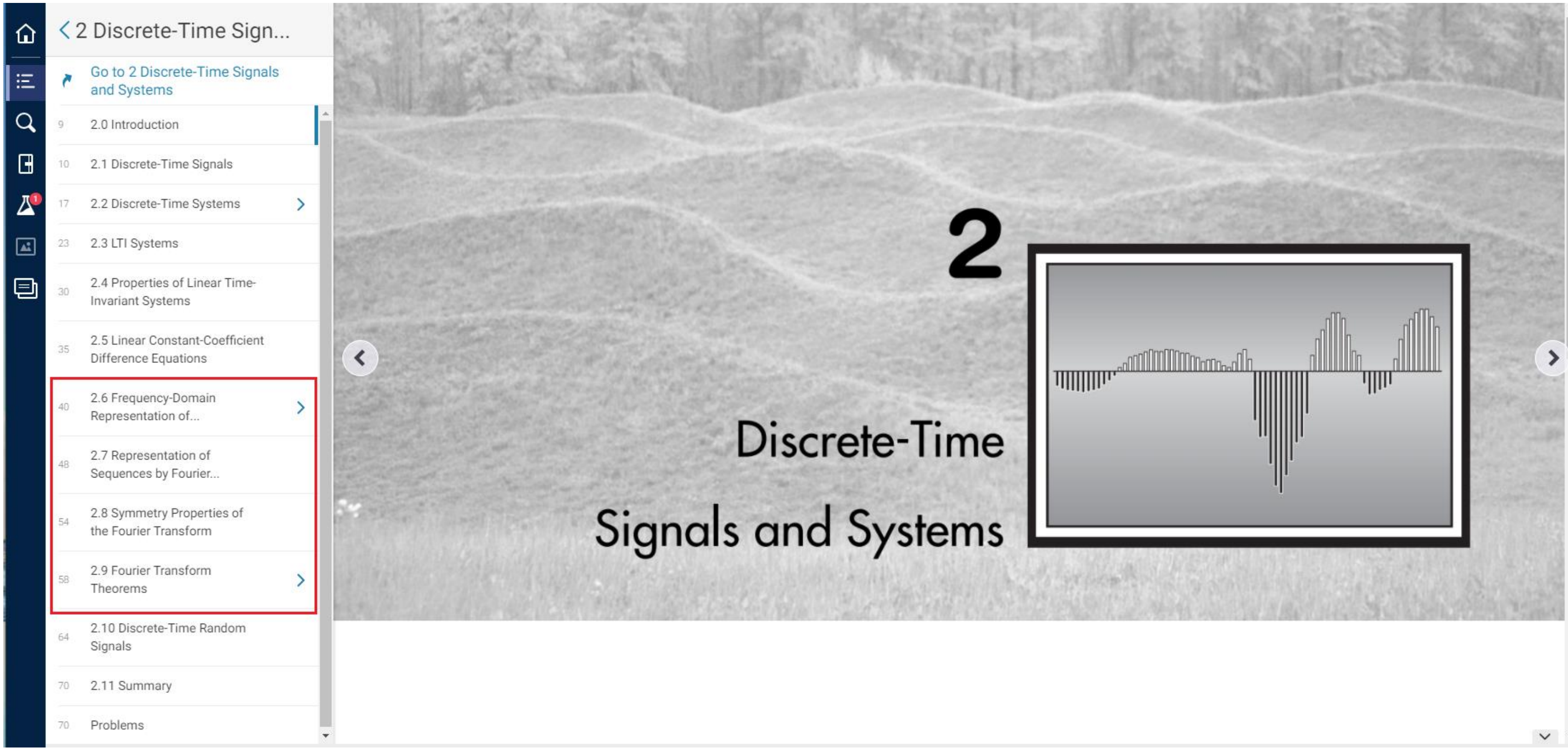
ELG4172 Digital Signal Processing

- Tutorial #2

Presented by: Hitham Jleed

Discrete-Time Fourier Transform DTFT

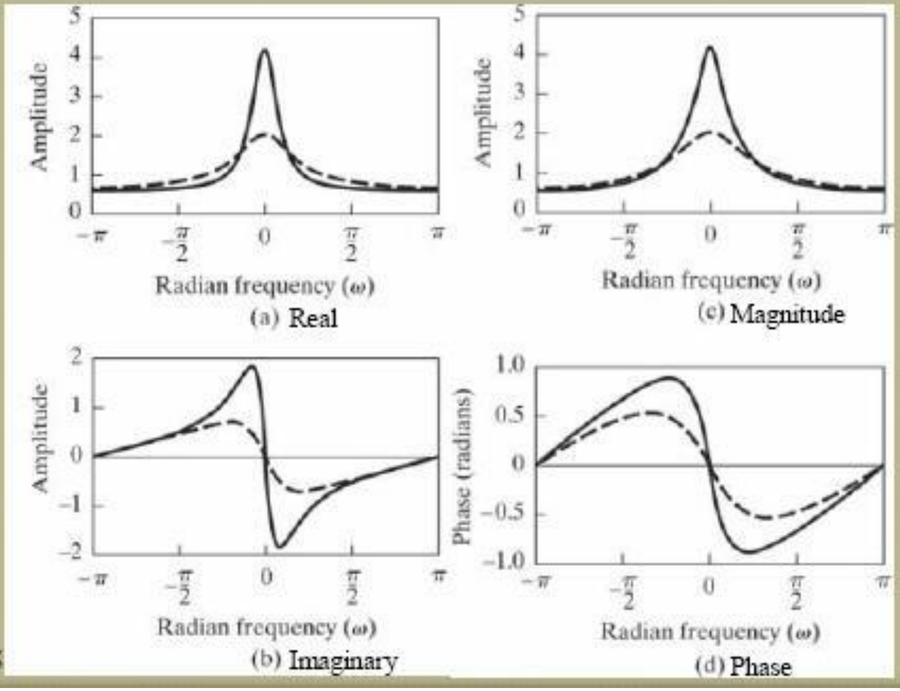


Review Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ **Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$ **Inverse Fourier Transform**

- Signal $x[n] = a^n u[n]$
  - Fourier transform $X(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$, if $|a|<1$



Solid curve: a=0.75
Dashed curve: a=0.5

uOttawa

2.6. (a) Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

(b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

https://bookshelf.vitalsource.com/#/books/9780133002287

2.6. **(a)** Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

(b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

(a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}.$$

(b) A system with frequency response:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

https://bookshelf.vitalsource.com/#/books/9780133002287

From Proakis & Manolakis Book

A linear time-invariant system is described by the following difference equation:

$$y(n) = ay(n-1) + bx(n), \qquad 0 < a < 1$$

(a) Determine the magnitude and phase of the frequency response $H(e^{j\omega})$ of the system.

(b) Choose the parameter b so that the maximum value of $|H(e^{j\omega})|$ is unity, and sketch $|H(e^{j\omega})|$ and $\measuredangle H(e^{j\omega})$ for $a = 0.9$.

(c) Determine the output of the system to the input signal

$$x(n) = 5 + 12 \sin \frac{\pi}{2} n - 20 \cos\left(\pi n + \frac{\pi}{4}\right)$$

From Proakis & Manolakis Book

A linear time-invariant system is described by the following difference equation:

$$y(n) = ay(n-1) + bx(n), \qquad 0 < a < 1$$

(a) Determine the magnitude and phase of the frequency response $H(e^{j\omega})$ of the system.

(b) Choose the parameter b so that the maximum value of $|H(e^{j\omega})|$ is unity, and sketch $|H(e^{j\omega})|$ and $\measuredangle H(e^{j\omega})$ for $a = 0.9$.

(c) Determine the output of the system to the input signal

$$x(n) = 5 + 12 \sin \frac{\pi}{2} n - 20 \cos \left(\pi n + \frac{\pi}{4} \right)$$

(a) The frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b}{1 - ae^{-j\omega}}$$

Since

$$1 - ae^{-j\omega} = (1 - a\cos\omega) + ja\sin\omega$$

it follows that

$$|1 - ae^{-j\omega}| = \sqrt{(1 - a\cos\omega)^2 + (a\sin\omega)^2}$$

$$= \sqrt{1 + a^2 - 2a\cos\omega}$$

$$|H(e^{j\omega})| = \frac{|b|}{|1 - ae^{-j\omega}|} = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

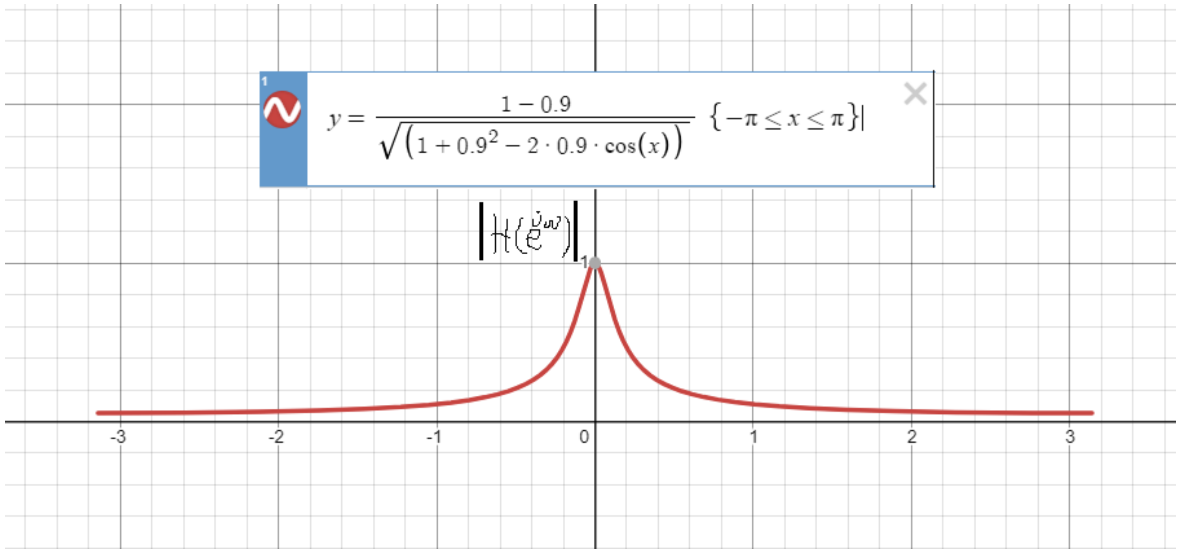
$$\measuredangle H(e^{j\omega}) = \Theta(e^{j\omega}) = \measuredangle b - \tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega}$$

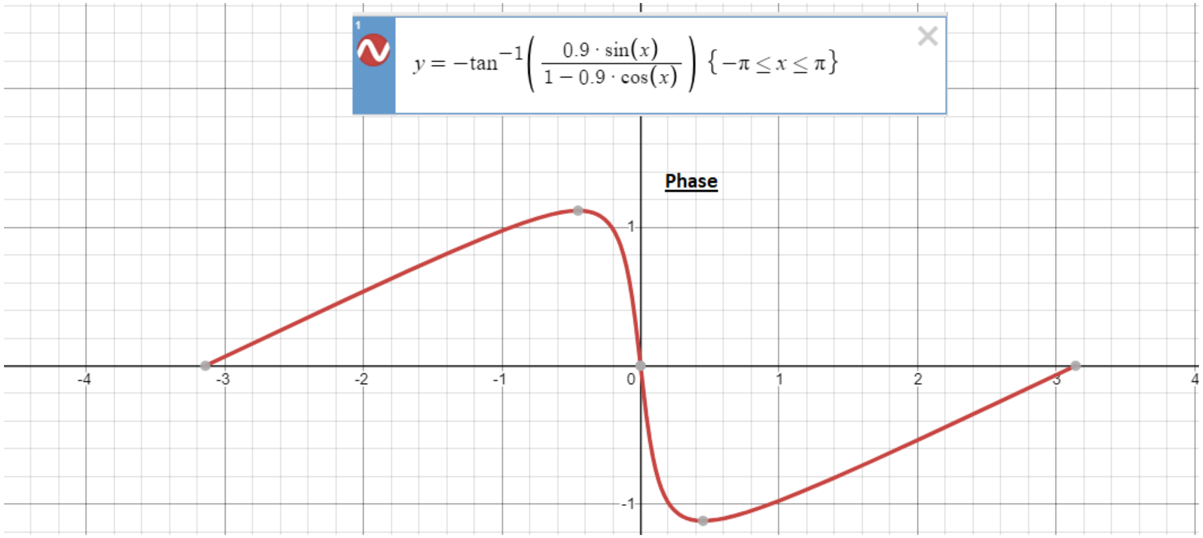
(b) Since the parameter a is positive, the denominator of $|H(e^{j\omega})|$ attains a minimum at $\omega = 0$. Therefore, $|H(e^{j\omega})|$ attains its maximum value at $\omega = 0$. At this frequency we have,

$$|H(0)| = \frac{|b|}{1 - a} = 1$$

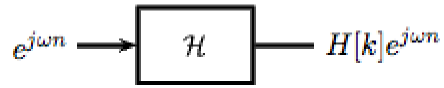
which implies that $b = \pm(1 - a)$. We choose $b = 1 - a$, so that

$$|H(e^{j\omega})| = \frac{1 - a}{\sqrt{1 + a^2 - 2a\cos\omega}} \quad \text{and} \quad \Theta(e^{j\omega}) = -\tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega}$$





https://www.desmos.com/calculator



(c) The input signal consists of components at frequencies $\omega = 0,\ \pi/2$, and π

For $\omega = 0$, $\Rightarrow$ $\left| H(e^{j\omega}) \right| = 1$ and $\Theta(0) = 0$. For $\omega = \pi/2$,

$$\omega = \pi/2 \quad \Rightarrow \quad \left| H(e^{j\omega}) \right| = \frac{1-a}{\sqrt{1+a^2}} = \frac{0.1}{\sqrt{1.81}} = 0.074$$

$$\Theta\left(\frac{\pi}{2}\right) = -\tan^{-1} a = -42^\circ$$

$$\text{For } \omega = \pi, \quad \Rightarrow \quad \left| H(e^{j\omega}) \right| = \frac{1-a}{1+a} = \frac{0.1}{1.9} = 0.053$$

$$\Theta(\pi) = 0$$

Therefore, the output of the system is

$$y(n) = 5|H(0)| + 12\left|H\left(\frac{\pi}{2}\right)\right| \sin\left[\frac{\pi}{2}n + \Theta\left(\frac{\pi}{2}\right)\right] - 20|H(\pi)| \cos\left[\pi n + \frac{\pi}{4} + \Theta(\pi)\right]$$

8. An LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

Q 2.8

$$y[n] = h[n] * x[n]$$

FT

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{5}{1 + \frac{1}{2} e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}} \cdot \frac{5}{1 + \frac{1}{2} e^{-j\omega}}$$

$$= \frac{3}{1 + \frac{1}{2} e^{-j\omega}} + \frac{2}{1 - \frac{1}{3} e^{-j\omega}}$$

$$y[n] = 2\left(\tfrac{1}{3}\right)^n u[n] + 3\left(-\tfrac{1}{2}\right)^n u[n]$$

uOttawa

2.9. Consider the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = \frac{1}{3}x[n-1].$$

(a) What are the impulse response, frequency response, and step response for the causal LTI system satisfying this difference equation?

2.9. (a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1 - \frac{1}{3}e^{-j\omega}} + \frac{2}{1 - \frac{1}{2}e^{-j\omega}} \\ h[n] &= -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

For the step response $s[n]$:

$$
\begin{aligned}
s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\
&= \sum_{k=-\infty}^{n} h[k] \\
&= -2\frac{1-(1/3)^{n+1}}{1-1/3}u[n] + 2\frac{1-(1/2)^{n+1}}{1-1/2}u[n] \\
&= (1+(\frac{1}{3})^n - 2(\frac{1}{2})^n)u[n]
\end{aligned}
$$

Frequency Response For Rational System Functions

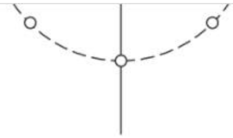


Figure 5.8 Pole–zero plot for Example 5.5.

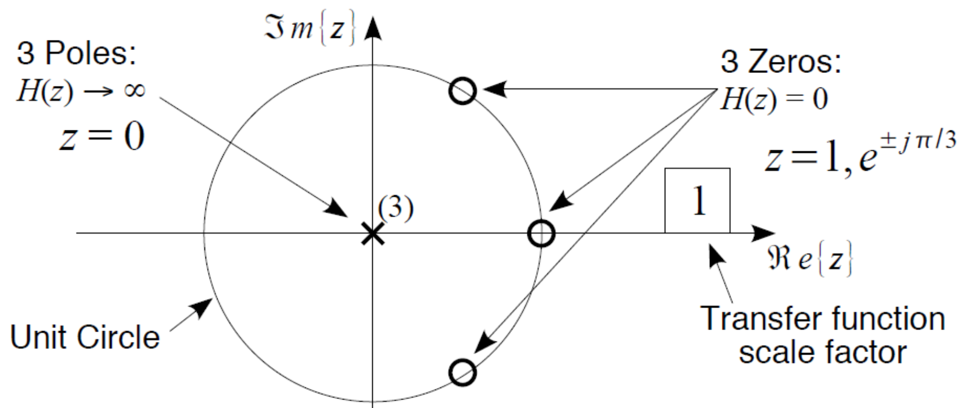
5.3 FREQUENCY RESPONSE FOR RATIONAL SYSTEM FUNCTIONS

If a stable LTI system has a rational system function, i.e., if its input and output satisfy a difference equation of the form of Eq. (5.19), then its frequency response (the system function of Eq. (5.20) evaluated on the unit circle) has the form

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}. \quad (5.45)$$

Poles & Zeros

A pole-zero plot is a useful way of expressing a transfer function. Consider the following for $H(z) = 1-2z^{-1} + 2z^{-2} - z^{-3}$.

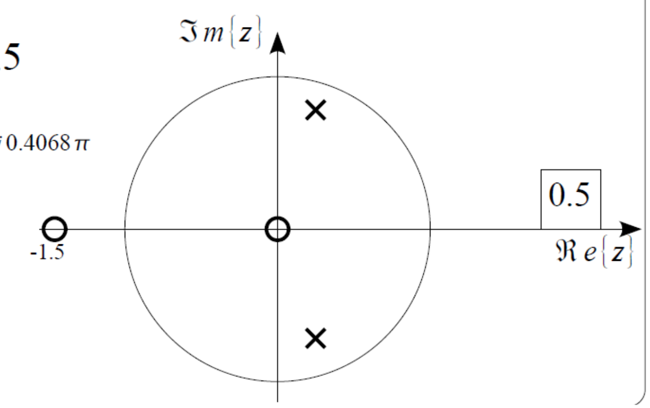


MATLAB: `zplane([1 -2 2 -1],[1])`
or `zplane([1 exp(j*pi/3) exp(-j*pi/3)].',[0 0 0].')`

$4y[n] = 2y[n-1] - 3y[n-2] + 2x[n] + 3x[n-1] \Rightarrow$ $H(z) = \dfrac{2+3z^{-1}}{4-2z^{-1}+3z^{-2}}$

Zeros:
$z = 0,\ z = -1.5$

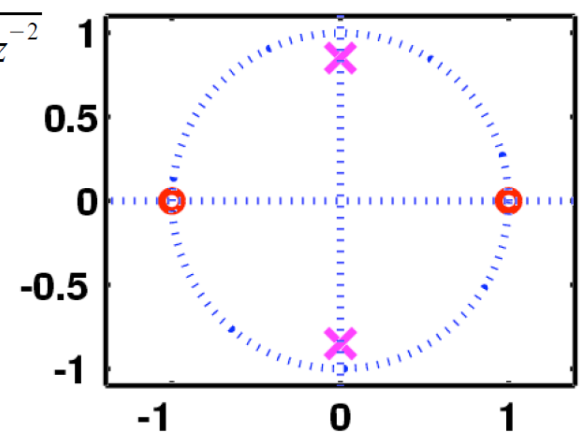
Poles:
$z = 0.886 e^{\pm j 0.4068\pi}$



$H(z) = \dfrac{1-z^{-2}}{1+0.7225z^{-2}}$

Zeros:
$z = \pm 1$

Poles:
$z = 0.85 e^{\pm j\pi/2}$



uOttawa

Second Order System

Example (page#296):

Example 5.6 2nd-Order IIR System

Consider the 2nd-order system

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}. \quad (5.61)$$

$$\Longleftrightarrow \quad H(e^{j\omega}) = \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}$$

The difference equation satisfied by the input and output of the system is

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n].$$

Using the partial fraction expansion technique, we can show that the impulse response of a causal system with this system function is

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]. \quad (5.62)$$

The system function in Eq. (5.61) has a pole at $z = re^{j\theta}$ and at the conjugate location, $z = re^{-j\theta}$, and two zeros at $z = 0$. The pole–zero plot is shown in Figure 5.12.

$$20\log_{10}|H(e^{j\omega})| = -10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)] - 10\log_{10}[1 + r^2 - 2r\cos(\omega + \theta)], \quad (5.63a)$$

$$\angle H(e^{j\omega}) = -\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right] - \arctan\left[\frac{r\sin(\omega+\theta)}{1 - r\cos(\omega+\theta)}\right] \quad (5.63b)$$

plotted for $r = 0.9$ and $\theta = \pi/4$.

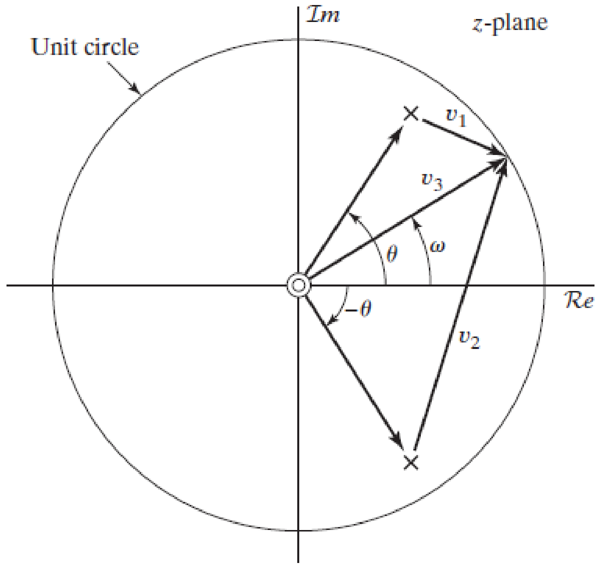


Figure 5.12 Pole–zero plot for Example 5.6.

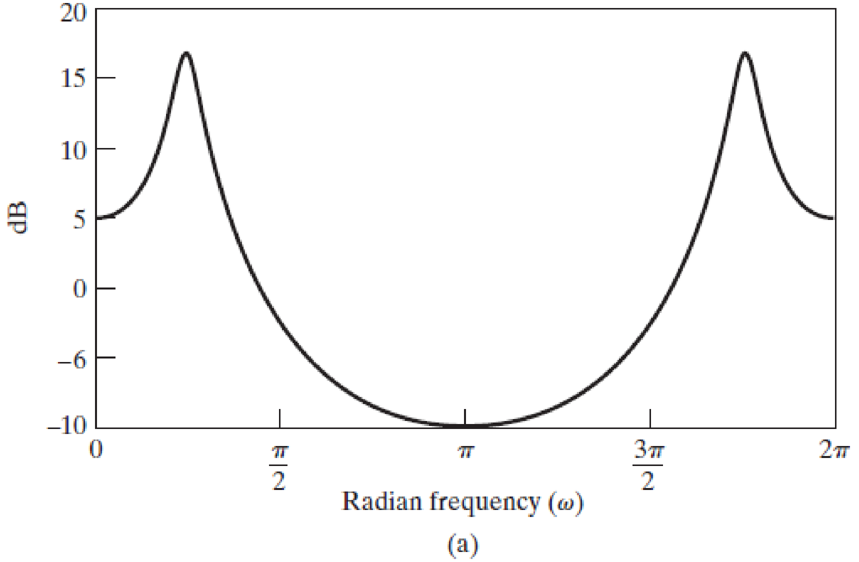


Figure 5.13 Frequency response for a complex-conjugate pair of poles as in Example 5.6, with $r = 0.9$, $\theta = \pi/4$. (a) Log magnitude.

Second Order System (2 poles & 2 zeroes)

$$H(z) = \frac{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}{(1 - Re^{j\omega_0}z^{-1})(1 - Re^{-j\omega_0}z^{-1})} = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

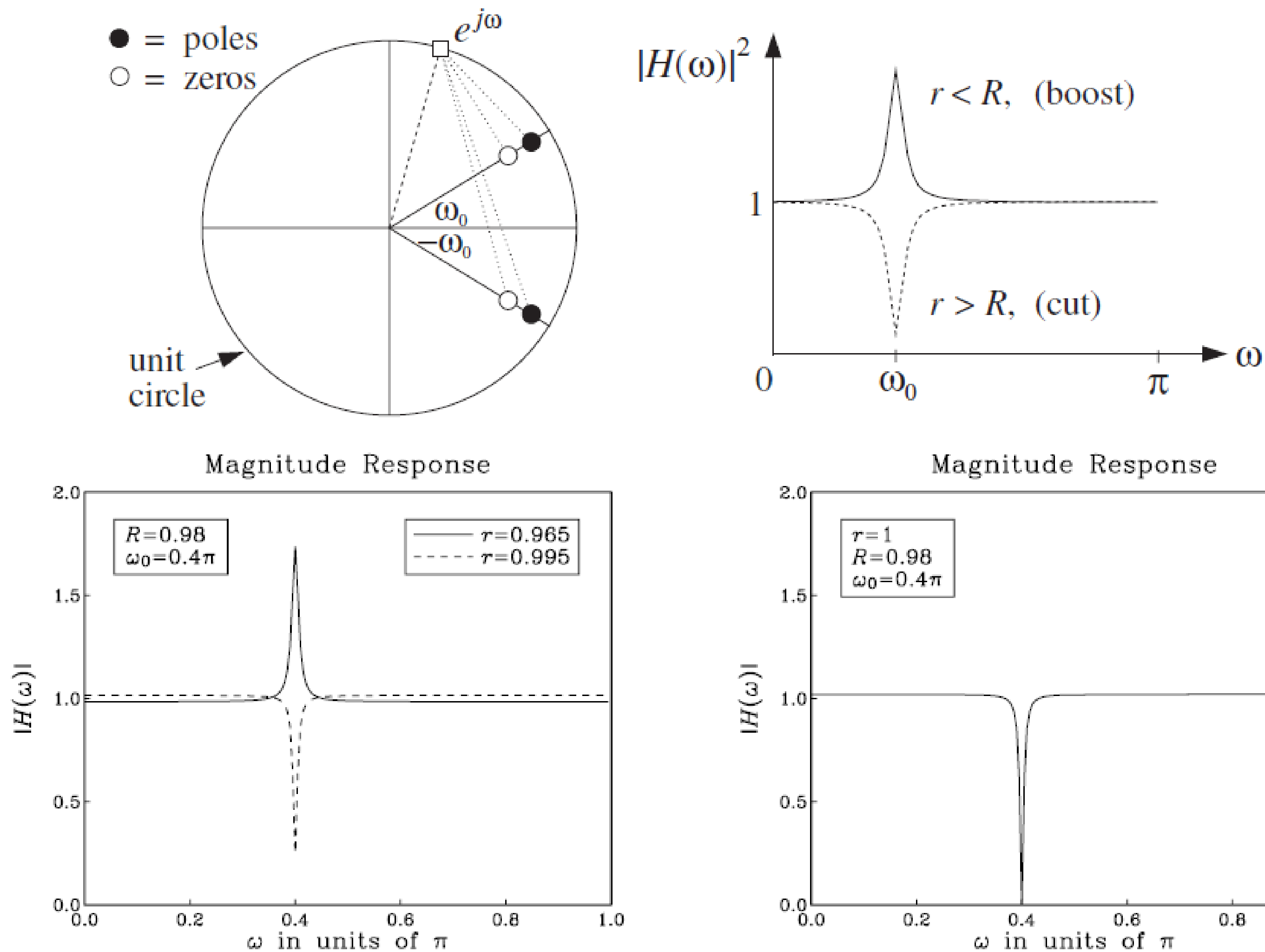
where the filter coefficients are given in terms of the parameters r, R, and ω_0:

$$b_1 = -2r\cos\omega_0 \qquad b_2 = r^2$$
$$a_1 = -2R\cos\omega_0 \qquad a_2 = R^2$$

The corresponding magnitude squared response is:

$$|H(e^{j\omega})| = \frac{\big(1 - 2r\cos(\omega - \omega_0) + r^2\big)\big(1 - 2r\cos(\omega + \omega_0) + r^2\big)}{\big(1 - 2R\cos(\omega - \omega_0) + R^2\big)\big(1 - 2R\cos(\omega + \omega_0) + R^2\big)}$$

Notch and Comb Filters



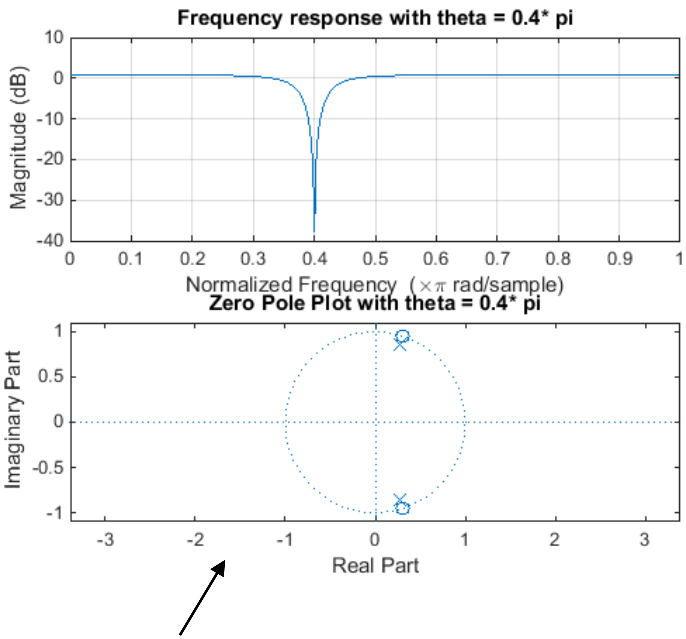
Notch & Comb Filters

Notch filter: If zeros are closer than poles to unit circle

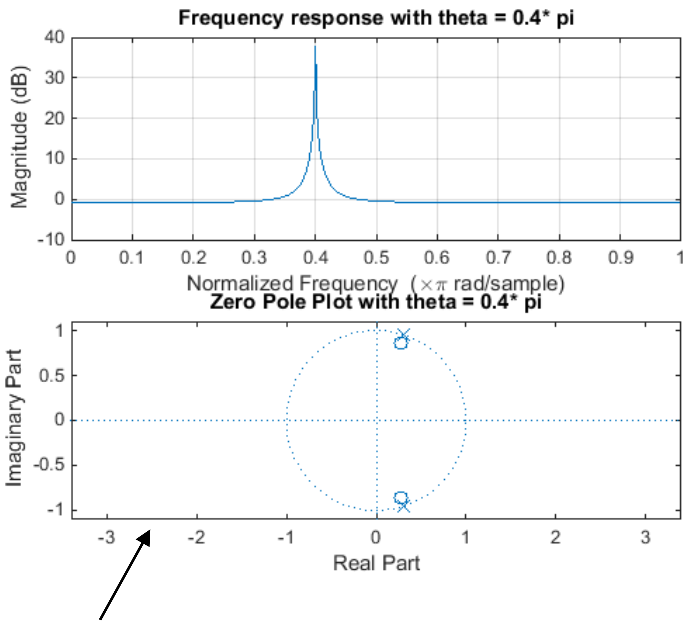
Comb Filter: If poles are closer than zeros to unit circle

$$H(z)=\frac{1-2\cos\theta\, z^{-1}+z^{-2}}{1-1.8\cos\theta\, z^{-1}+0.81z^{-2}}$$

$$H(z)=\frac{1-1.8\cos\theta\, z^{-1}+0.81z^{-2}}{1-2\cos\theta\, z^{-1}+z^{-2}}$$



Notch

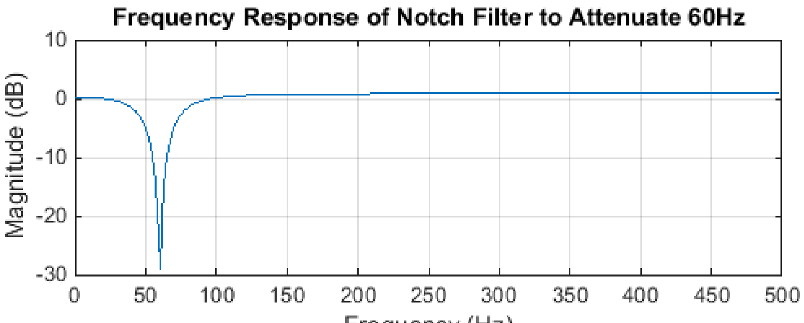


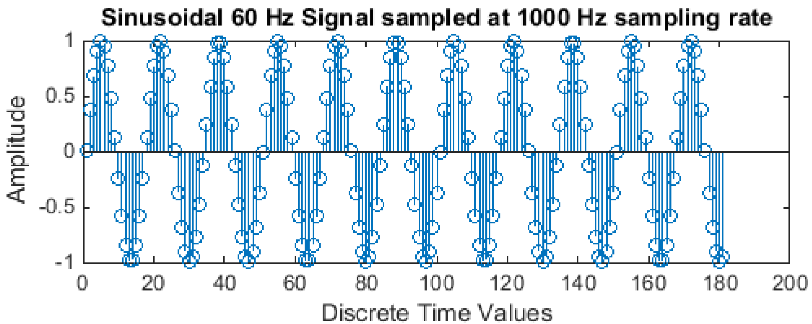
Comb

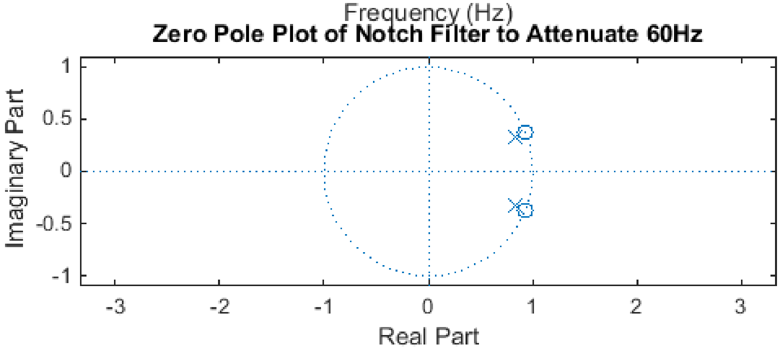
From Lab2: Remove certain frequency via Notch filter

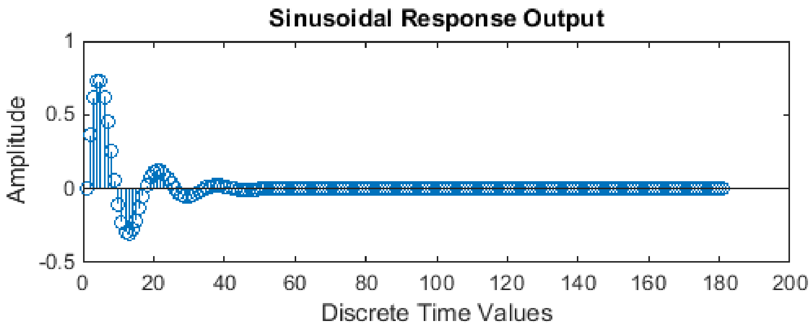
Removing 60 Hz

$$\theta = \frac{f}{fs}.2\pi = 60/1000*2\pi$$









uOttawa

Parametric Resonators

As another example, consider the design of a simple second-order "resonator" filter whose frequency response is dominated by a single narrow pole peak at some frequency ω0.

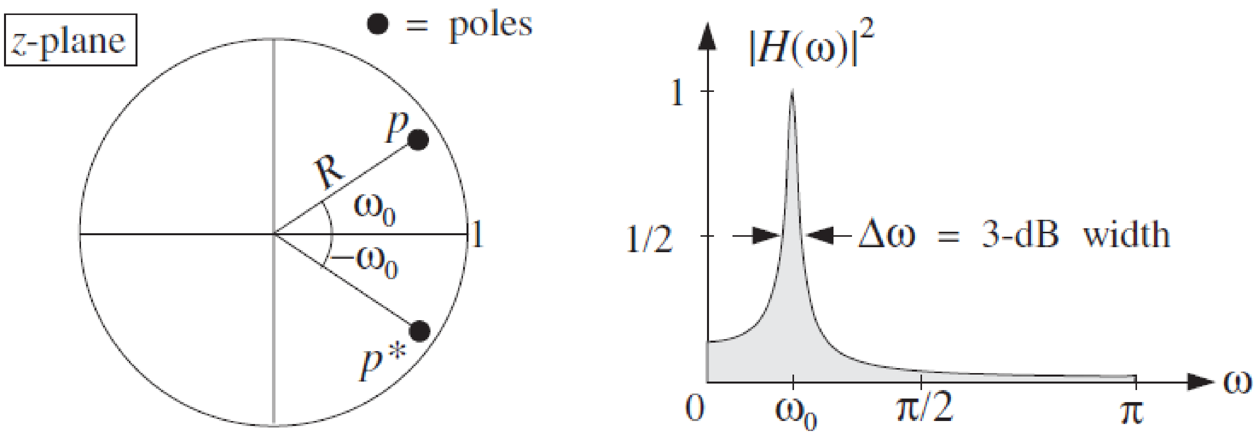


Fig. 6.4.2 Pole/zero pattern and frequency response of resonator filter.

$$H(z) = \frac{G}{(1 - Re^{j\omega_0}z^{-1})(1 - Re^{-j\omega_0}z^{-1})} = \frac{G}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{6.4.3}$$

where a_1 and a_2 are related to R and ω_0 by

$$a_1 = -2R\cos\omega_0, \quad a_2 = R^2$$

The gain G may be fixed so as to normalize the filter to unity at ω_0, that is, $|H(e^{j\omega})| = 1$. The frequency response of the filter is obtained by the substitution $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{G}{(1 - Re^{j\omega_0}e^{-j\omega})(1 - Re^{-j\omega_0}e^{-j\omega})} = \frac{G}{1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}}$$

The normalization requirement $|H(\omega_0)| = 1$ gives the condition:

$$|H(\omega_0)| = \frac{G}{|(1 - Re^{j\omega_0}e^{-j\omega_0})(1 - Re^{-j\omega_0}e^{-j\omega_0})|} = 1$$

which can be solved for G:

$$G = (1 - R)\sqrt{1 - 2R\cos(2\omega_0) + R^2}$$

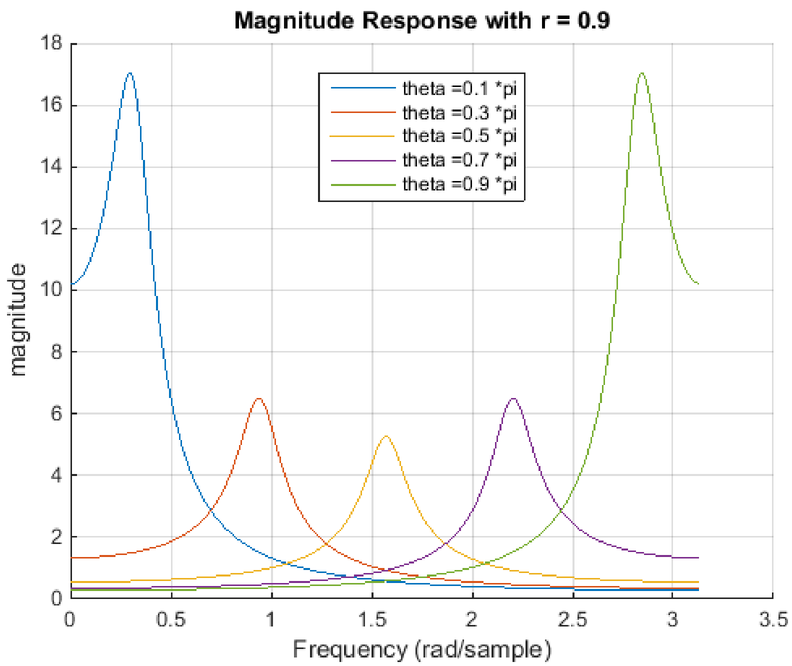
MATLAB

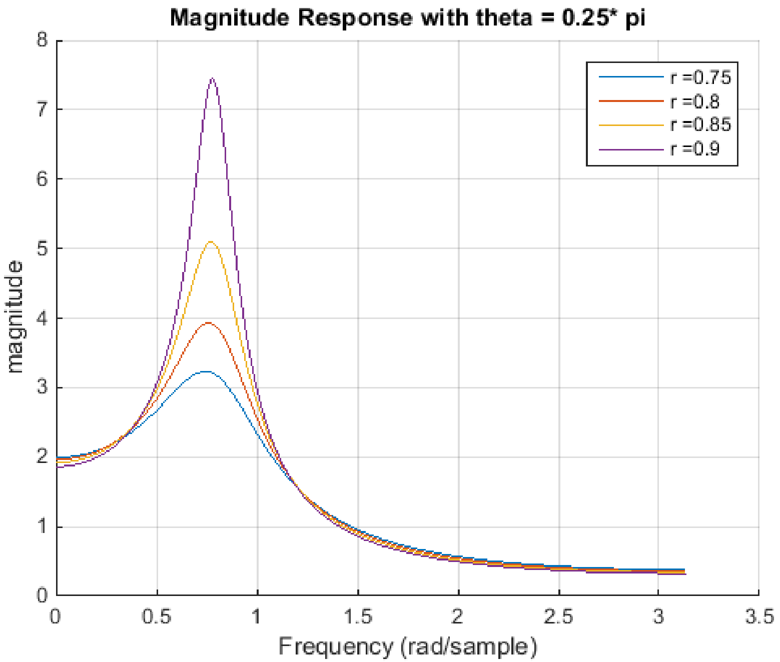
Introduction to Lab#2

FILTERS AND RESONATORS

From Lab2: Resonators

$$H(z) = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$



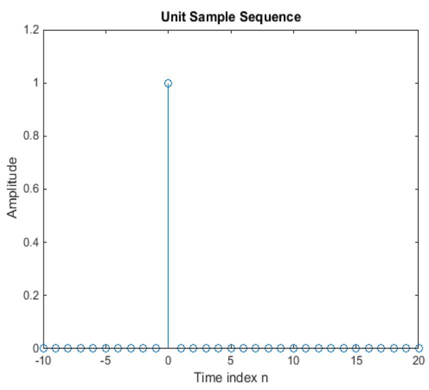


$$\text{The Gain } \; G = \left|H(e^{j\omega})\right| = \frac{1}{1 - r\sqrt{1 - 2r\cos(2\theta) + r^2}}$$

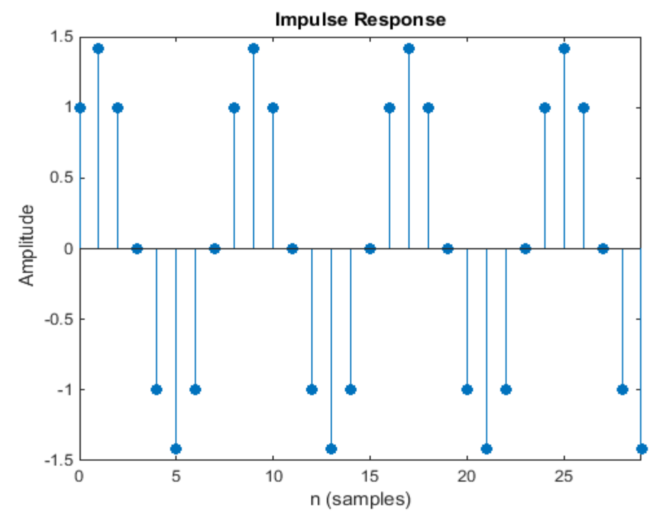
Sinusoid Generator

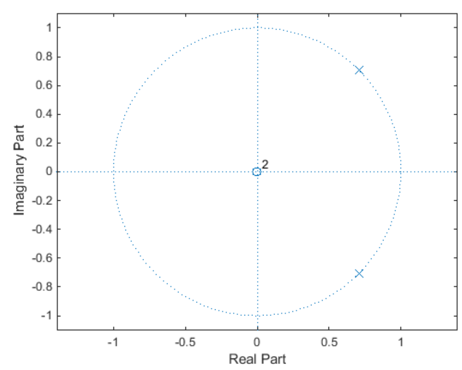
$$H(z)=\frac{1}{1-2r\cos\theta\, z^{-1}+r^2z^{-2}}$$

Let r=1



$$\frac{1}{1-2\cos\theta\, z^{-1}+z^{-2}}$$





$$\text{Frequency} = f = \frac{\theta}{2\pi}.fs$$

THE END