ELG4177 - DIGITAL SIGNAL PROCESSING
Tutorial 5

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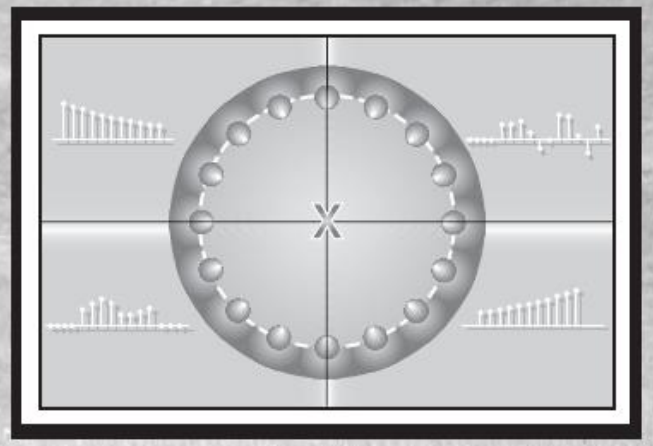
5

Transform Analysis of Linear Time-Invariant Systems



Problems: 5.2, 5.9, 5.10, 5.52, 5.28, 5.8, 5.24, 5.36

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Q 5.2

2. Consider a stable LTI system with input $x[n]$ and output $y[n]$. The input and output satisfy the difference equation

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n].$$

(a) Plot the poles and zeros of the system function in the z-plane.
(b) Determine the impulse response $h[n]$.

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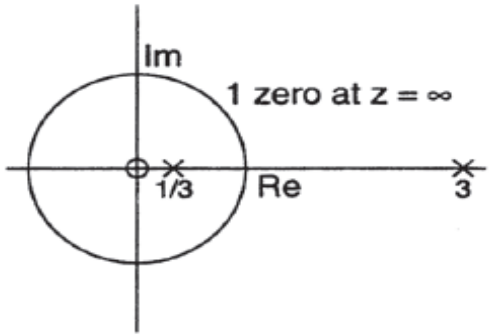
5.2. We have $y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$ or $z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$. So,

$$\begin{aligned} H(z) &= \frac{1}{z^{-1} - \frac{10}{3} + z} \\ &= \frac{z}{(z - \frac{1}{3})(z-3)} \\ &= \frac{-\frac{1}{8}}{z - \frac{1}{3}} + \frac{\frac{9}{8}}{z-3} \end{aligned}$$

(a)



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1 - 3z^{-1}}$$

Stable $\Rightarrow$ ROC is $\frac{1}{3} \leq |z| \leq 3$. Therefore,

$$h[n] = -\frac{1}{8}\left(\frac{1}{3}\right)^{n-1} u[n-1] - \frac{9}{8}(3)^{n-1}u[-n]$$

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Q 5.9

9. Consider an LTI system with input $x[n]$ and output $y[n]$ for which

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n].$$

The system may or may not be stable or causal. By considering the pole–zero pattern associated with this difference equation, determine three possible choices for the impulse response of the system. Show that each choice satisfies the difference equation. Indicate which choice corresponds to a stable system and which choice corresponds to a causal system.

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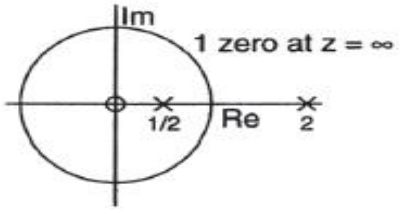
5.9.

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

$$z^{-1}Y(z) - \frac{5}{2}Y(z) + zY(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})} \\ &= \frac{\frac{2}{3}}{1 - 2z^{-1}} - \frac{\frac{2}{3}}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$



Three regions of convergence:

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

Includes $|z| = 1$, so this is stable.

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}(2)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

ROC outside of largest pole, so this is causal.

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Q 5.10

10. If the system function $H(z)$ of an LTI system has a pole–zero diagram as shown in Figure P10 and the system is causal, can the inverse system $H_i(z)$, where $H(z)H_i(z) = 1$, be both causal and stable? Clearly justify your answer.

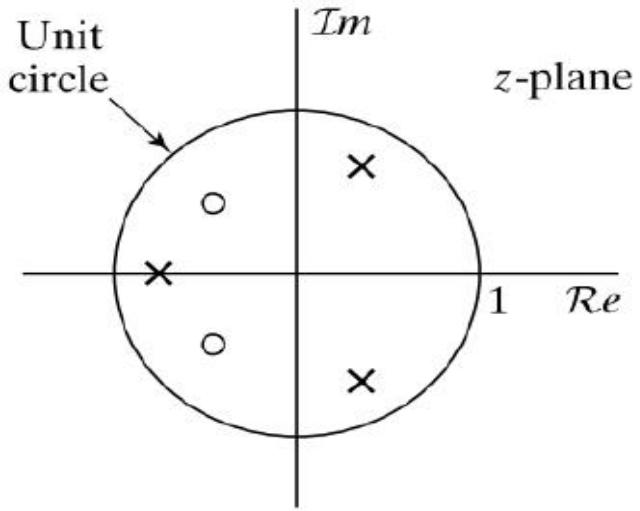

Figure P10

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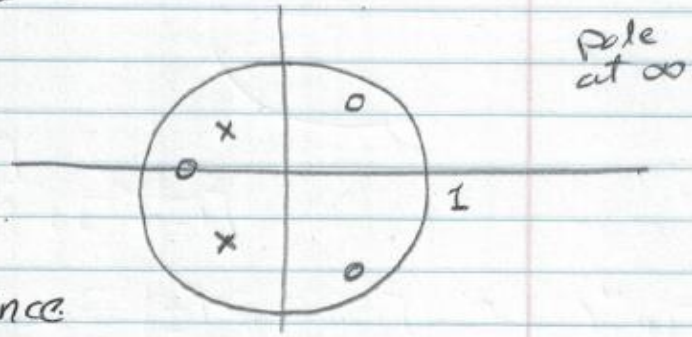
Ce 5.10

$H(z)\ H_i(z) = 1$

* The Pole-zero plot shows two zeros & three poles. since number of poles is equal to number of zeros, there must be another zero, which is at $z = \infty$

* The system $H(z)$ is causal, therefore the RoC extends from the outermost pole

* In the inverse system $H_i(z)$, poles & zeros are switched →



The system can be stable if the RoC includes the unit circle. However, it cannot be causal, since there is a pole at $z = \infty$.

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Q 5.52

5.52. Consider a causal sequence $x[n]$ with the z-transform

$$X(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{5}z\right)}{\left(1 - \frac{1}{6}z\right)}.$$

For what values of α is $\alpha^n x[n]$ a real, minimum-phase sequence?

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@5.52

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z^{+1})}{(1-\frac{1}{6}z)}$$

$$= \frac{6}{5} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-5z^{-1})}{(1-6z^{-1})}$$

$$\alpha^n x[n] \xrightarrow{z-T} X(\alpha^{-1}z)$$

$$X(\alpha^{-1}z) = \frac{6}{5} \cdot \frac{(1-\frac{1}{2}\alpha z^{-1})(1-\frac{1}{4}\alpha z^{-1})(1-5\alpha z^{-1})}{(1-6\alpha z^{-1})}$$

* A min. phase sequence has all poles and zeros inside the unit circle.:

$|\alpha/2| < 1 \longrightarrow |\alpha| < 2$
$|\alpha/4| < 1 \longrightarrow |\alpha| < 4$
$|5\alpha| < 1 \longrightarrow |\alpha| < 1/5$
$|6\alpha| < 1 \longrightarrow |\alpha| < 1/6$

∴ To have a min phase sequence $(\alpha^n x[n])$

$|\alpha| < 1/6$

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Q 5.28

5.28. A causal LTI system has the system function

$$H(z) = \frac{(1 - e^{j\pi/3}z^{-1})(1 - e^{-j\pi/3}z^{-1})(1 + 1.1765z^{-1})}{(1 - 0.9e^{j\pi/3}z^{-1})(1 - 0.9e^{-j\pi/3}z^{-1})(1 + 0.85z^{-1})}.$$

(a) Write the difference equation that is satisfied by the input $x[n]$ and output $y[n]$ of this system.
(b) Plot the pole–zero diagram and indicate the ROC for the system function.
(c) Make a carefully labeled sketch of $|H(e^{j\omega})|$. Use the pole–zero locations to explain why the frequency response looks as it does.

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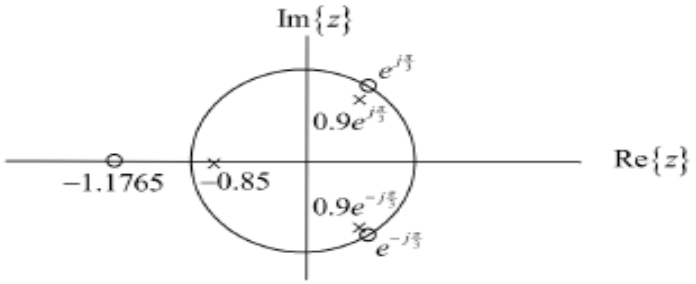
5.28. A.

$$H(z)=\frac{\left(1-e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+1.1765z^{-1}\right)}{\left(1-0.9e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+0.85z^{-1}\right)}$$
$$=\frac{1+0.1765z^{-1}-0.1765z^{-2}+1.1765z^{-3}}{1-0.05z^{-1}+0.045z^{-2}+0.6885z^{-3}}$$
$$=\frac{Y(z)}{X(z)}.$$

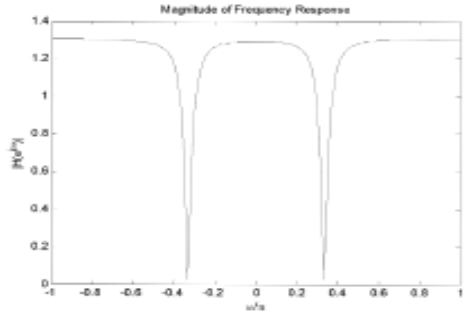
$$y[n]=0.05y[n-1]-0.45y[n-2]-0.6885y[n-3]$$
$$+x[n]+0.1765x[n-1]-0.1765x[n-2]+1.1765x[n-3].$$

B.



Since the system is causal, the ROC is the region outside the outermost pole. That is, $|z|>0.9$.

C.



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Q 5.8

5.8. A causal LTI system is described by the difference equation

$$y[n] = \frac{3}{2}y[n-1] + y[n-2] + x[n-1].$$

(a) Determine the system function $H(z) = Y(z)/X(z)$ for this system. Plot the poles and zeros of $H(z)$, and indicate the ROC.
(b) Determine the impulse response of the system.
(c) You should have found the system to be unstable. Determine a stable (noncausal) impulse response that satisfies the difference equation.

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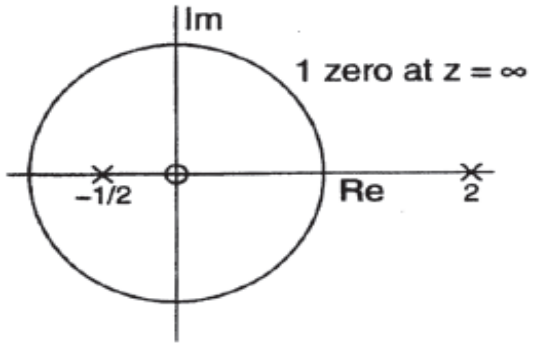
$$y[n] = \frac{3}{2}y[n-1] + y[n-2] + x[n-1]$$
$$Y(z) = \frac{3}{2}z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{3}{2}z^{-1} - z^{-2}} = \frac{z^{-1}}{(1 - 2z^{-1})(1 + \frac{1}{2}z^{-1})}, \quad |z| > 2$$



(b)

$$H(z) = \frac{z^{-1}}{(1 - 2z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{A}{(1 - 2z^{-1})} + \frac{B}{(1 + \frac{1}{2}z^{-1})}, \quad |z| > 2$$
$$A = \left.\frac{z^{-1}}{(1 + \frac{1}{2}z^{-1})}\right|_{z^{-1}=\frac{1}{2}} = \frac{2}{5}$$
$$B = \left.\frac{z^{-1}}{(1 - 2z^{-1})}\right|_{z^{-1}=-2} = -\frac{2}{5}$$
$$h[n] = \frac{2}{5}\left[(2)^n - \left(-\frac{1}{2}\right)^n\right]u[n]$$

(c) Use ROC of $\frac{1}{2} < |z| < 2$ since the ROC must include $|z| = 1$ for a stable system.

$$h[n] = -\frac{2}{5}(2)^n u[-n-1] - \frac{2}{5}\left(-\frac{1}{2}\right)^n u[n]$$

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Q 5.24

24. A stable system with system function $H(z)$ has the pole–zero diagram shown in Figure P24. It can be represented as the cascade of a stable minimum-phase system $H_{min}(z)$ and a stable all-pass system $H_{ap}(z)$.

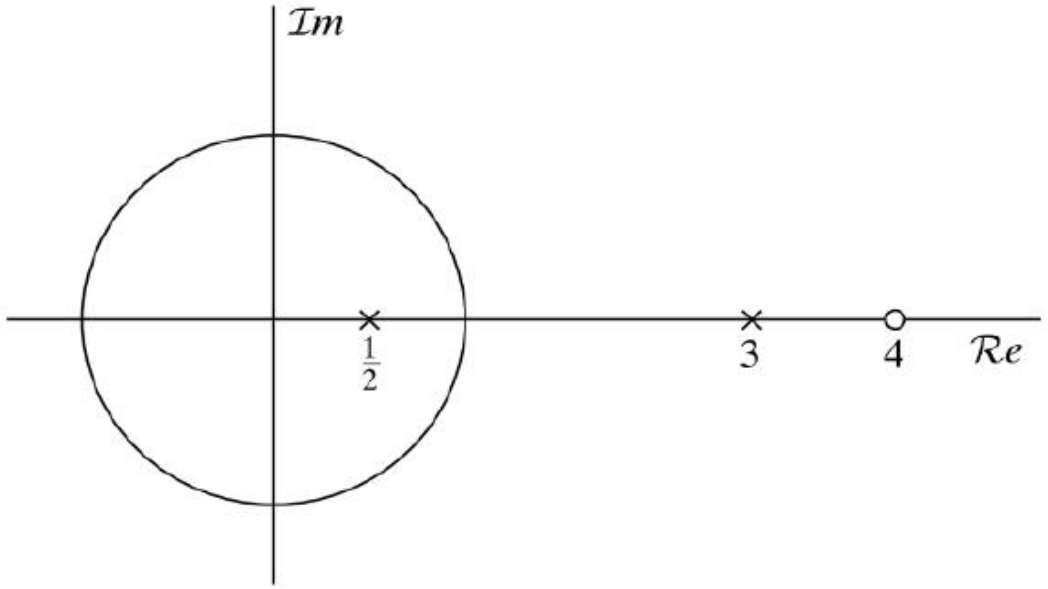

Figure P24 Pole–zero diagram for $H(z)$.

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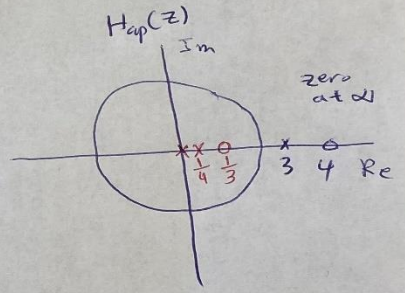
* $H(z) = H_{min}(z)\, H_{ap}(z)$

$H_{ap}(z) \Rightarrow \quad H_{ap}(z) = \dfrac{z^{-1} - a^*}{1 - a z^{-1}}$

$H_{min}(z) = \dfrac{H(z)}{H_{ap}(z)}$ & $H_{ap}(z) = \dfrac{H(z)}{H_{min}(z)}$

$H_{ap}(z) = \dfrac{(z^{-1} - \frac{1}{4})}{(1 - \frac{1}{4} z^{-1})} \dfrac{(z^{-1} - 3)}{(1 - 3 z^{-1})} \dfrac{(z^{-1} - \frac{1}{\alpha})}{(1 - \frac{1}{\alpha} z^{-1})}$



$H_{ap}(z) = \dfrac{(z^{-1} - \frac{1}{4})(z^{-1} - 3)\, z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - 3 z^{-1})}$

$H_{min}(z) = \dfrac{H(z)}{H_{ap}(z)} = \dfrac{z^{-1}(1 - 4 z^{-1})\frac{4}{4}}{(1 - 3 z^{-1})(1 - \frac{1}{2} z^{-1})} \dfrac{(1 - \frac{1}{4} z^{-1})(1 - 3 z^{-1})}{(z^{-1} - \frac{1}{4})(z^{-1} - 3) z^{-1}}$

$= \dfrac{4(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1})(z^{-1} - 3)} = \boxed{\dfrac{4}{3} \dfrac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})}}$

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Solution

Notice the pole/zero plot specifies a system function

$$H(z) = \frac{(z-4)}{(z-3)(z-1/2)} = \frac{z^{-1}(1-4z^{-1})}{(1-3z^{-1})(1-1/2z^{-1})}$$

To find the minimum phase system $H_{min}(z)$ the poles and zeros outside of the unit circle must be moved inside to conjugate reciprocal locations.

$$H_{min}(z) = K\frac{(1-1/4z^{-1})}{(1-1/3z^{-1})(1-1/2z^{-1})}$$

Notice that there is a zero at the origin because the order of the denominator is greater than the numerator. The factor K accounts for the unknown scale in $H(z)$.

The all-pass system $H_{ap}(z)$ is selected to move the poles and zeros from the min-phase system back to the original locations.

$$H_{ap}(z) = z^{-1}\left(\frac{z^{-1}-3}{1-3z^{-1}}\right)\left(\frac{z^{-1}-1/4}{1-1/4z^{-1}}\right)$$

This is a unique decomposition up to a scale factor since there are no additional all-pass terms that could be included that will be canceled by the min-phase system. (Extra poles and zeros would appear outside of the unit circle).

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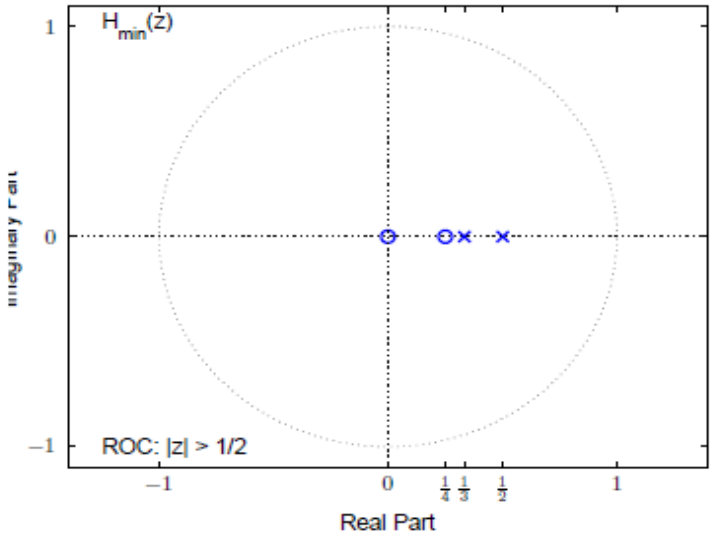
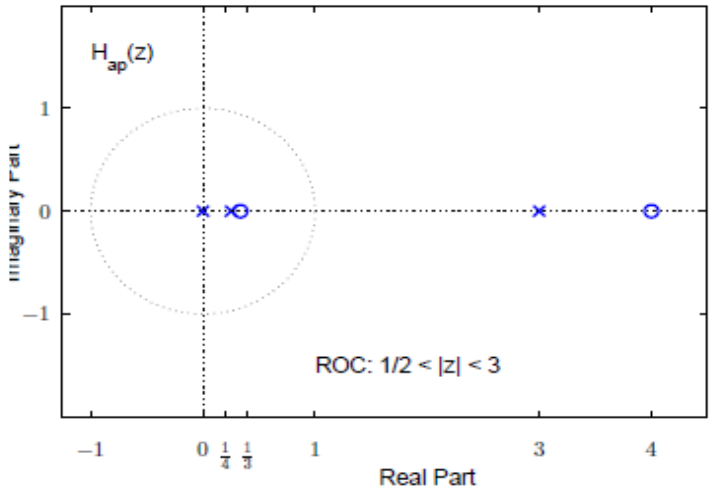
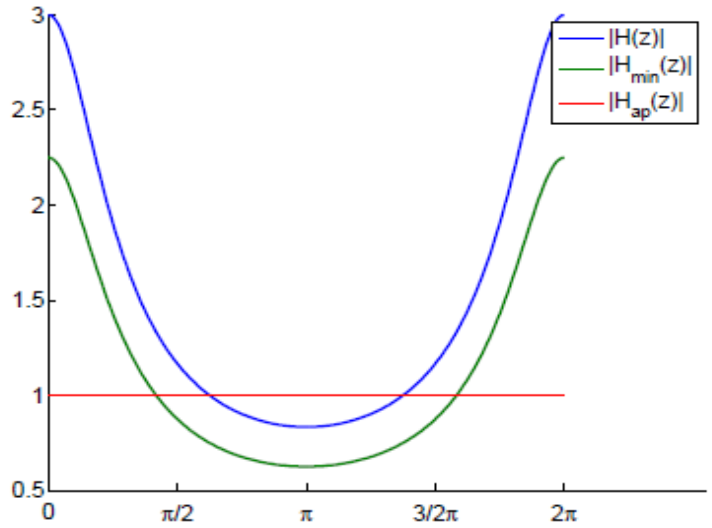
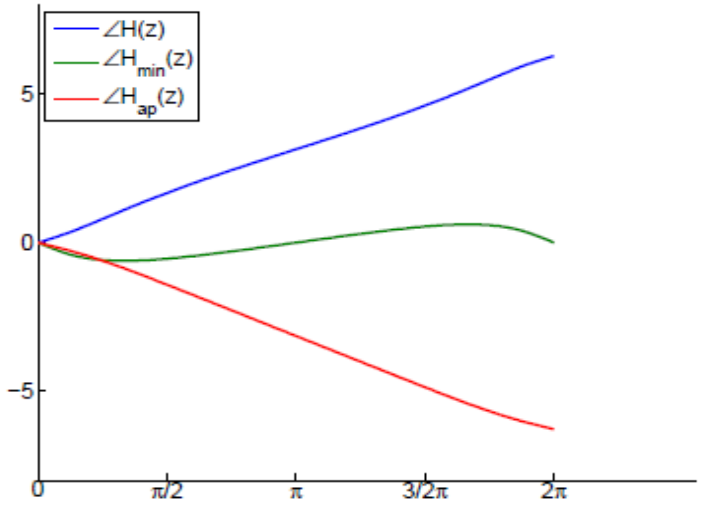




The following Figure shows plots of $H(z), H_{min}(z)$, and $H_{ap}(z)$. Notice that H_{min} is offset from $H(z)$ by a constant factor $K = 4/3$.





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5.36. Consider the stable LTI system with system function

$$H(z) = \frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

The system function $H(z)$ can be factored such that

$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z),$$

where $H_{\text{min}}(z)$ is a minimum-phase system, and $H_{\text{ap}}(z)$ is an all-pass system, i.e.,

$$|H_{\text{ap}}(e^{j\omega})| = 1.$$

Sketch the pole–zero diagrams for $H_{\text{min}}(z)$ and $H_{\text{ap}}(z)$. Be sure to label the positions of all the poles and zeros. Also, indicate the ROC for $H_{\text{min}}(z)$ and $H_{\text{ap}}(z)$.

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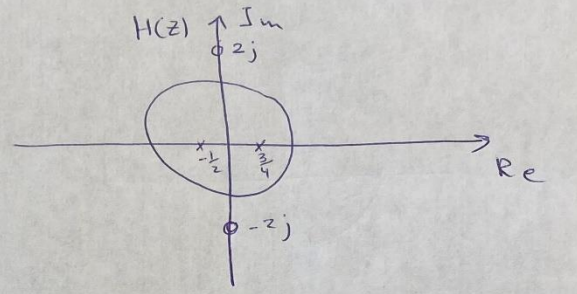
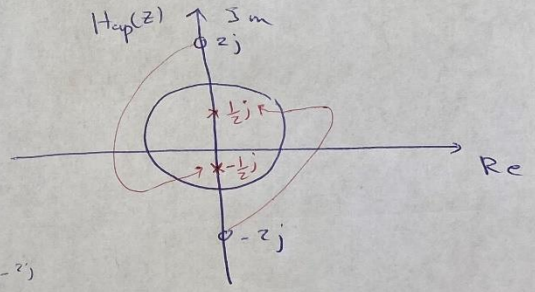
$$H(z) = \frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{(1 - 2j\,z^{-1})(1 + 2j\,z^{-1})}{(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}$$





$\frac{1}{2j} = -\frac{1}{2}j$ zero at $2j$

$\frac{1}{-2j} = \frac{1}{2}j$ zero at $-2j$

$$H_{ap}(z) = \frac{(z^{-1} + \frac{1}{2}j)\,(z^{-1} - \frac{1}{2}j)}{(1 + \frac{1}{2}j\,z^{-1})\,(1 - \frac{1}{2}j\,z^{-1})}$$

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$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$$

$$H_{min}(z) = \frac{(1-2jz^{-1})(1+2jz^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})} \cdot \frac{(1+\frac{1}{2}jz^{-1})(1-\frac{1}{2}jz^{-1})}{(z^{-1}+\frac{1}{2}j)(z^{-1}-\frac{1}{2}j)}$$

$$H_{min}(z) = \frac{\frac{4j}{4j}(1-2jz^{-1})(1+2jz^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})} \frac{(1+\frac{1}{2}jz^{-1})(1-\frac{1}{2}jz^{-1})}{(z^{-1}+\frac{1}{2}j)(z^{-1}-\frac{1}{2}j)}$$

$$H_{min}(z) = \frac{4(z^{-1}+\frac{1}{2}j)(z^{-1}-\frac{1}{2}j)}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})} \frac{(1+\frac{1}{2}jz^{-1})(1-\frac{1}{2}jz^{-1})}{(z^{-1}+\frac{1}{2}j)(z^{-1}-\frac{1}{2}j)}$$

$$H_{min}(z) = \frac{4(1+\frac{1}{2}jz^{-1})(1-\frac{1}{2}jz^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

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Solution

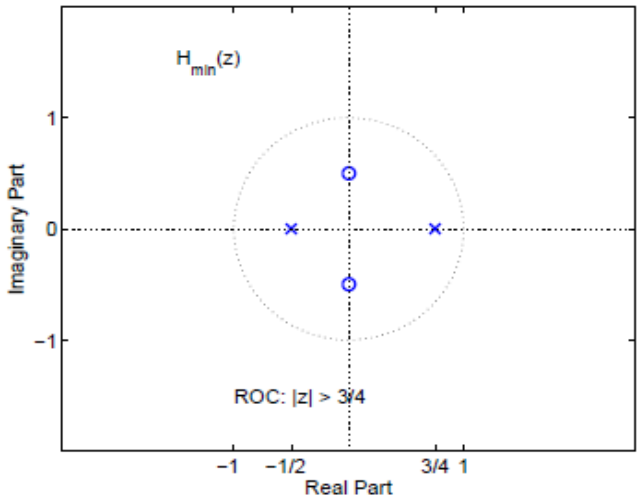
$$H(z) = \frac{1+4z^{-2}}{1-\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{(1-2jz^{-1})(1+2jz^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

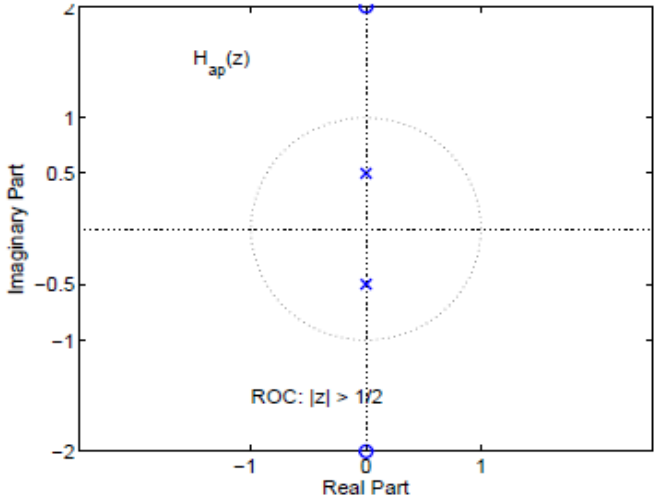
Select the AP section from zeros outside the unit circle as

$$H_{ap}(z) = \frac{(1-2jz^{-1})(1+2jz^{-1})}{(1-\frac{1}{2}jz^{-1})(1+\frac{1}{2}jz^{-1})}.$$

Then $H_{min}(z)$ can be found by solving

$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)} = \frac{(1-\frac{1}{2}jz^{-1})(1+\frac{1}{2}jz^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$





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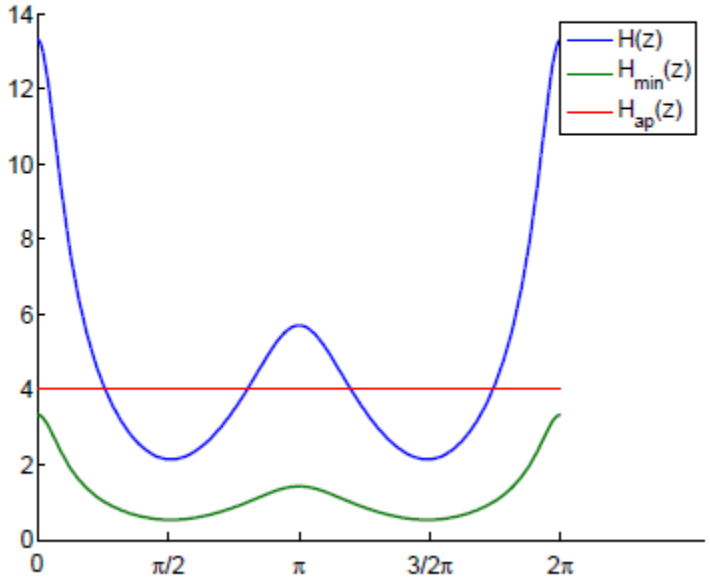
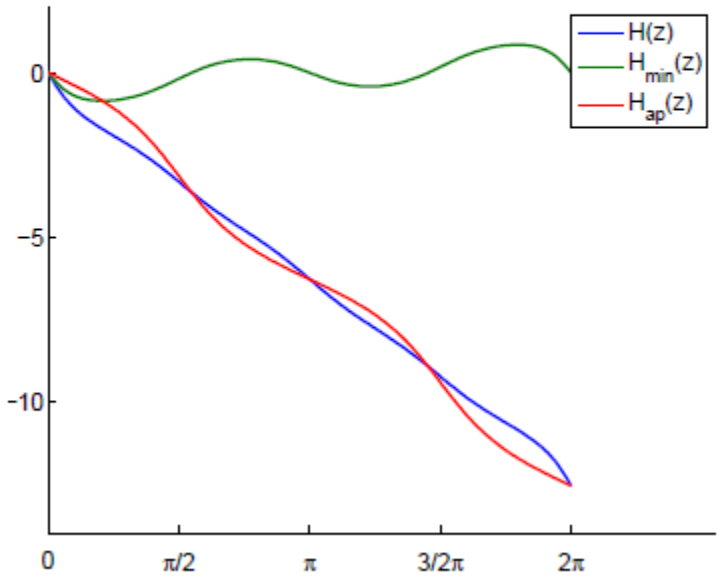
The following Figure shows plots of $H(z), H_{min}(z)$, and $H_{ap}(z)$. Notice that H_{min} is offset from $H(z)$ by a constant factor $K = 4$ which is the magnitude of the all-pass system.





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Introduction to Lab#4

FIR filter design

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There two methods for FIR Design

- **(A) Design with frequency sampling**
- **(B) Design with window functions**

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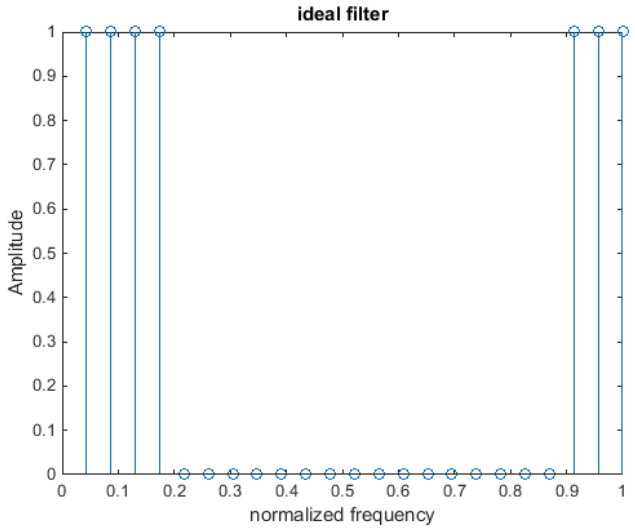
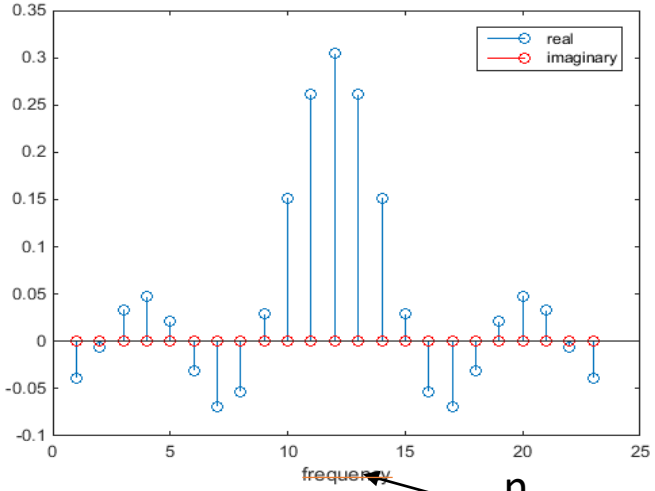
Design FIR (Type I)

Illustration of (a) &(b)




```
wo=0.3*pi;       % Passband edge
len=23;          % Length of filter
pass_len=fix(wo*len/(2*pi))+1;  % Passband
length
Ad=[ones(1,pass_len),zeros(1,len-
2*pass_len+1),ones(1,pass_len-1)];
stem((1:len)/len,Ad);
xlabel('normalized frequency');
ylabel('Amplitude');
title('ideal filter')
```




```
M=(len-1)/2;   % Type 1
k=0:(len-1);
pd=exp(2*pi*j*(-M)*k/len);
Hd=Ad.*pd;
h=real(ifft(Hd));
h2=imag(ifft(Hd));
figure; stem(h);hold on;
stem(h2, 'r');
legend('real','imaginary');
xlabel('frequency');
```

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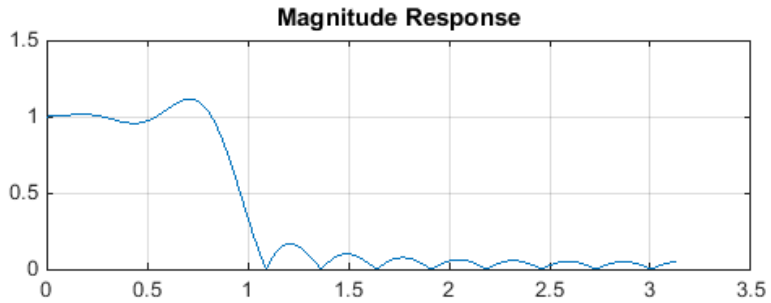
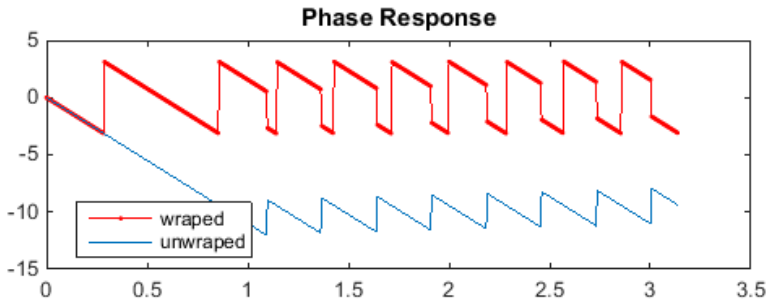
cont

```
[resp,omeg]=freqz(h);
figure;
subplot(211);plot(omeg,abs(resp));grid;
title('Magnitude Response');
subplot(212); plot(omeg,angle(resp), 'r.-');
hold on;
plot(omeg,unwrap(angle(resp))); title('Phase
Response');
hold
off;legend('wraped','unwraped','location','be
st')
figure;
plot(omeg,abs(resp)); title('Magnitude
Response');
hold on; plot(omeg,unwrap(angle(resp))/10,
'k'); title('Phase Response');
```

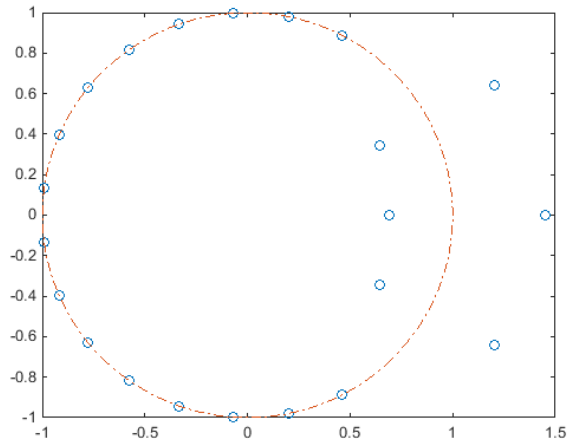




$$\theta(\omega) = -M\omega$$

$$M = \frac{N-1}{2}.$$

M=11



ringing $\overset{\mathcal{F}}{\longleftrightarrow}$ truncation (Gibbs in time-domain)

truncation $\overset{\mathcal{F}}{\longleftrightarrow}$ ringing ($H(\omega)$ pass/stopband rippling)

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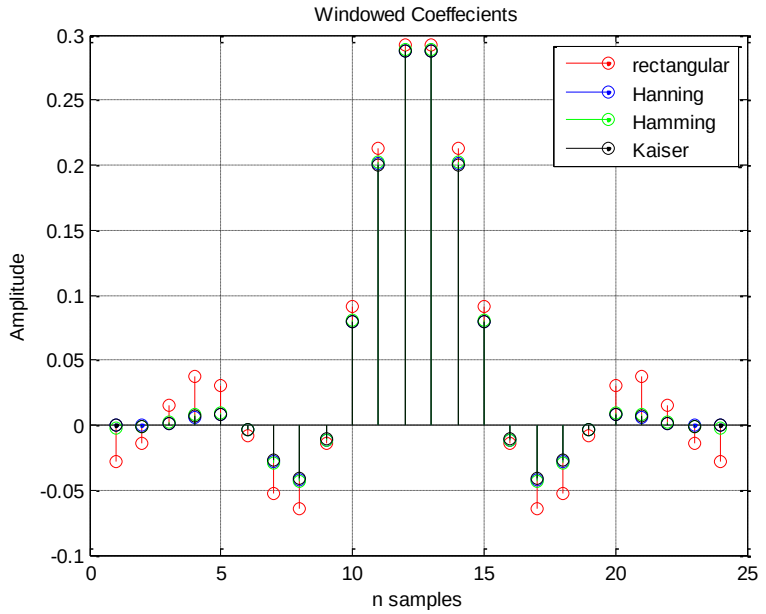
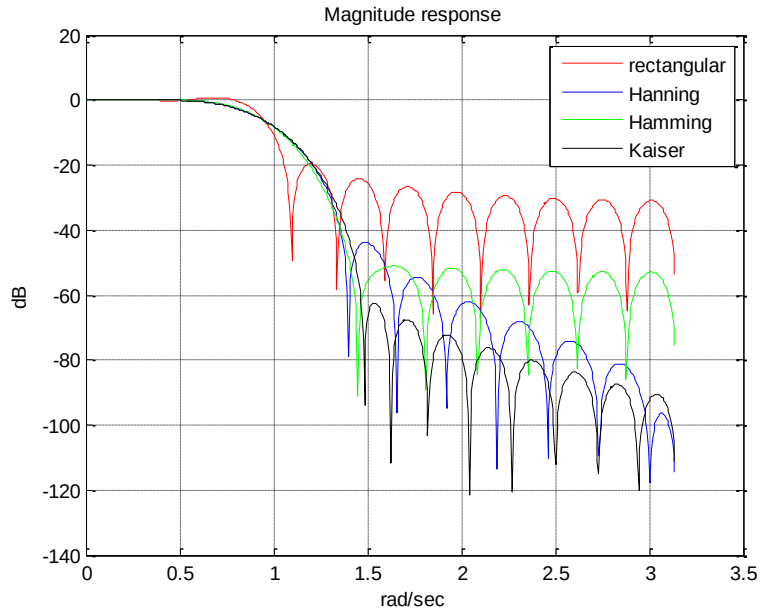
h) Design a length-23 linear-phase FIR low-pass filter with a passband edge of $\beta = 6$. radians/sample using a window approach. Do not use a frequency sampling technique as in a). Use the windowing approach with the following windows: rectangular, Hanning, Hamming and Kaiser with $\omega_0 = 0.3\pi$ Find the frequency response of the resulting filters, and compare with the filter found in a).

```
b1=fir1(len,wn,rectwin(N+1));
b2=fir1(len,wn,hann(N+1));
b3=fir1(len,wn,hamming(N+1));
b4=fir1(len,wn,kaiser(N+1,beta));
```





The cut-off frequency "wn" must be normalized between 0 < Wn < 1.0,

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