ELG4172 Digital Signal Processing

- Tutorial-3

Presented by: Hitham Jleed


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Alan V. Oppenhei
Discrete-Time Signal Processing
Alan V Oppenheim; Ronald W. Schafer
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The z-Transform

z-transform

The z-transform operator $\mathcal{Z}\{\cdot\}$:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

The unique correspondence between a sequence and its z-transform can be indicated by:

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

Two-sided or bilateral z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

One-sided or unilateral z-transform:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Type	$X(z)$	$x[n]$
polynomial in z	$\sum_k c_k z^{-k}$	$\sum_k c_k \delta[n-k]$
single real pole	$\dfrac{1}{1-pz^{-1}}$	$p^n\, u[n]$
double real pole	$\dfrac{pz^{-1}}{(1-pz^{-1})^2}$	$np^n\, u[n]$
double real pole	$\dfrac{1}{(1-pz^{-1})^2}$	$(n+1)p^n\, u[n]$
triple real pole	$\dfrac{1}{(1-pz^{-1})^3}$	$\dfrac{(n+2)(n+1)}{2}p^n\, u[n]$
complex conjugate pair	$\dfrac{az\sin\omega_0}{(z-a\,e^{\jmath\omega_0})(z-a\,e^{-\jmath\omega_0})}$	$a^n\sin(\omega_0 n)\, u[n]$
complex conjugate pair $p = \|p\|\, e^{\jmath\omega_0}$	$\dfrac{r}{1-pz^{-1}} + \dfrac{r^*}{1-p^*z^{-1}}$	$2\,\|r\|\,\|p\|^n\cos(\omega_0 n + \angle r)\, u[n]$

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1. Determine the z-transform, including the ROC, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

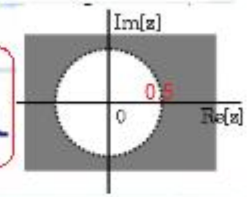
(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n]-u[n-10])$.

Q.3.1

(a) $$\mathcal{Z}\{(1/2)^n u[n]\} = \sum_{n=0}^{\infty} (1/2)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \left|\frac{1}{2z}\right|<1 \Rightarrow |z|>1/2$$



(b) $$\mathcal{Z}\{-(1/2)^n u[-n-1]\} = -\sum_{n=-\infty}^{-1} (1/2)^n z^{-n}$$

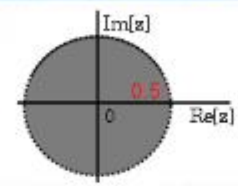
replace $n = -n$

$$= -\sum_{n=1}^{\infty} (1/2)^{-n} z^{n} = -\sum_{n=1}^{\infty} (2z)^n$$

$$= -\left[\sum_{n=0}^{\infty} (2z)^n - (2z)^0\right]$$

$$= -\left[\frac{1}{1-2z} - 1\right] \qquad |2z|<1 \Rightarrow |z|<1/2$$

$$= \frac{-2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z|<1/2$$



replace n=-n

(c) $$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[-n]\right] = \sum_{n=0}^{\infty} (2z)^n = \frac{1}{1-2z} \qquad |z| < \frac{1}{2}$$

(d) $$\mathcal{Z}[\delta[n]] = z^0 = 1 \qquad \text{all } z$$

(e) $$\mathcal{Z}[\delta[n-1]] = z^{-1} \qquad |z| > 0$$

f) $\mathcal{Z}\{\delta[n+1]\} = z$ (2)

RoC: $0 \le |z| < \infty$

g) $\mathcal{Z}\{(\frac{1}{2})^n(u[n]-u[n-10])\} = \sum_{n=0}^{9} (\frac{1}{2})^n z^{-n}$

$= \sum_{n=0}^{9} \left(\frac{1}{2z}\right)^n = \frac{1-(2z)^{-10}}{1-(2z)^{-1}}$

RoC: $|z| > 0$

Note: $\sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}}$

Note: There are finite number of terms in the summation, hence the sum will be finite for $|a| < \infty$ and $|z| \neq 0$ only. Therefore, the

RoC is $|z| > 0$

3.7. The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}.$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

(b) What is the ROC for $Y(z)$?

(c) Determine $y[n]$.

3.7. (a)

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow \qquad X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1+z^{-1}}$$

$H(z)$ causal $\Rightarrow$ ROC $|z| > 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the z-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)

$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \qquad |z| > 1$$

Therefore,

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

Inverse z-transform

- Inspection method
- Partial fraction expansion
- Power series expansion

Inspection Method

This "method" is to basically become familiar with the z-transform pair tables and then "reverse engineer".

Example 1

When given

$$X(z) = \frac{z}{z - \alpha}$$

with an ROC of

$$|z| > \alpha$$

we could determine "by inspection" that

$$x[n] = \alpha^n u[n]$$

Power Series Expansion Method

When the z-transform is defined as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

then each term of the sequence $x[n]$ can be determined by looking at the coefficients of the respective power of z^{-n}.

Example 3.11 Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2 \left(1 - \frac{1}{2} z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \tag{3.52}$$

Although $X(z)$ is obviously a rational function of z, it is really not a rational function in the form of Eq. (3.39). Its only poles are at $z = 0$, so a partial fraction expansion according to the technique of Section 3.3.2 is not appropriate. However, by multiplying the factors of Eq. (3.52), we can express $X(z)$ as

$$X(z) = z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1].$$

Example 3.13 Power Series Expansion by Long Division

Consider the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.55}$$

Since the ROC is the exterior of a circle, the sequence is a right-sided one. Furthermore, since $X(z)$ approaches a finite constant as z approaches infinity, the sequence is causal. Thus, we divide, so as to obtain a series in powers of z^{-1}. Carrying out the long division, we obtain

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \quad \cdots \end{array},$$

$$x[n] = \delta[n] + a\delta[n-1] + a^2\delta[n-2] + a^3\delta[n-3] + \ldots$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots .$$

Hence, $x[n] = a^n u[n]$.

Partial-Fraction Expansion Method

When dealing with linear time-invariant systems the z-transform is often of the form

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

This can also expressed as

$$X(z) = \frac{a_0}{b_0} \frac{\prod_{k=1}^{M} 1 - c_k z^{-1}}{\prod_{k=1}^{N} 1 - d_k z^{-1}}$$

where c_k represents the nonzero zeros of $X(z)$ and d_k represents the nonzero poles.

If $M < N$ then $X(z)$ can be represented as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

This form allows for easy inversions of each term of the sum using the inspection method and the transform table. If the numerator is a polynomial, however, then it becomes necessary to use partial-fraction expansion to put $X(z)$ in the above form. If $M \geq N$ then $X(z)$ can be expressed as

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example 3.10 Inverse by Partial Fractions

To illustrate the case in which the partial fraction expansion has the form of Eq. (3.45), consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1. \tag{3.48}$$

The pole–zero plot for $X(z)$ is shown in Figure 3.11. From the ROC and Property 5, Section 3.2, it is clear that $x[n]$ is a right-sided sequence. Since $M = N = 2$ and the poles are all 1st-order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

The constant B_0 can be found by long division:

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad 5z^{-1} - 1 \end{array}$$

Since the remainder after one step of long division is of degree 1 in the variable z^{-1}, it is not necessary to continue to divide. Thus, $X(z)$ can be expressed as

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}. \tag{3.49}$$

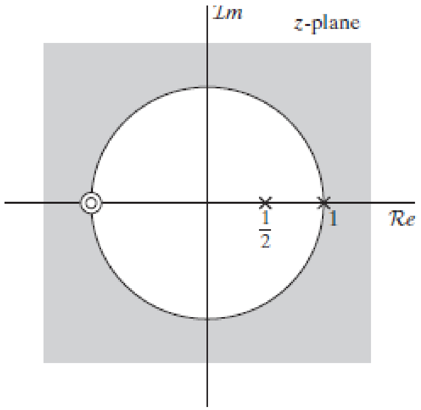


Figure 3.11 Pole–zero plot for the z-transform in Example 3.10.

Now the coefficients A_1 and A_2 can be found by applying Eq. (3.43) to Eq. (3.48) or, equivalently, Eq. (3.49). Using Eq. (3.49), we obtain

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)(1 - z^{-1})\right]_{z=1} = 8.$$

Therefore,

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}. \tag{3.50}$$

From Table 3.1, we see that since the ROC is $|z| > 1$,

$$2 \overset{\mathcal{Z}}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

$$\frac{1}{1 - z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} u[n].$$

Thus, from the linearity of the z-transform,

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

If the poles $p_1, \ldots, p_N$ are all different (distinct) then the expansion we seek has the form

$$X(z) = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}},$$

where the r_k's are real or complex numbers called **residues**.

For distinct roots:

$$\boxed{r_k = (1 - p_k z^{-1})\, X(z) \Big|_{z=p_k}}$$

Proof:

$$(1 - p_k z^{-1})\, X(z) = (1 - p_k z^{-1}) \frac{r_1}{1 - p_1 z^{-1}} + \cdots + r_k + \cdots + (1 - p_k z^{-1}) \frac{r_N}{1 - p_N z^{-1}},$$

and evaluate the LHS and RHS at $z = p_k$.

Example.

$$X(z) = \frac{r}{1 - p z^{-1}} + \frac{r^*}{1 - p^* z^{-1}}$$

thus

$$x[n] = [r p^n + r^* (p^*)^n]\, u[n]\,.$$

Since this is of the form $a + a^*$, it must be real, so it is useful to express it using real quantities.

$$x[n] = 2\,\mathrm{real}(r p^n)\, u[n] = 2\,\mathrm{real}\big(|r|\, e^{\jmath\phi}\, |p|^n\, e^{\jmath\omega_0 n}\big)\, u[n] = 2\,|r|\,|p|^n \cos(\omega_0 n + \phi)\, u[n]$$

where $p = |p|\, e^{\jmath\omega_0}$ and $r = |r|\, e^{\jmath\phi}$. Note the different roles of $\angle p = \omega_0$ (frequency) and $\angle r = \phi$ (phase).

MATLAB's function: residuez

3.6. Following are several z-transforms. For each, determine the inverse z-transform using both methods—partial fraction expansion and power series expansion—discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

(b) $$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$$

(c) $$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \qquad |z| > \frac{1}{2}$$

(d) $$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \qquad |z| > \frac{1}{2}$$

(e) $$X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \qquad |z| > |1/a|$$

3.6. (a)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

Partial fractions: one pole → inspection, $x[n] = (-\frac{1}{2})^n u[n]$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{1}{2}z^{-1} & +\frac{1}{4}z^{-2} & + \dots \\ \hline 1+\frac{1}{2}z^{-1} & 1 & & & \\ & 1 & +\frac{1}{2}z^{-1} & & \\ \hline & & -\frac{1}{2}z^{-1} & & \\ & & -\frac{1}{2}z^{-1} & -\frac{1}{4}z^{-2} & \\ \hline & & & +\frac{1}{4}z^{-2} & \\ & & & +\frac{1}{4}z^{-2} & +\frac{1}{8}z^{-3} \\ \hline \end{array}$$

$$\Longrightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(b)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}$$

Partial Fractions: one pole → inspection,

$$x[n] = -(-\tfrac{1}{2})^n u[-n-1]$$

Long division:

$$\begin{array}{r|llll} & 2z & -4z^2 & +8z^3 & + \dots \\ \hline \frac{1}{2}z^{-1}+1 & 1 & & & \\ & 1 & +2z & & \\ \hline & & -2z & & \\ & & -2z & -4z^2 & \\ \hline & & & +4z^2 & \\ & & & +4z^2 & +8z^3 \\ \hline \end{array}$$

$$\Longrightarrow x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

(c)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$X(z) = \frac{-3}{1 + \frac{1}{4}z^{-1}} + \frac{4}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$$x[n] = \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right]u[n]$$

Long division:

$$\begin{array}{r|llll}
 & 1 & + (-\frac{3}{4} - \frac{1}{2})z^{-1} & + (-\frac{3}{16} + 1)z^{-2} & + \ldots \\
\hline
1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} & 1 & - \frac{1}{2}z^{-1} & & \\
 & 1 & + \frac{3}{4}z^{-1} & + \frac{1}{8}z^{-2} & \\
\hline
 & & (-\frac{3}{4} - \frac{1}{2})z^{-1} & - \frac{1}{8}z^{-2} & \\
 & & (-\frac{3}{4} - \frac{1}{2})z^{-1} & + \frac{3}{4}(-\frac{3}{4} - \frac{1}{2})z^{-2} & + \frac{1}{8}(-\frac{3}{4} - \frac{1}{2})z^{-3} \\
\hline
 & & & [-\frac{1}{8} + \frac{3}{4}(\frac{3}{4} + \frac{1}{2})]z^{-2} & + \frac{1}{8}(\frac{3}{4} + \frac{1}{2})z^{-3}
\end{array}$$

$$\Longrightarrow x[n] = \left[-3(\left(-\frac{1}{4}\right)^n + \left(-\frac{1}{2}\right)^{n-2}\right]u[n]$$

(d)
$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}} = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$
$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

Long division: see part (i) above.

(e)
$$X(z) = \frac{1-az^{-1}}{z^{-1}-a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1-a^2)}{1-a^{-1}z^{-1}} \qquad |z| > |a^{-1}|$$
$$x[n] = -a\delta[n] - (1-a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|l} & -\frac{1}{a} \quad -\left(\frac{a^{-1}-a}{a}\right)z^{-1} \quad -\left(\frac{a^{-1}-a}{a^2}\right)z^{-2} \quad + \ \dots \\ \hline -a+z^{-1} & 1 \qquad\qquad -az^{-1} \\ & 1 \qquad\qquad -az^{-1} \\ \hline & \qquad (a^{-1}-a)z^{-1} \qquad\qquad \dots \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1-a^2)a^{-(n+1)}u[n]$$

3.8. The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output $y[n]$.
(c) Is the system stable? That is, is $h[n]$ absolutely summable?

3.8. The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})} \qquad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ causal $\Rightarrow$

$$h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$$

(b)

$$\begin{aligned} Y(z) &= X(z)H(z) = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{3}{4}z^{-1})} \qquad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1 + \frac{3}{4}z^{-1}} \end{aligned}$$

Therefore the output is

$$y[n] = -\frac{8}{13}\left(\frac{1}{3}\right)^n u[n] + \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n]$$

(c) For $h[n]$ to be causal the ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

3.9. A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1 + z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}.$$

(a) What is the ROC of $H(z)$?
(b) Is the system stable? Explain.
(c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1].$$

(d) Find the impulse response $h[n]$ of the system.

3.9.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} - \frac{1}{(1+\frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow |z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1 \Rightarrow$ stable.

(c)

$$\begin{aligned} y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\ Y(z) &= \frac{-\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} \\ &= \frac{1+z^{-1}}{(1+\frac{1}{4}z^{-1})(1-2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\ X(z) &= \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-2z^{-1})} \qquad |z| < 2 \\ x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1} u[-n] \end{aligned}$$

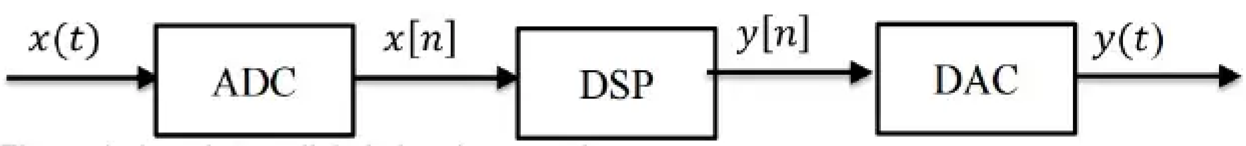
(d)

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

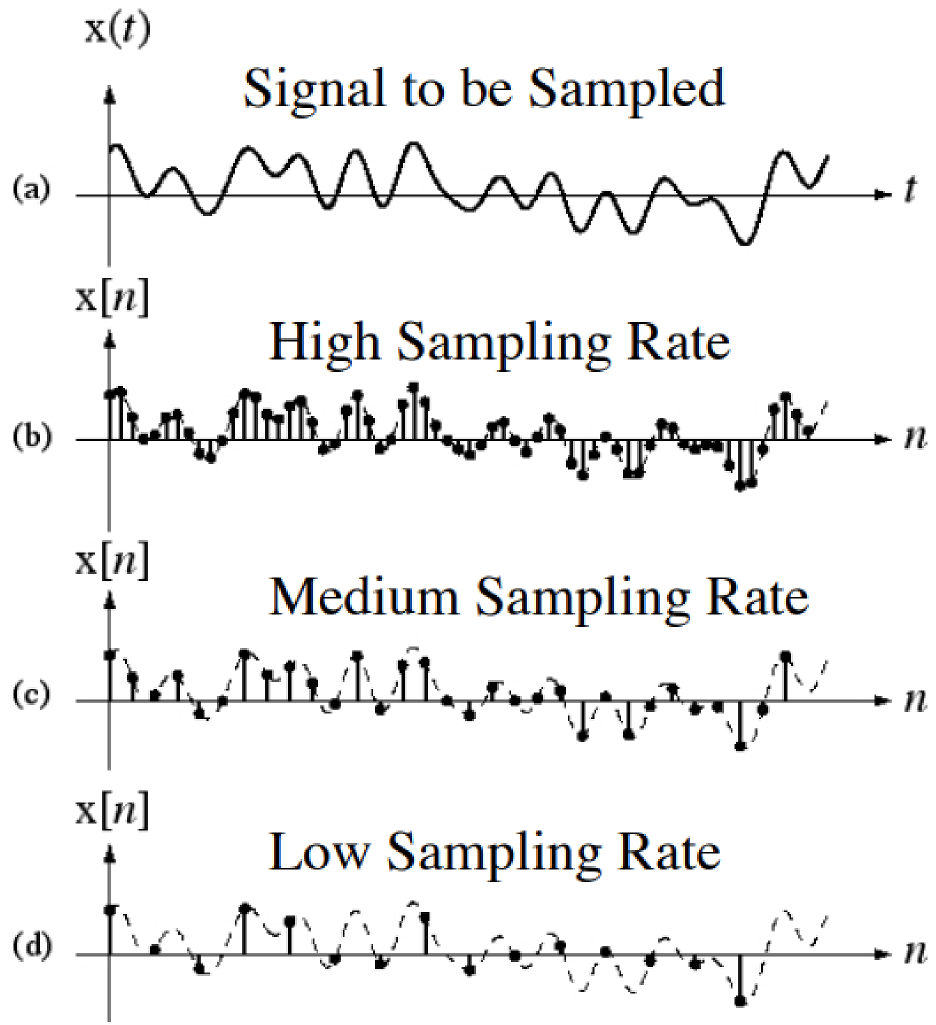
Introduction to Lab#3

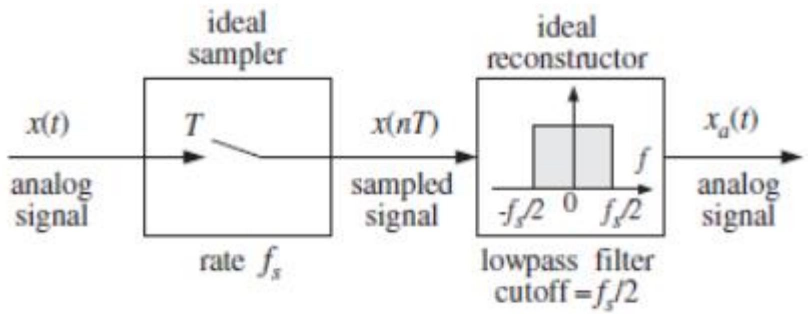
SAMPLING, A/D CONVERSION AND D/A CONVERSION

Most of the signals around us such as pressure, temperature, Light Intensity etc. Most of our contemporary modern equipment and systems are entirely digital. Therefore, to enable these systems to interact with these physical signals, then conversion from physical form to their equivalent electrical analogue form and eventually conversion from analogue to digital form and vice versa using ADC and DAC converters respectively becomes necessary in order to analyse, comprehend their properties and processes them using areal time digital processing system as shown below.



Step 1 Sampling Process





The fundamental consideration in sampling is how fast to sample a signal to be able to reconstruct it.

The sampling theorem

Suppose you sample a signal in some way. If you can exactly reconstruct the signal from the samples, then you have done a proper sampling and captured the key signal information

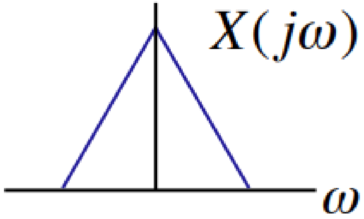
Among the frequencies, there is a unique one that lies within the Nyquist interval. It is obtained by reducing the original f modulo-fs, that is, adding to or subtracting from f enough multiples of fs until it lies within the symmetric Nyquist interval [−fs/2, fs/2].

Sampling in the Frequency Domain

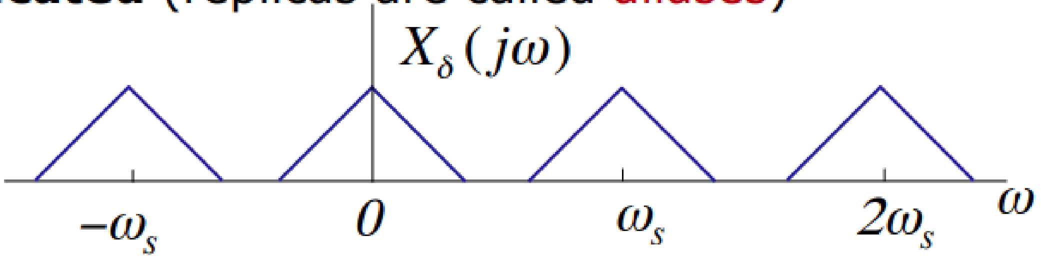
Graphical interpretation of the formula

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big), \quad \omega_s = 2\pi f_s, \quad f_s = \frac{1}{T_s}$$

Continuous-time spectrum



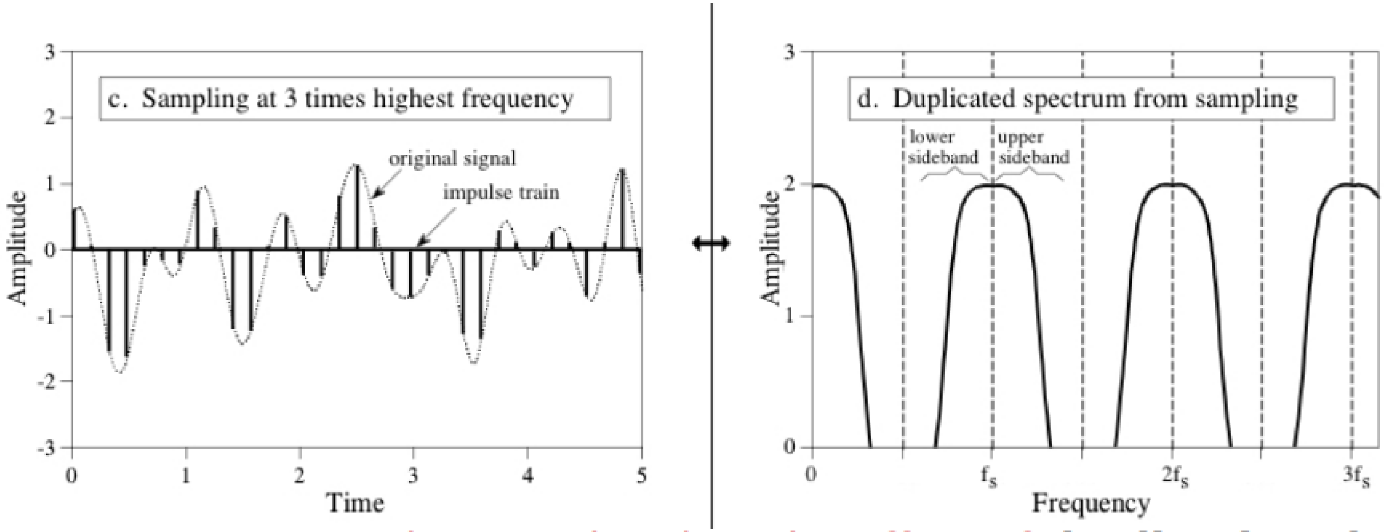
is **scaled** and **replicated** (replicas are called aliases)



Aliasing: If signal bandwith $f_B \geq \frac{f_s}{2}$ then spectrum overlaps!

Aliasing in the frequency domain

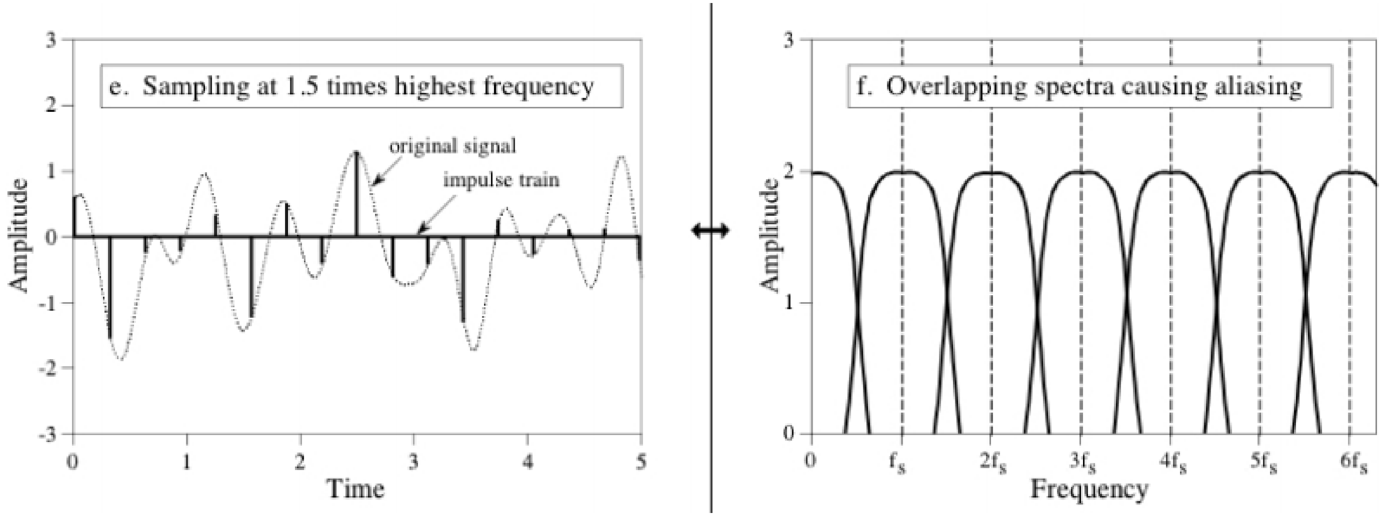
The sampling of the signal corresponds to doing a **convolution of the signal with a delta impulse train**. We call the resulting signal **impulse train**. The magnitude of this impulse train is shown on the right



As you can see, the sampling has the effect of **duplicating the spectrum** of the original signal an infinite number of times. In other words, **sampling introduces new frequencies**

Aliasing in the frequency domain

Compare the previous plot with the following one for a different sampling:



Here, $f = 0.66 \times f_s > 0.5 \times f_s$ so this is **an improper sampling**. In the FD this means that **the repeated spectra overlap**. Since there is no way to separate the overlap, the **signal information is lost**

Sinusoidal Signal

Example 4.1 Sampling and Reconstruction of a Sinusoidal Signal

If we sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = 1/6000$, we obtain $x[n] = x_c(nT) = \cos(4000\pi Tn) = \cos(\omega_0 n)$, where $\omega_0 = 4000\pi T = 2\pi/3$. In this case, $\Omega_s = 2\pi/T = 12000\pi$, and the highest frequency of the signal is $\Omega_0 = 4000\pi$, so the conditions of the Nyquist sampling theorem are satisfied and there is no aliasing. The Fourier transform of $x_c(t)$ is

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi).$$

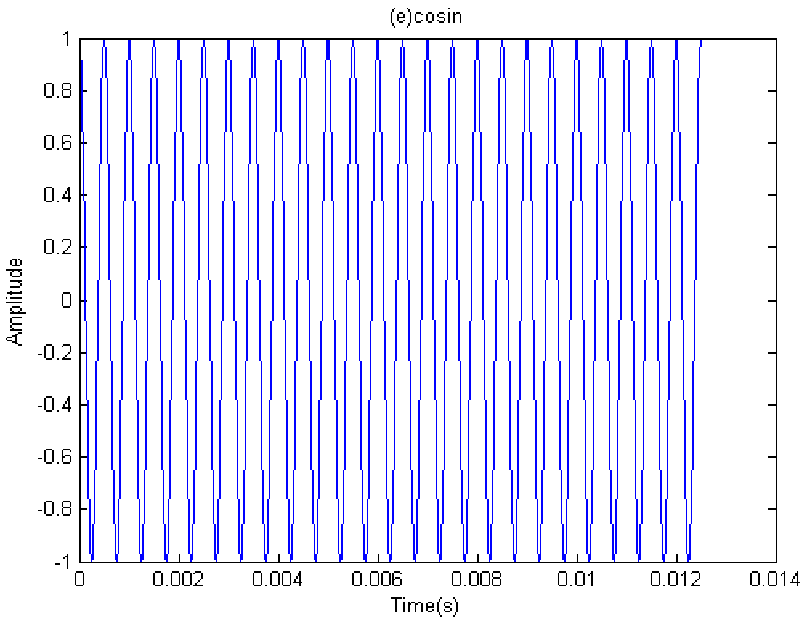
Figure 4.6(a) shows

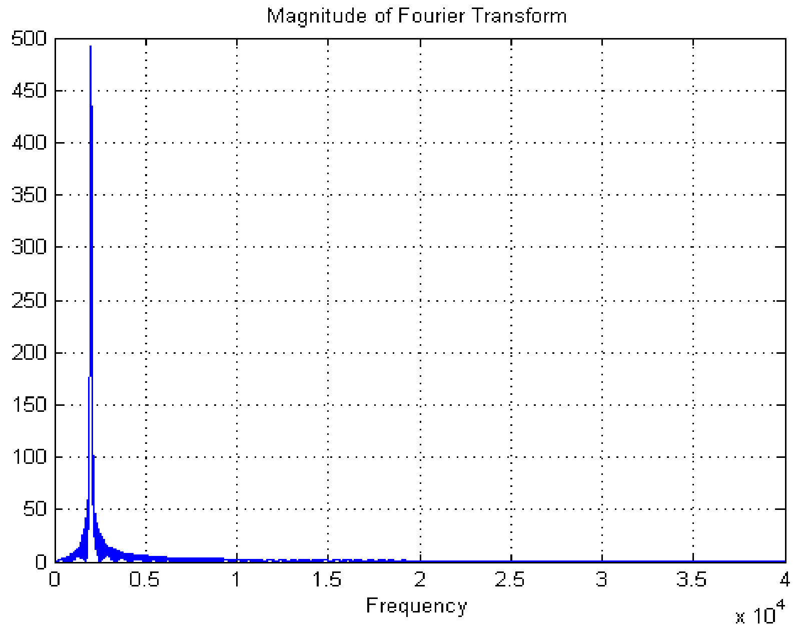
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c[j(\Omega - k\Omega_s)] \tag{4.21}$$

for $\Omega_s = 12000\pi$. Note that $X_c(j\Omega)$ is a pair of impulses at $\Omega = \pm 4000\pi$, and we see shifted copies of this Fourier transform centered on $\pm\Omega_s, \pm 2\Omega_s$, etc. Plotting $X(e^{j\omega}) = X_s(j\omega/T)$ as a function of the normalized frequency $\omega = \Omega T$ results in Figure 4.6(b), where we have used the fact that scaling the independent variable of an impulse also scales its area, i.e., $\delta(\omega/T) = T\delta(\omega)$ (Oppenheim and Willsky, 1997). Note that the original frequency $\Omega_0 = 4000\pi$ corresponds to the normalized frequency $\omega_0 = 4000\pi T = 2\pi/3$, which satisfies the inequality $\omega_0 < \pi$, corresponding to the fact that $\Omega_0 = 4000\pi < \pi/T = 6000\pi$. Figure 4.6(a) also shows the frequency response of an ideal reconstruction filter $H_r(j\Omega)$ for the given sampling rate of $\Omega_s = 12000\pi$. This figure shows that the reconstructed signal would have frequency $\Omega_0 = 4000\pi$, which is the frequency of the original signal $x_c(t)$.

Simulate a continuous sinusoid

$$x(t) = \cos(2\pi f_0 t) \approx \cos(2\pi f_0 \, n \, \Delta t) \qquad n=0,1,2,... \qquad f_0=2\text{kHz} \qquad \Delta t = 1/80000$$





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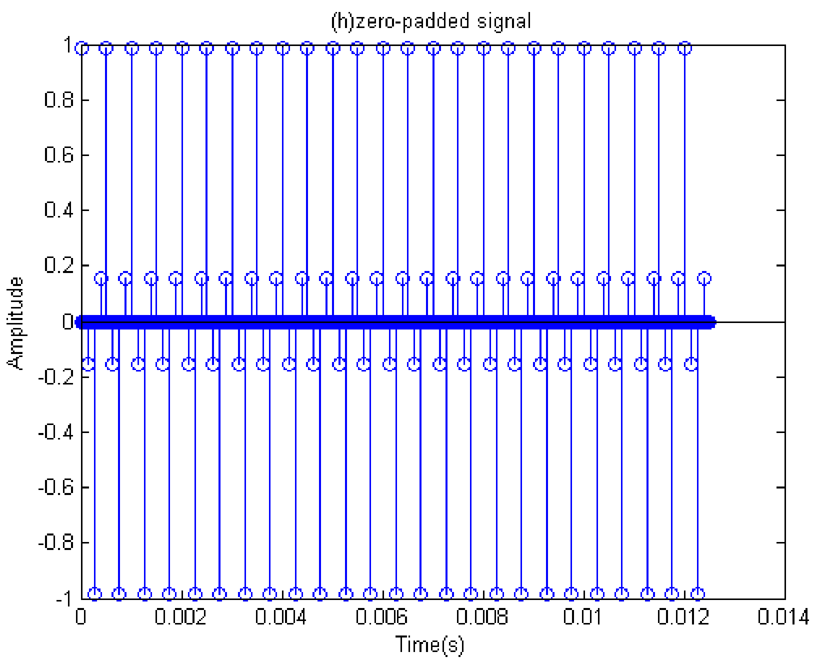
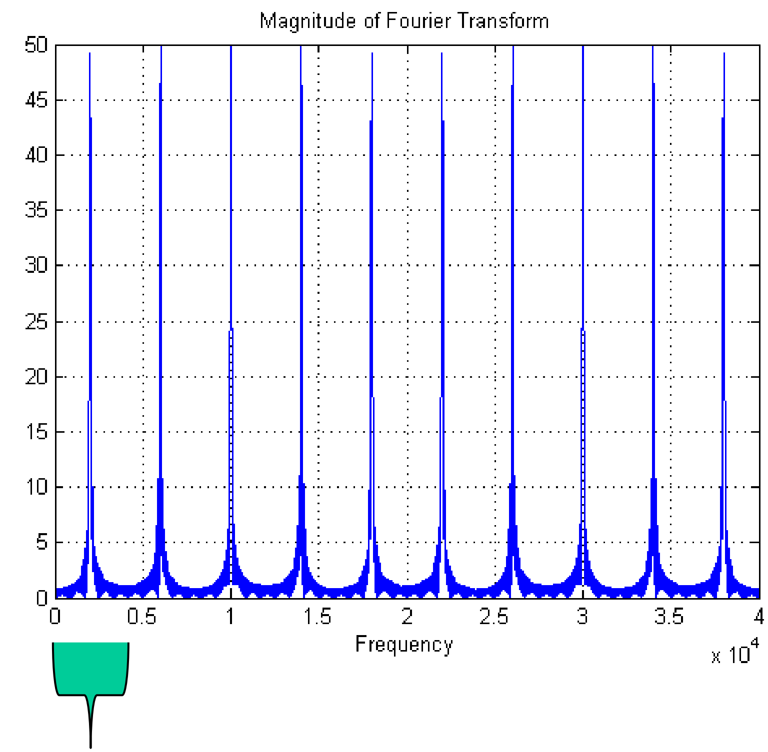
sample and hold (padding zeros)





Filter this part

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THE END