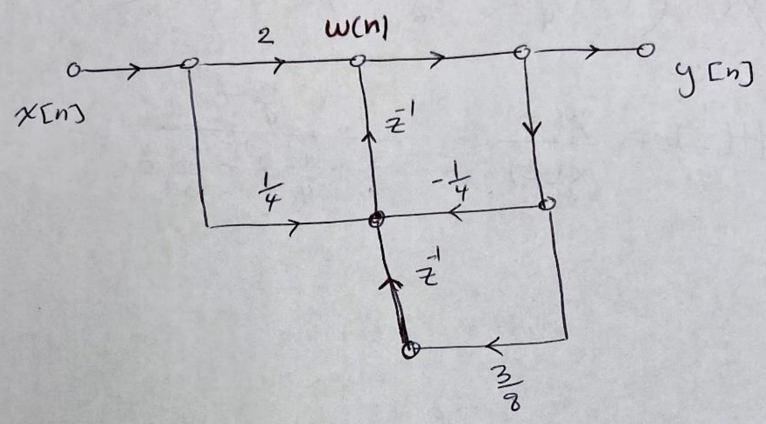
(b) Write the difference equation that is satisfied by the input sequence $x[n]$ and the output sequence $y[n]$.



(d)

Figure P6.3

6.4



(a)

This is transposed signal flow graph of direct form II structure

$$w(n) = 2x(n) + \frac{1}{4}x(n-1) - \frac{1}{4}w(n-1) + \frac{3}{8}w(n-2)$$

$$y(n) = w(n)$$

$$\therefore \; y(n) = 2x(n) + \frac{1}{4}x(n-1) - \frac{1}{4}y(n-1) + \frac{3}{8}y(n-2)$$

$$Y(z) = 2X(z) + \frac{1}{4}z^{-1}X(z) - \frac{1}{4}z^{-1}Y(z) + \frac{3}{8}z^{-2}Y(z)$$

$$Y(z)\left(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}\right) = X(z)\left(2 + \frac{1}{4}z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

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(a) From the flow graph, we have:

$$Y(z) = 2X(z) + (\frac{1}{4}X(z) - \frac{1}{4}Y(z) + \frac{3}{8}Y(z)z^{-1})z^{-1}$$

That is:

$$Y(z)(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}) = X(z)(2 + \frac{1}{4}z^{-1}).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

(b) To get the difference equation, we just inverse Z−transform the equation in a. We get:

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = 2x[n] + \frac{1}{4}x[n-1].$$

6.5. An LTI system is realized by the flow graph shown in Figure P6.5.

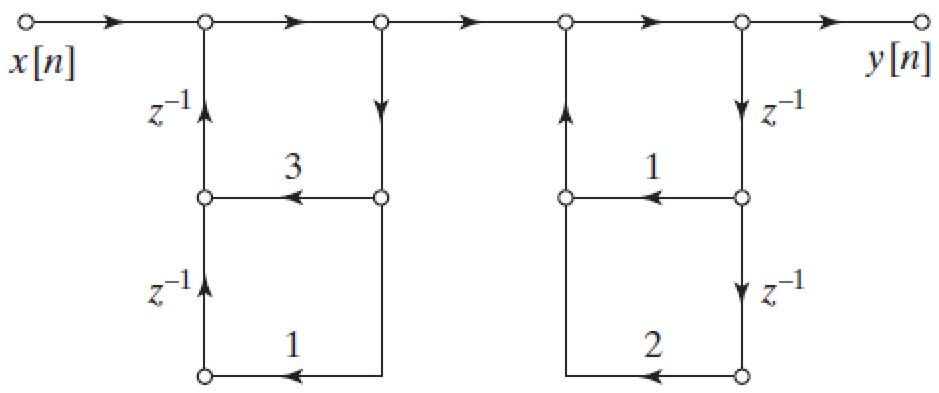


Figure P6.5

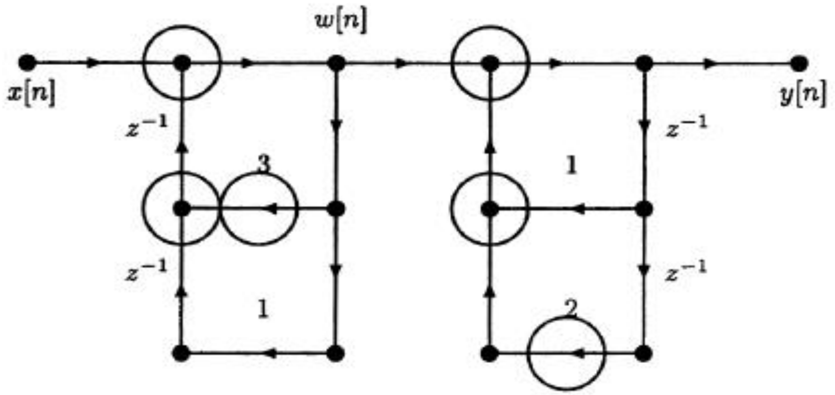
(a) Write the difference equation relating $x[n]$ and $y[n]$ for this flow graph.

(b) What is the system function of the system?

(c) In the realization of Figure P6.5, how many real multiplications and real additions are required to compute each sample of the output? (Assume that $x[n]$ is real, and assume that multiplication by 1 does not count in the total.)

(d) The realization of Figure P6.5 requires four storage registers (delay elements). Is it possible to reduce the number of storage registers by using a different structure? If so, draw the flow graph; if not, explain why the number of storage registers cannot be reduced.

The flow graph for this system is drawn below.



(a)

$$w[n] = x[n] + 3w[n-1] + w[n-2]$$

$$y[n] = w[n] + y[n-1] + 2y[n-2]$$

(b)

$$W(z) = X(z) + 3z^{-1}W(z) + z^{-2}W(z)$$

$$Y(z) = W(z) + z^{-1}Y(z) + 2z^{-2}Y(z)$$

So

$$\begin{aligned} \frac{Y(z)}{X(z)} &= H(z) \\ &= \frac{1}{(1 - z^{-1} - 2z^{-2})(1 - 3z^{-1} - z^{-2})} \\ &= \frac{1}{1 - 4z^{-1} + 7z^{-3} + 2z^{-4}}. \end{aligned}$$

(c) Adds and multiplies are circled above: 4 real adds and 2 real multiplies per output point.

(d) It is not possible to reduce the number of storage registers. Note that implementing $H(z)$ above in the canonical direct form II (minimum storage registers) also requires 4 registers.

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6.6. Determine the impulse response of each of the systems in Figure P6.6.

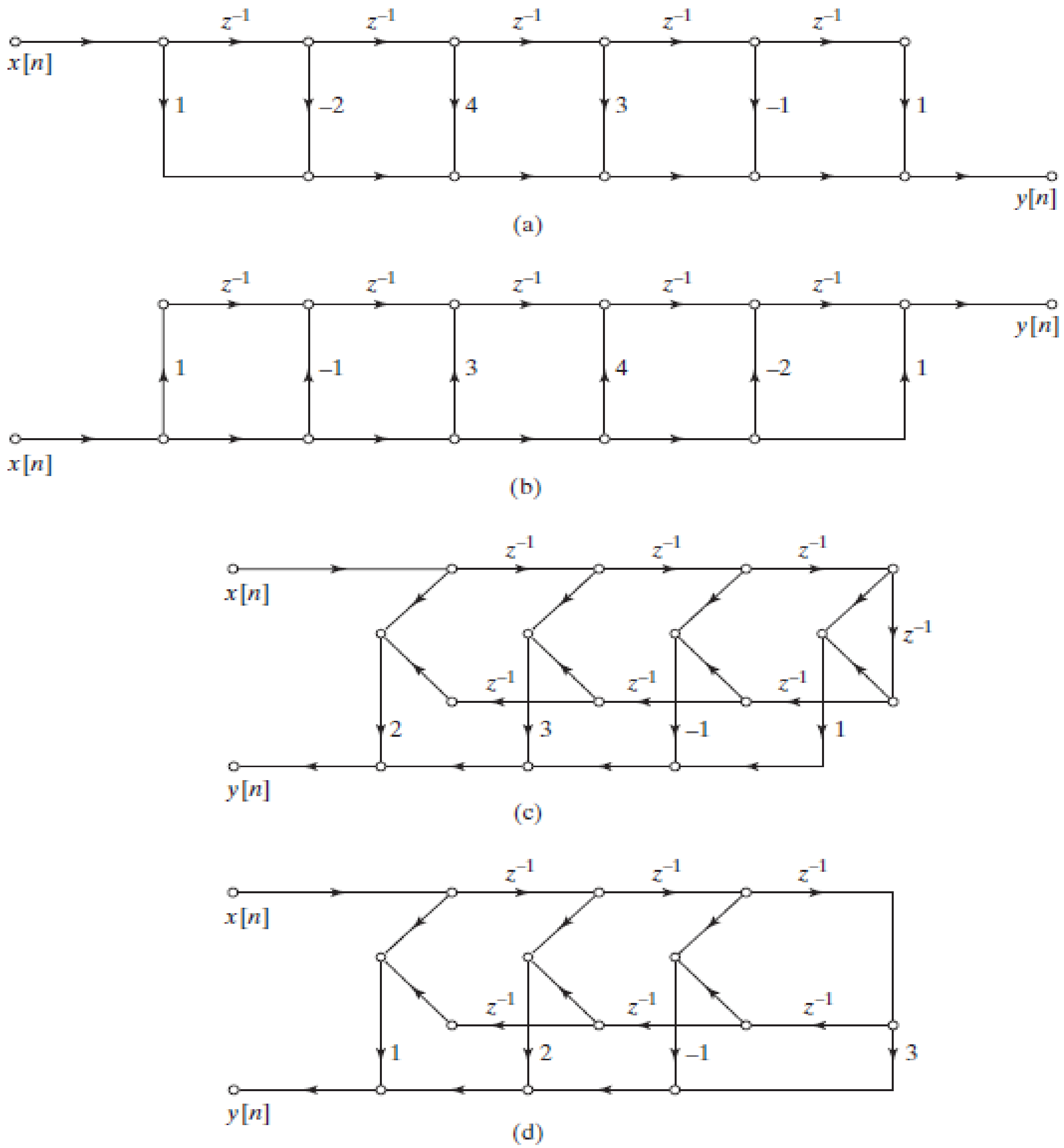
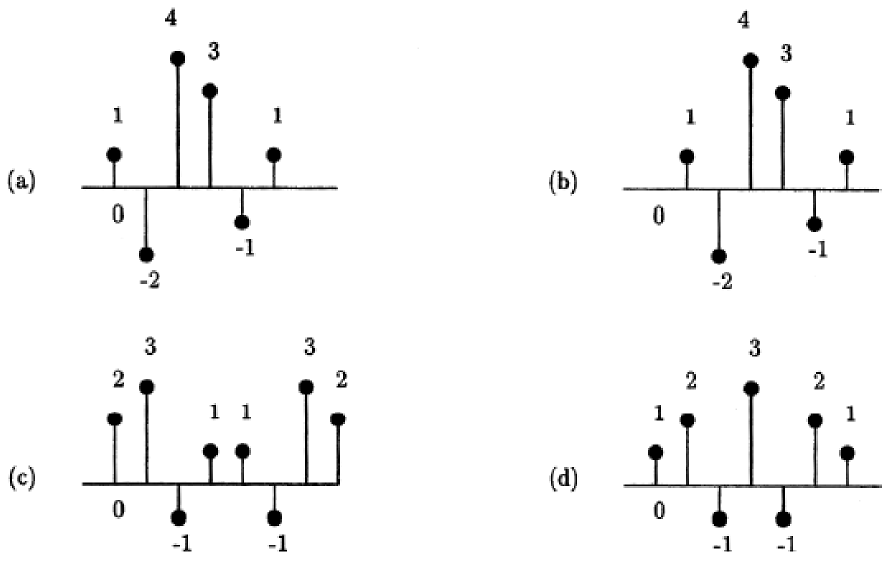
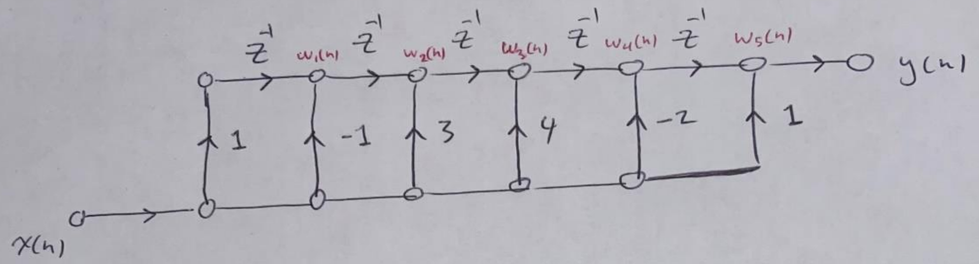


Figure P6.6

The impulse responses of each system are shown below.



6.6 (B)



$w_1(n) = x(n-1) - x(n)$ — ①

$w_2(n) = w_1(n-1) + 3x(n)$ — ②

$w_3(n) = w_2(n-1) + 4x(n)$ — ③

$w_4(n) = w_3(n-1) - 2x(n)$ — ④

$w_5(n) = w_4(n-1) + x(n)$ — ⑤

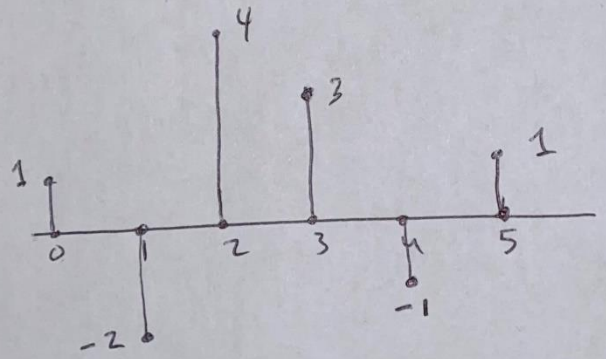
$y(n) = w_5(n)$

$\therefore y(n) = w_3(n-2) - 2x(n-1) + x(n)$, from ④

$y(n) = w_2(n-3) + 4x(n-2) - 2x(n-1) + x(n)$ from ③

$y(n) = w_1(n-4) + 3x(n-3) + 4x(n-2) - 2x(n-1) + x(n)$, from ②

$y(n) = x(n-5) - x(n-4) + 3x(n-3) + 4x(n-2) - 2x(n-1) + x(n)$, from ①



6.9. Figure P6.9 shows the signal flow graph for a causal discrete-time LTI system. Branches without gains explicitly indicated have a gain of unity.

(a) By tracing the path of an impulse through the flowgraph, determine $h[1]$, the impulse response at $n = 1$.

(b) Determine the difference equation relating $x[n]$ and $y[n]$.

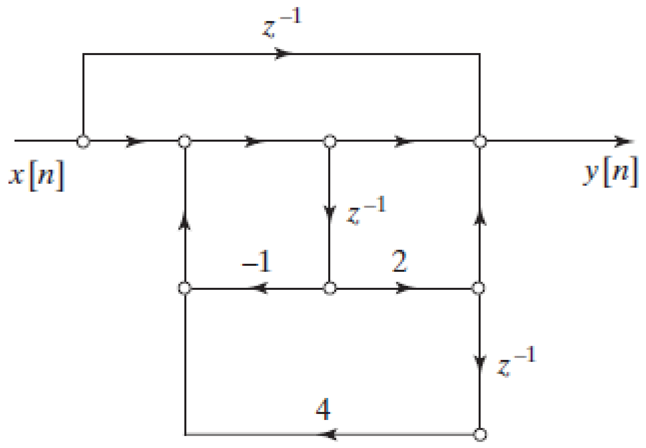
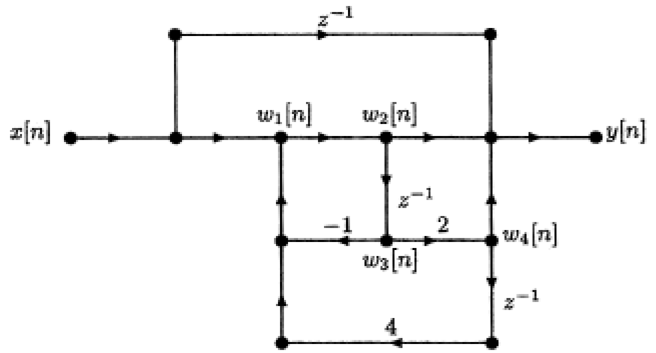


Figure P6.9



(a) First we need to determine the transfer function. We have

$$
\begin{aligned}
w_1[n] &= x[n] - w_3[n] + 4w_4[n-1] \\
w_2[n] &= w_1[n] \\
w_3[n] &= w_2[n-1] \\
w_4[n] &= 2w_3[n] \\
y[n] &= w_2[n] + x[n-1] + w_4[n].
\end{aligned}
$$

Taking the $Z-$transform of the above equations, rearranging and substituting terms, we get:

$$H(z) = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}}.$$

The difference equation is thus given by:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

The impulse response is the response to an impulse, therefore:

$$h[n] + h[n-1] - 8h[n-2] = \delta[n] + 3\delta[n-1] + \delta[n-2] - 8\delta[n-3].$$

From the above equation, we have:

$$h[0] = 1$$
$$h[1] = 3 - h[0] = 2.$$

(b) From part (a) we have:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

(a)

$$x(n) = w_1(n) + w_3(n) - 4w_4(n-1)$$

$$x(n) = w_2(n) + w_2(n-1) - 8w_2(n-2) \quad — ①$$

$$y(n) = w_2(n) + w_2(n-1) + w_2(n-2) - 8w_2(n-3) + 2w_2(n-1) \quad — ②$$

By taking Z-transform, for ① and ②

$$X(z) = W_2(z)\left(1 + z^{-1} - 8z^{-2}\right)$$

$$Y(z) = W_2(z)\left(1 + 3z^{-1} + z^{-2} - 8z^{-3}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}}$$

6.10. Consider the signal flow graph shown in Figure P6.10.

(a) Using the node variables indicated, write the set of difference equations represented by this flow graph.

(b) Draw the flow graph of an equivalent system that is the cascade of two 1st-order systems.

(c) Is the system stable? Explain.

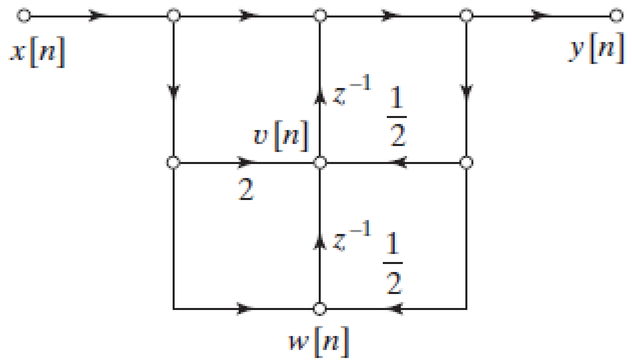


Figure P6.10

(a)

$$w[n] = \frac{1}{2}y[n] + x[n]$$
$$v[n] = \frac{1}{2}y[n] + 2x[n] + w[n-1]$$
$$y[n] = v[n-1] + x[n].$$

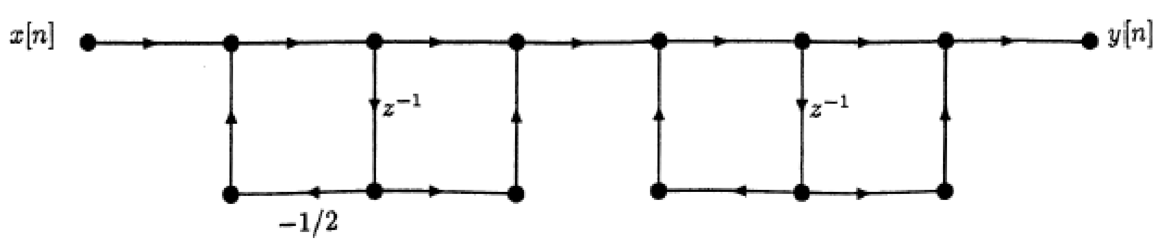
(b) Using the Z−transform of the difference equations in part (a), we get the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

We can rewrite it as :

$$H(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

We thus get the following cascade form:



(c) The system function has poles at $z = -\frac{1}{2}$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.

(a)

$$v(n) = \frac{1}{2}y(n) + \frac{1}{2}y(n-1) + x(n-1) + 2x(n)$$

Substitute $v(n)$ in $y(n)$

$$y(n) = x(n) + \frac{1}{2}y(n-1) + \frac{1}{2}y(n-2) + x(n-2) + 2x(n-1)$$

By taking z-transform for both sides

$$Y(z)\left(1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}\right) = X(z)\left(1 + 2z^{-1} + z^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$$

6.11. Consider a causal LTI system with impulse response $h[n]$ and system function

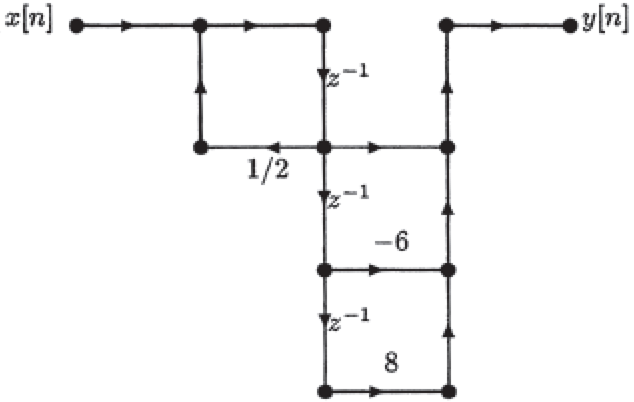
$$H(z) = \frac{(1 - 2z^{-1})(1 - 4z^{-1})}{z\left(1 - \frac{1}{2}z^{-1}\right)}$$

(a) Draw a direct form II flow graph for the system.
(b) Draw the transposed form of the flow graph in part (a).

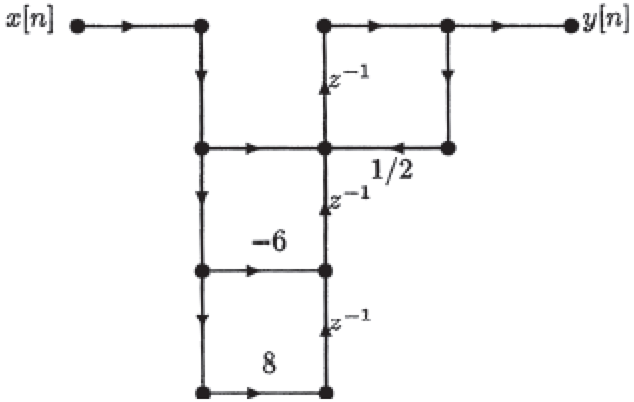
(a) $H(z)$ can be rewritten as:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}.$$

We thus get the following direct from II flow graph :



(b) To get the transposed form, we just reverse the arrows and exchange the input and the ouput. The graph can then be redrawn as:



(b) To check the difference equations of the transposed form

$$y(n) = x(n-1) - 6\,x(n-2) + 8\,x(n-3) + \frac{1}{2}\,y(n-1)$$

By taking z-transform for both sides

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)\left(z^{-1} - 6z^{-2} + 8z^{-3}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

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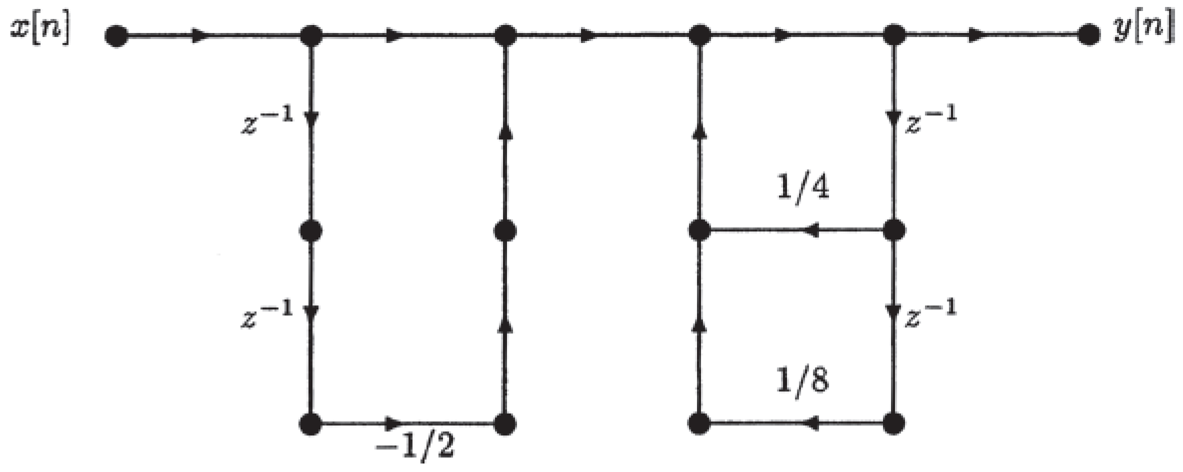
6.13. Draw the signal flow graph for the direct form I implementation of the LTI system with system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

The solution

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

The direct form I implementation is:



6.18. For some nonzero choices of the parameter a, the signal flow graph in Figure P6.18 can be replaced by a 2nd-order direct form II signal flow graph implementing the same system function. Give one such choice for a and the system function $H(z)$ that results.

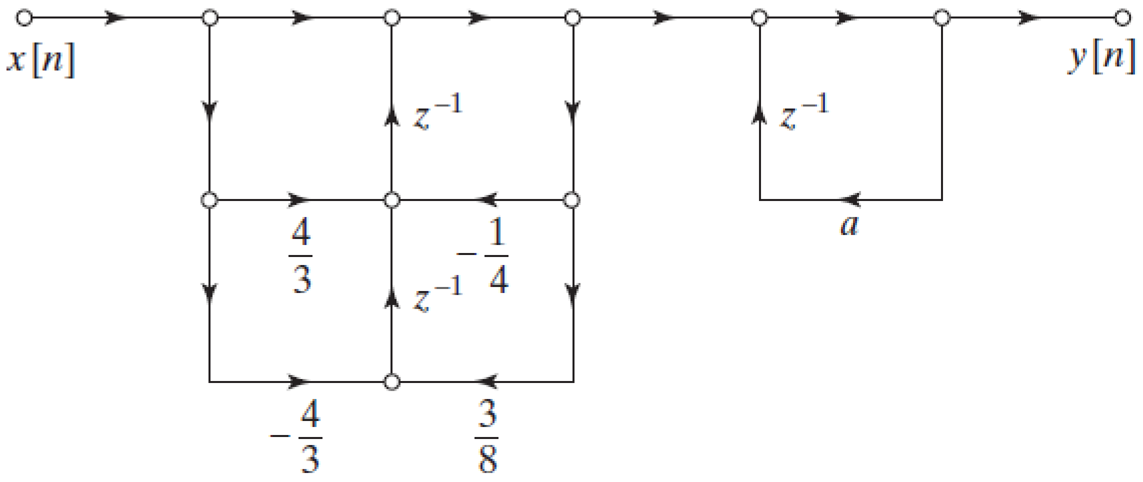


Figure P6.18

The flow graph is just a cascade of two transposed direct form II structures, the system function is thus given by:

$$H(z) = \left(\frac{1+\frac{4}{3}z^{-1}-\frac{4}{3}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}\right)\left(\frac{1}{1-az^{-1}}\right).$$

Which can be rewritten as:

$$H(z) = \frac{(1+2z^{-1})(1-\frac{2}{3}z^{-1})}{(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})(1-az^{-1})}.$$

In order to implement this system function with a second-order direct form II signal flow graph, a pole-zero cancellation has to occur, this happens if $a = \frac{2}{3}$, $a = -2$ or $a = 0$. If $a = \frac{2}{3}$, the overall system function is:

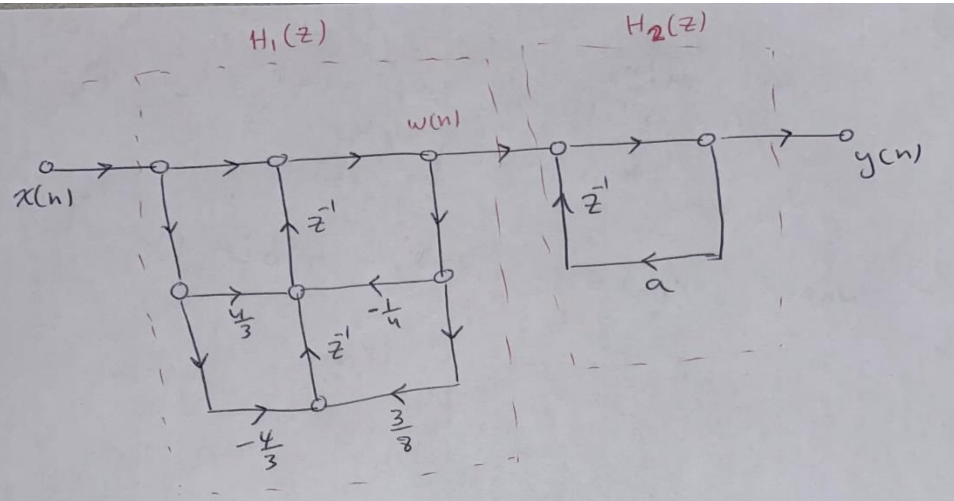
$$H(z) = \frac{1+2z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is:

$$H(z) = \frac{1-\frac{2}{3}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$

And finally if $a = 0$, the overall system function is:

$$H(z) = \frac{(1+2z^{-1})(1-\frac{2}{3}z^{-1})}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}.$$



$$w(n) = x(n) + \frac{4}{3}x(n-1) - \frac{4}{3}x(n-2) - \frac{1}{4}w(n-1) + \frac{3}{8}w(n-2) \quad (1)$$

$$W(z)\left(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}\right) = X(z)\left(1+\frac{4}{3}z^{-1}-\frac{4}{3}z^{-2}\right)$$

$$H_1(z) = \frac{1+\frac{4}{3}z^{-1}-\frac{4}{3}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}$$

$$y(n) = w(n) + a\,y(n-1) \quad (2)$$

$$Y(z)(1-az^{-1}) = W(z) \Rightarrow H_2(z) = \frac{1}{1-az^{-1}}$$

$$H(z) = H_1(z)\,H_2(z)$$

6.19. Consider the causal LTI system with the system function

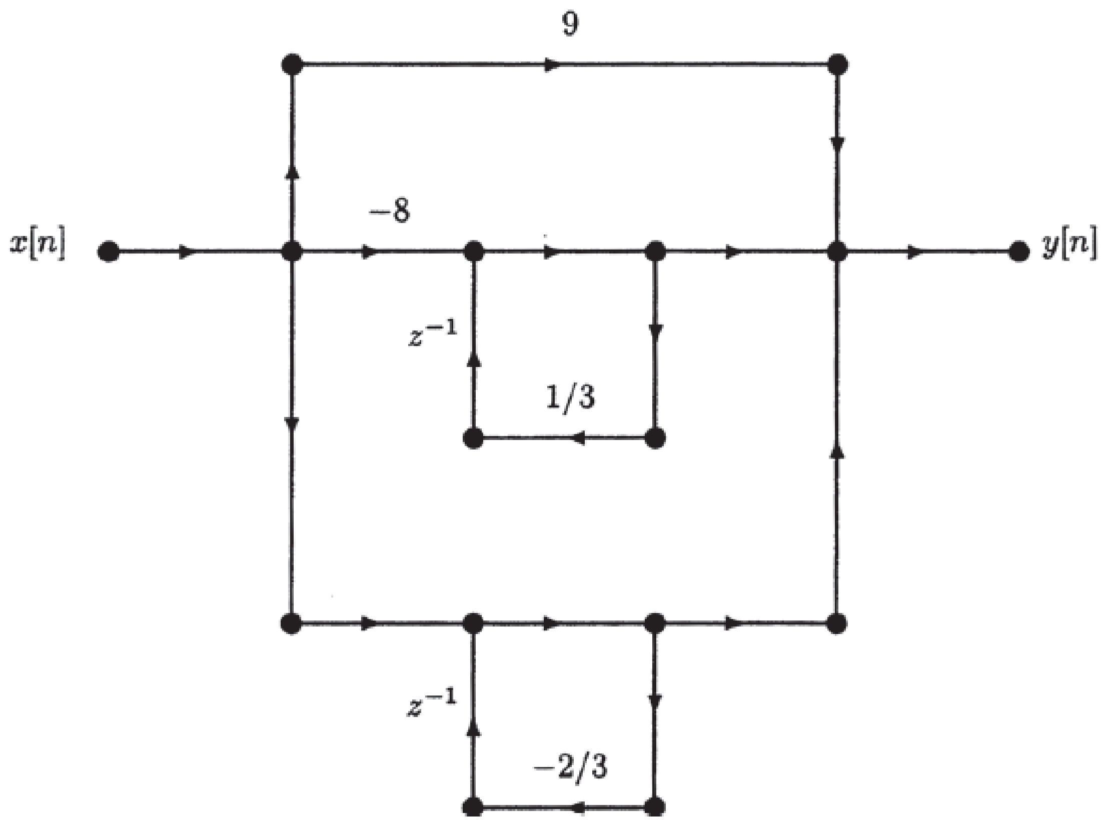
$$H(z) = \frac{2 - \frac{8}{3}z^{-1} - 2z^{-2}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{2}{3}z^{-1}\right)}.$$

Draw a signal flow graph that implements this system as a parallel combination of 1^{st}-order transposed direct form II sections.

Using partial fraction expansion, the system function can be rewritten as:

$$H(z) = \frac{-8}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 + \frac{2}{3}z^{-1}} + 9.$$

Now we can draw the flow graph that implements this system as a parallel combination of first-order transposed direct form II sections:



6.27. An LTI system with system function

$$H(z) = \frac{0.2(1+z^{-1})^6}{\left(1-2z^{-1}+\frac{7}{8}z^{-2}\right)\left(1+z^{-1}+\frac{1}{2}z^{-2}\right)\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)}$$

is to be implemented using a flow graph of the form shown in Figure P6.27.

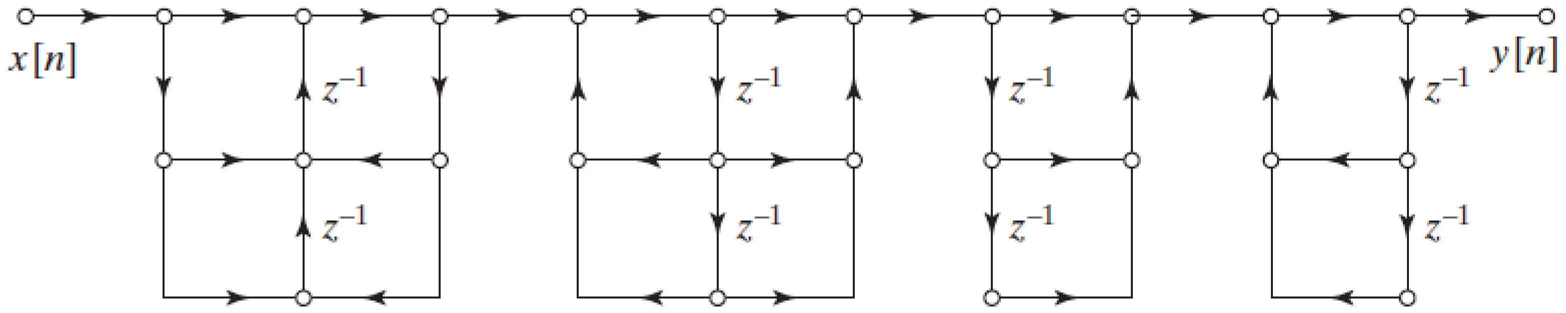


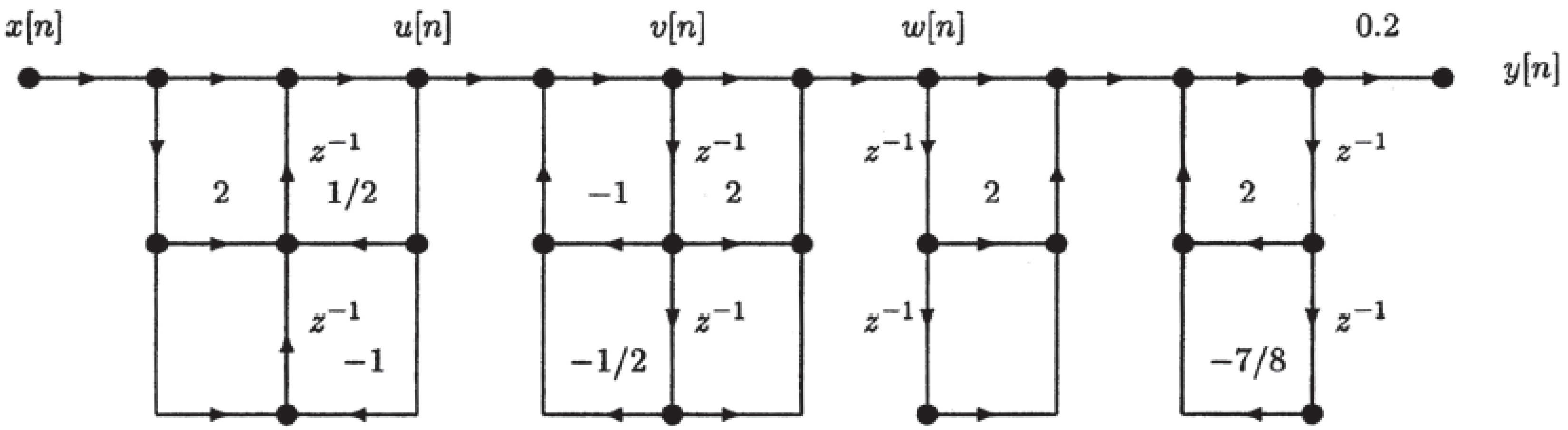
Figure P6.27

(a) Fill in all the coefficients in the diagram of Figure P6.27. Is your solution unique?

(b) Define appropriate node variables in Figure P6.27, and write the set of difference equations that is represented by the flow graph.

(a) We can rearrange $H(z)$ this way:

$$H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2$$



The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

(b)

$$\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}$$

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6.37. The flow graph shown in Figure P6.37 is noncomputable; i.e., it is not possible to compute the output using the difference equations represented by the flow graph because it contains a closed loop having no delay elements.

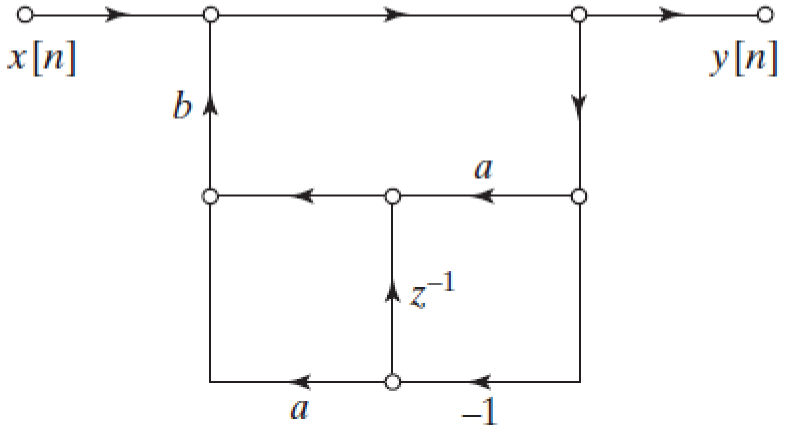
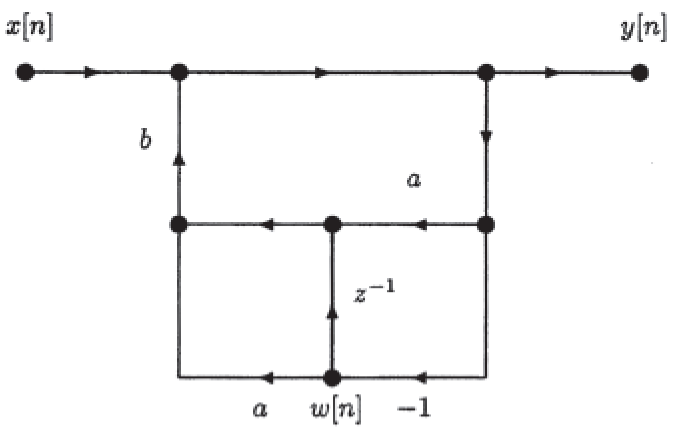


Figure P6.37

(a) Write the difference equations for the system of Figure P6.37, and from them, find the system function of the flow graph.

(b) From the system function, obtain a flow graph that is computable.



(a)

$$\begin{aligned} y[n] &= x[n] + abw[n] + bw[n-1] + aby[n] \\ w[n] &= -y[n]. \end{aligned}$$

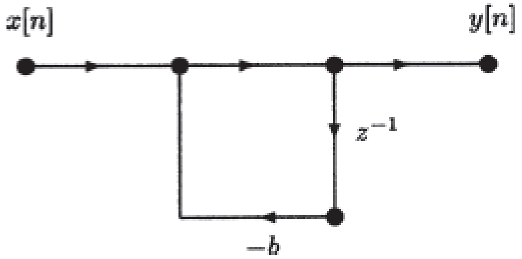
Eliminate w[n]:

$$\begin{aligned} y[n] &= x[n] - aby[n] - by[n-1] + aby[n] \\ y[n] &= x[n] - by[n-1] \end{aligned}$$

So:

$$H(z) = \frac{1}{1 + bz^{-1}}.$$

(b)



6.38. The impulse response of an LTI system is

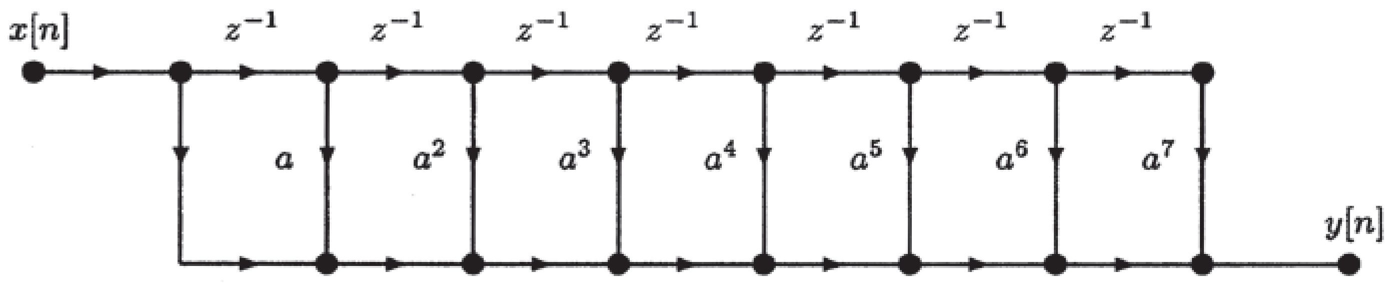
$$h[n] = \begin{cases} a^n, & 0 \le n \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Draw the flow graph of a direct form nonrecursive implementation of the system.

(b) Show that the corresponding system function can be expressed as

$$H(z) = \frac{1 - a^8 z^{-8}}{1 - a z^{-1}}, \quad |z| > |a|.$$

(c) Draw the flow graph of an implementation of $H(z)$, as expressed in part (b), corresponding to a cascade of an FIR system (numerator) with an IIR system (denominator).



(b) From

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

it follows that

$$\sum_{n=0}^{7} a^n z^{-n} = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

(c)