

ELG4172 Digital Signal Processing

- Tutorial #2

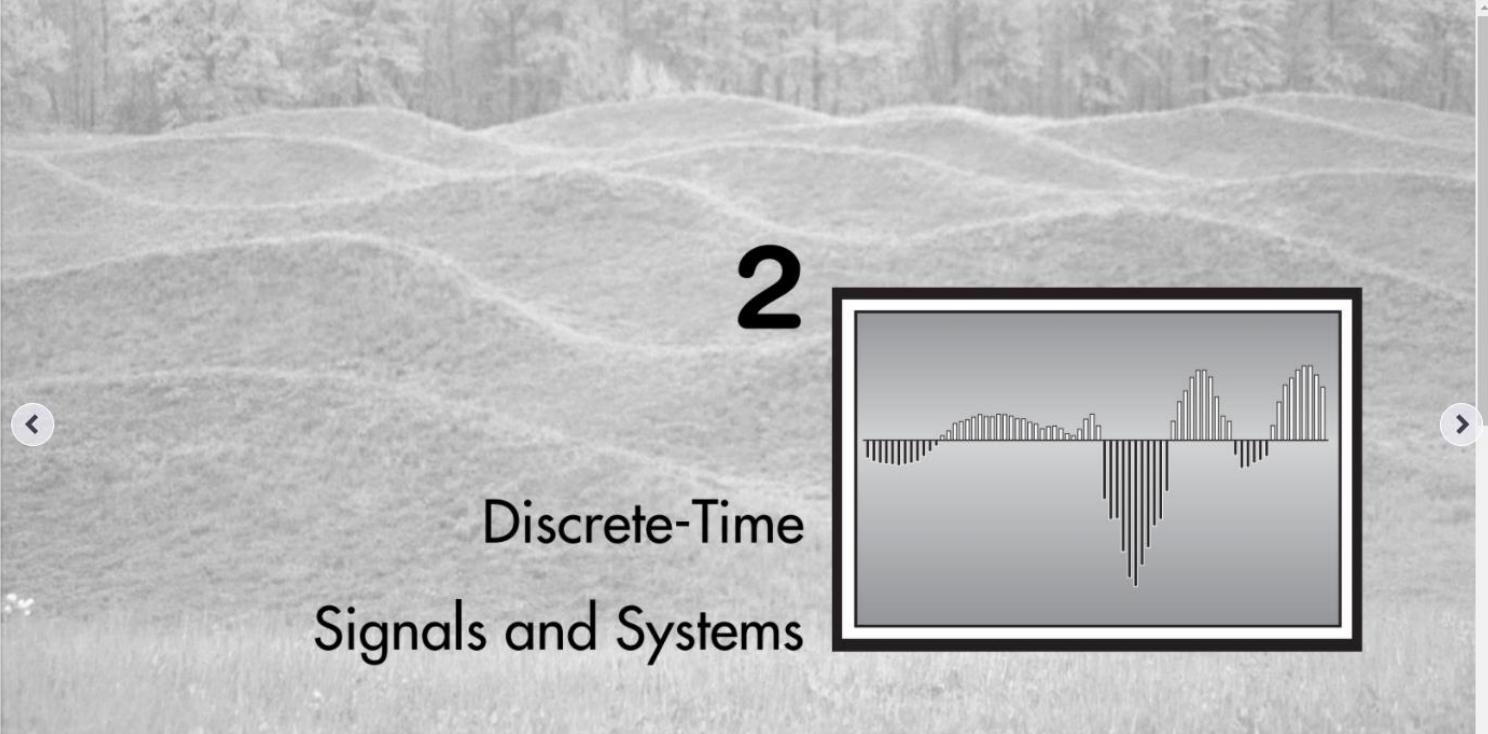
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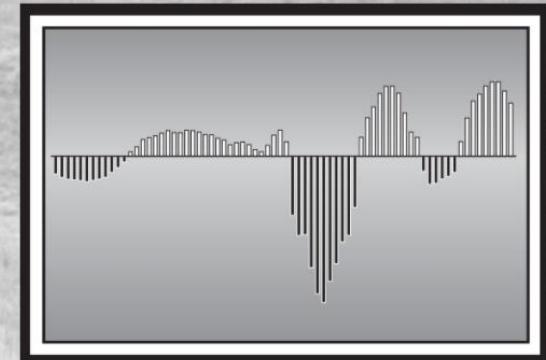
Discrete-Time Fourier Transform DTFT



The background of the slide features a grayscale photograph of a rolling, grassy hillside under a hazy sky.

2

Discrete-Time Signals and Systems



A rectangular frame contains a visualization of a signal spectrum. The plot shows a series of vertical bars of varying heights, representing the magnitude of different frequency components. The bars are concentrated in two distinct regions, suggesting periodic or impulsive signals. The entire visualization is set against a dark gray background.

Navigation icons: back arrow, forward arrow, and a scroll bar at the bottom right.

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Review Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

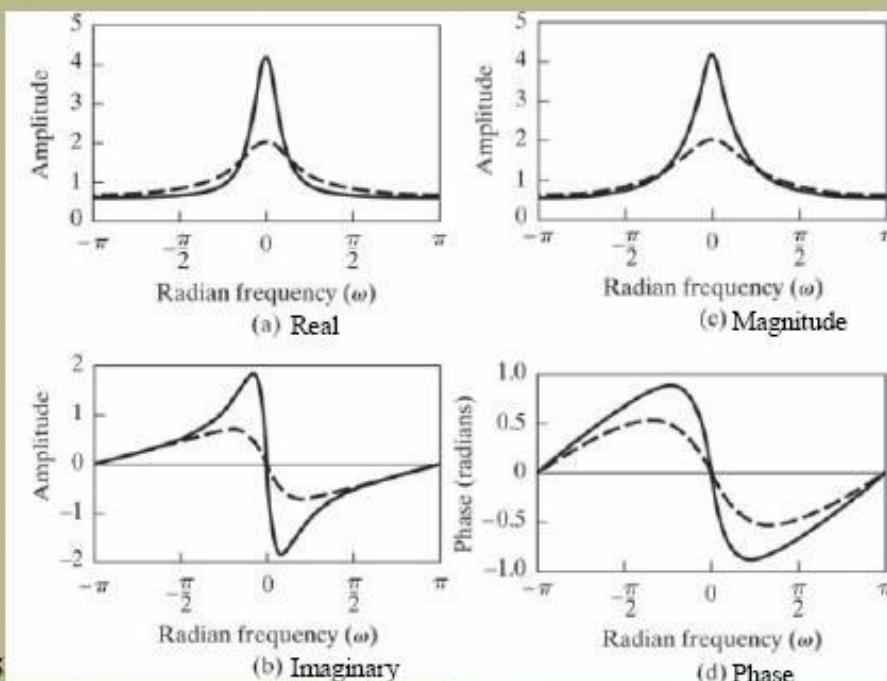
Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

Inverse Fourier Transform

Signal $x[n] = a^n u[n]$

- Fourier transform $X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$, if $|a| < 1$



- 2.6. (a)** Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

- (b)** Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

- 2.6. (a)** Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

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(a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}.$$

(b) A system with frequency response:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

A linear time-invariant system is described by the following difference equation:

$$y(n) = ay(n-1) + bx(n), \quad 0 < a < 1$$

- (a) Determine the magnitude and phase of the frequency response $H(e^{j\omega})$ of the system.
- (b) Choose the parameter b so that the maximum value of $|H(e^{j\omega})|$ is unity, and sketch $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ for $a = 0.9$.
- (c) Determine the output of the system to the input signal

$$x(n) = 5 + 12 \sin \frac{\pi}{2}n - 20 \cos \left(\pi n + \frac{\pi}{4} \right)$$

A linear time-invariant system is described by the following difference equation:

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- (c) Determine the output of the system to the input signal

$$x(n) = 5 + 12 \sin \frac{\pi}{2}n - 20 \cos \left(\pi n + \frac{\pi}{4} \right)$$

- (a) The frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b}{1 - ae^{-j\omega}}$$

Since

$$1 - ae^{-j\omega} = (1 - a \cos \omega) + ja \sin \omega$$

it follows that

$$\begin{aligned} |1 - ae^{-j\omega}| &= \sqrt{(1 - a \cos \omega)^2 + (a \sin \omega)^2} \\ &= \sqrt{1 + a^2 - 2a \cos \omega} \end{aligned}$$

$$|H(e^{j\omega})| = \frac{|b|}{|1 - ae^{-j\omega}|} = \frac{|b|}{\sqrt{1 + a^2 - 2a \cos \omega}}$$

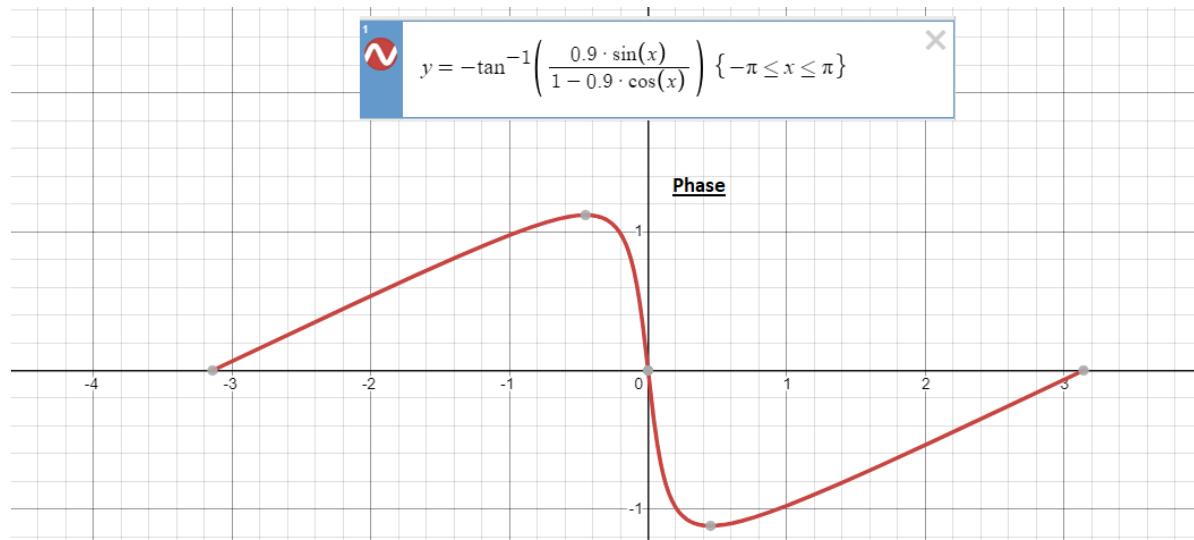
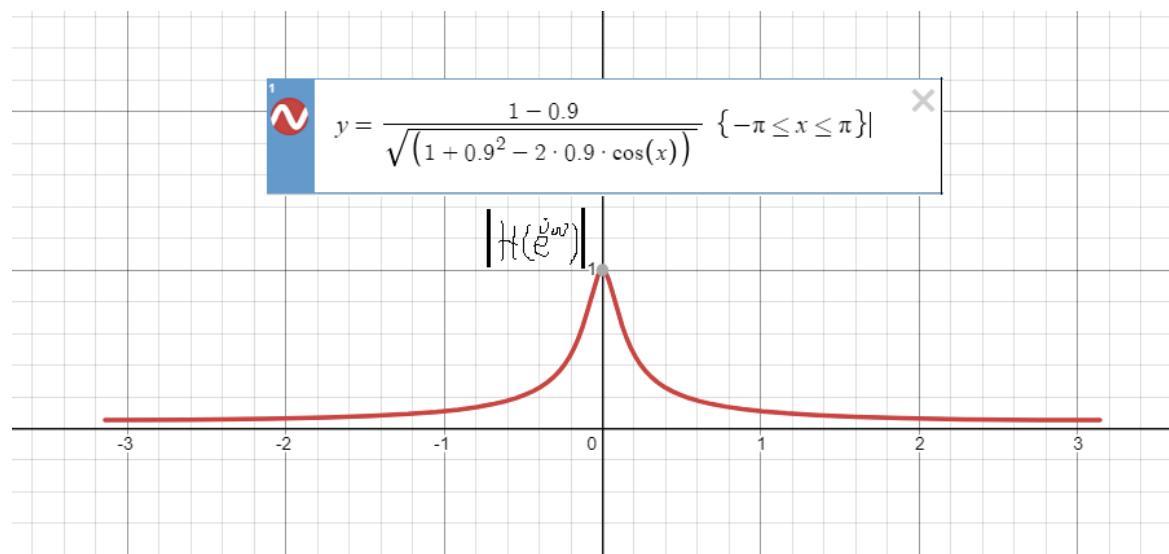
$$\angle H(e^{j\omega}) = \Theta(e^{j\omega}) = \angle b - \tan^{-1} \frac{a \sin \omega}{1 - a \cos \omega}$$

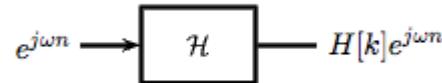
- (b) Since the parameter a is positive, the denominator of $|H(e^{j\omega})|$ attains a minimum at $\omega = 0$. Therefore, $|H(e^{j\omega})|$ attains its maximum value at $\omega = 0$. At this frequency we have,

$$|H(0)| = \frac{|b|}{1 - a} = 1$$

which implies that $b = \pm(1 - a)$. We choose $b = 1 - a$, so that

$$|H(e^{j\omega})| = \frac{1 - a}{\sqrt{1 + a^2 - 2a \cos \omega}} \quad \text{and} \quad \Theta(e^{j\omega}) = -\tan^{-1} \frac{a \sin \omega}{1 - a \cos \omega}$$





(c) The input signal consists of components at frequencies $\omega = 0$, $\pi/2$, and π

For $\omega = 0$, $\Rightarrow |H(e^{j\omega})| = 1$ and $\Theta(0) = 0$. For $\omega = \pi/2$,

$$\omega = \pi/2 \Rightarrow |H(e^{j\omega})| = \frac{1-a}{\sqrt{1+a^2}} = \frac{0.1}{\sqrt{1.81}} = 0.074$$

$$\Theta\left(\frac{\pi}{2}\right) = -\tan^{-1} a = -42^\circ$$

$$\text{For } \omega = \pi, \Rightarrow |H(e^{j\omega})| = \frac{1-a}{1+a} = \frac{0.1}{1.9} = 0.053$$

$$\Theta(\pi) = 0$$

Therefore, the output of the system is

$$y(n) = 5|H(0)| + 12 \left| H\left(\frac{\pi}{2}\right) \right| \sin \left[\frac{\pi}{2}n + \Theta\left(\frac{\pi}{2}\right) \right] - 20|H(\pi)| \cos \left[\pi n + \frac{\pi}{4} + \Theta(\pi) \right]$$

8. An LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

Q 2.8

$$\begin{aligned} y[n] &= h[n] * x[n] \\ \text{FT} \curvearrowleft y(e^{j\omega}) &= H(e^{j\omega}) X(e^{j\omega}) \end{aligned}$$

$$H(e^{j\omega}) = \frac{5}{1 + \frac{1}{2} e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

$$\begin{aligned} Y(e^{j\omega}) &= \frac{4}{1 + \frac{1}{3} e^{-j\omega}} \cdot \frac{5}{1 + \frac{1}{2} e^{-j\omega}} \\ &= \frac{3}{1 + \frac{1}{2} e^{-j\omega}} + \frac{2}{1 - \frac{1}{3} e^{-j\omega}} \end{aligned}$$

$$y[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n]$$

2.9. Consider the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = \frac{1}{3}x[n-1].$$

- (a) What are the impulse response, frequency response, and step response for the causal LTI system satisfying this difference equation?

2.9. (a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1 - \frac{1}{3}e^{-j\omega}} + \frac{2}{1 - \frac{1}{2}e^{-j\omega}} \\ h[n] &= -2\left(\frac{1}{3}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

For the step response $s[n]$:

$$\begin{aligned}s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\&= \sum_{k=-\infty}^n h[k] \\&= -2 \frac{1 - (1/3)^{n+1}}{1 - 1/3} u[n] + 2 \frac{1 - (1/2)^{n+1}}{1 - 1/2} u[n] \\&= (1 + (\frac{1}{3})^n - 2(\frac{1}{2})^n)u[n]\end{aligned}$$

Frequency Response For Rational System Functions

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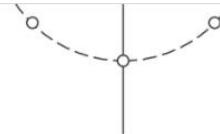


Figure 5.8 Pole–zero plot for Example 5.5.

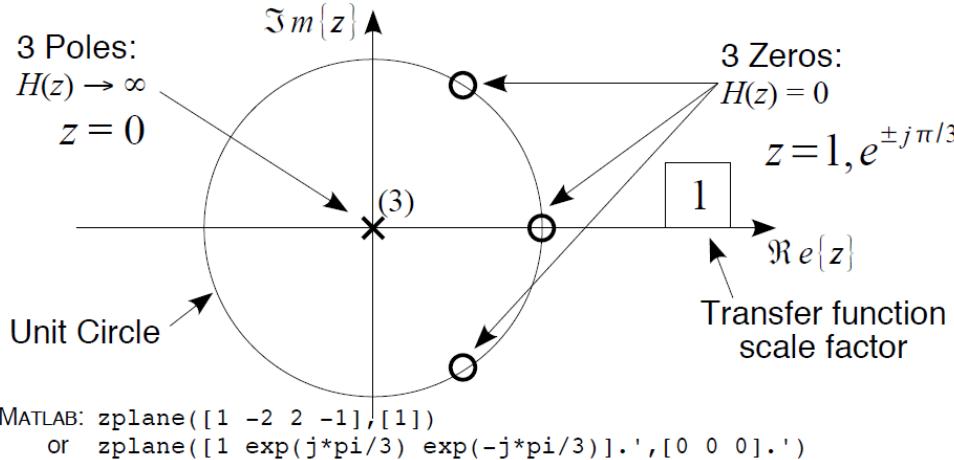
5.3 FREQUENCY RESPONSE FOR RATIONAL SYSTEM FUNCTIONS

If a stable LTI system has a rational system function, i.e., if its input and output satisfy a difference equation of the form of Eq. (5.19), then its frequency response (the system function of Eq. (5.20) evaluated on the unit circle) has the form

$$H(e^{j\omega}) = \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{\sum_{k=0}^N a_k e^{-j\omega k}}. \quad (5.45)$$

A pole-zero plot is a useful way of expressing a transfer function. Consider the following for $H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$.

Poles & Zeros



$$4y[n] = 2y[n-1] - 3y[n-2] + 2x[n] + 3x[n-1] \Rightarrow$$

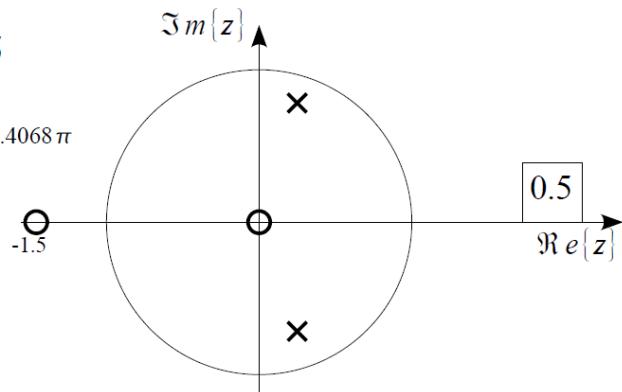
$$H(z) = \frac{2+3z^{-1}}{4-2z^{-1}+3z^{-2}}$$

Zeros:

$$z = 0, z = -1.5$$

Poles:

$$z = 0.886e^{\pm j0.4068\pi}$$



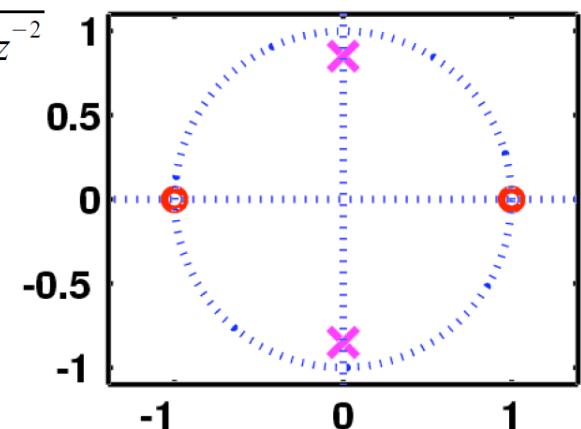
$$H(z) = \frac{1-z^{-2}}{1+0.7225z^{-2}}$$

Zeros:

$$z = \pm 1$$

Poles:

$$z = 0.85e^{\pm j\pi/2}$$



Example (page#296):**Second Order System****Example 5.6 2nd-Order IIR System**

Consider the 2nd-order system

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}.$$

The difference equation satisfied by the input and output of the system is

$$y[n] - 2r \cos \theta y[n-1] + r^2 y[n-2] = x[n].$$

Using the partial fraction expansion technique, we can show that the impulse response of a causal system with this system function is

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin \theta} u[n]. \quad (5.62)$$

The system function in Eq. (5.61) has a pole at $z = re^{j\theta}$ and at the conjugate location, $z = re^{-j\theta}$, and two zeros at $z = 0$. The pole-zero plot is shown in Figure 5.12.

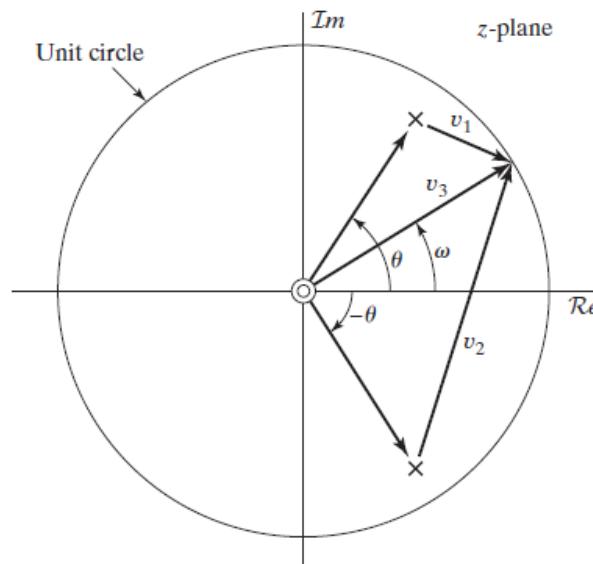


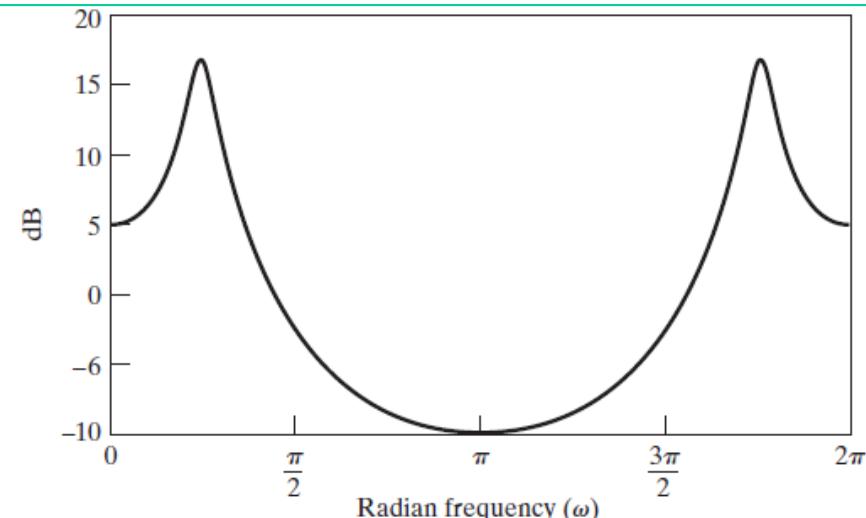
Figure 5.12 Pole-zero plot for Example 5.6.

$$(5.61) \Leftrightarrow H(e^{j\omega}) = \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}$$

$$\begin{aligned} 20 \log_{10} |H(e^{j\omega})| &= -10 \log_{10}[1 + r^2 - 2r \cos(\omega - \theta)] \\ &\quad - 10 \log_{10}[1 + r^2 - 2r \cos(\omega + \theta)], \end{aligned} \quad (5.63a)$$

$$\angle H(e^{j\omega}) = -\arctan \left[\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right] - \arctan \left[\frac{r \sin(\omega + \theta)}{1 - r \cos(\omega + \theta)} \right] \quad (5.63b)$$

plotted for $r = 0.9$ and $\theta = \pi/4$.



(a)

Figure 5.13 Frequency response for a complex-conjugate pair of poles as in Example 5.6, with $r = 0.9$, $\theta = \pi/4$. (a) Log magnitude.

Second Order System (2 poles & 2 zeroes)

$$H(z) = \frac{(1 - re^{j\omega_0}Z^{-1})(1 - re^{-j\omega_0}Z^{-1})}{(1 - Re^{j\omega_0}Z^{-1})(1 - Re^{-j\omega_0}Z^{-1})} = \frac{1 + b_1Z^{-1} + b_2Z^{-2}}{1 + a_1Z^{-1} + a_2Z^{-2}}$$

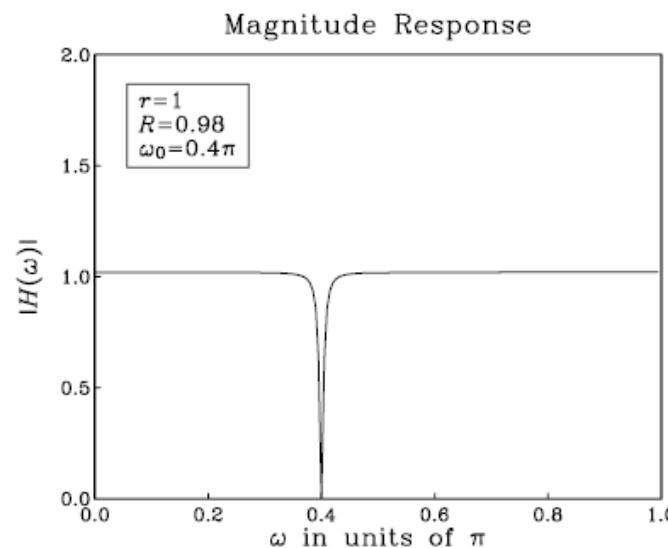
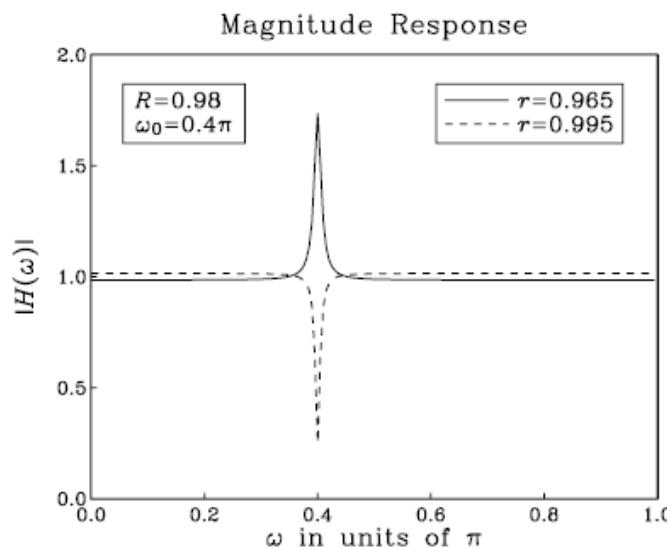
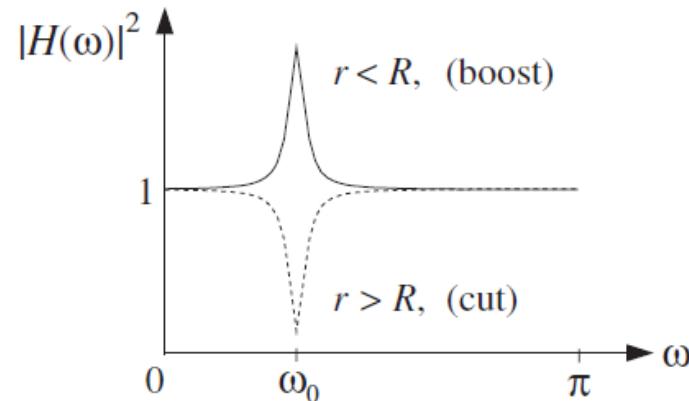
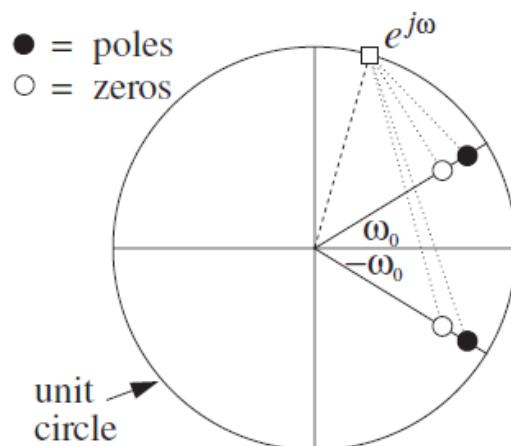
where the filter coefficients are given in terms of the parameters r , R , and ω_0 :

$$\begin{aligned} b_1 &= -2r \cos \omega_0 & b_2 &= r^2 \\ a_1 &= -2R \cos \omega_0 & a_2 &= R^2 \end{aligned}$$

The corresponding magnitude squared response is:

$$|H(e^{j\omega})|^2 = \frac{(1 - 2r \cos(\omega - \omega_0) + r^2)(1 - 2r \cos(\omega + \omega_0) + r^2)}{(1 - 2R \cos(\omega - \omega_0) + R^2)(1 - 2R \cos(\omega + \omega_0) + R^2)}$$

Notch and Comb Filters



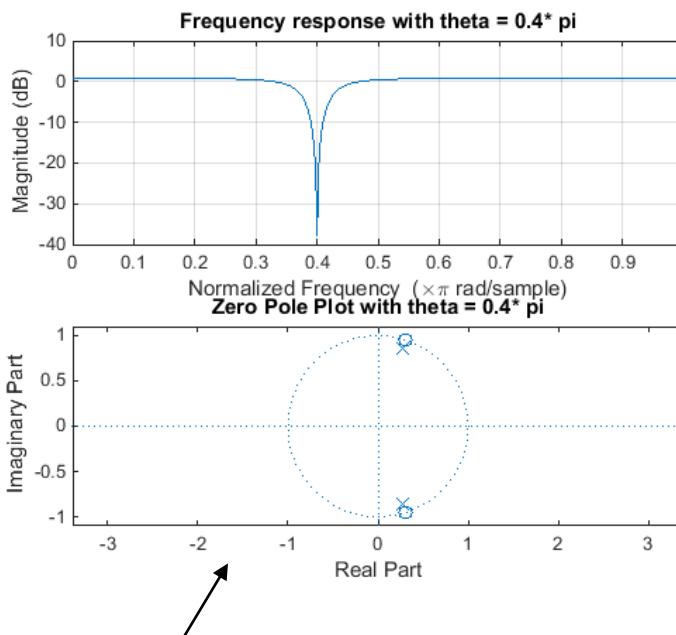
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Notch & Comb Filters

Notch filter: If zeros are closer than poles to unit circle

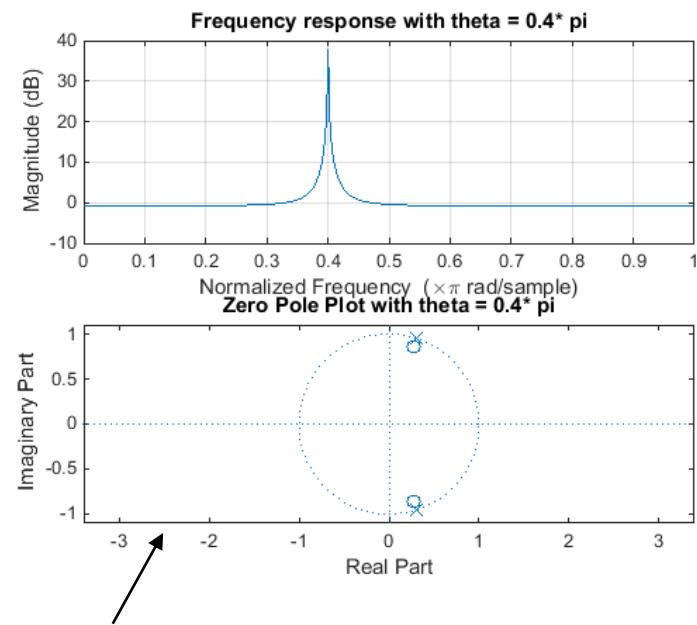
Comb Filter: If poles are closer than zeros to unit circle

$$H(z) = \frac{1 - 2\cos\theta z^{-1} + z^{-2}}{1 - 1.8\cos\theta z^{-1} + 0.81z^{-2}}$$



uOttawa.ca Notch

$$H(z) = \frac{1 - 1.8\cos\theta z^{-1} + 0.81z^{-2}}{1 - 2\cos\theta z^{-1} + z^{-2}}$$



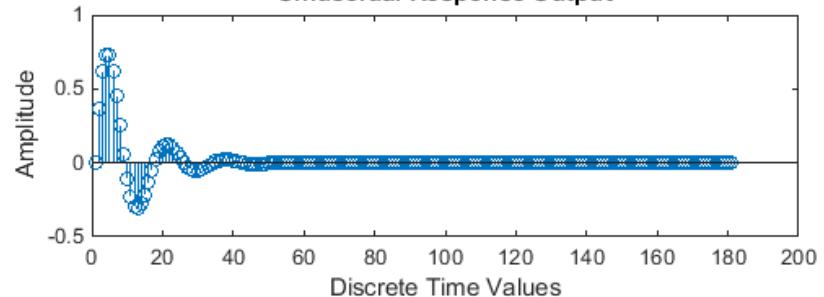
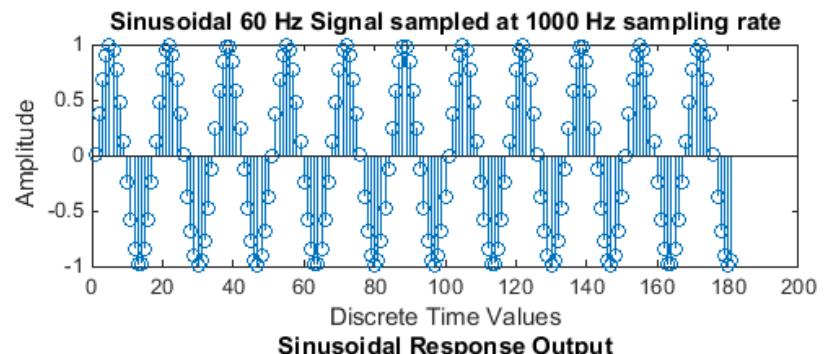
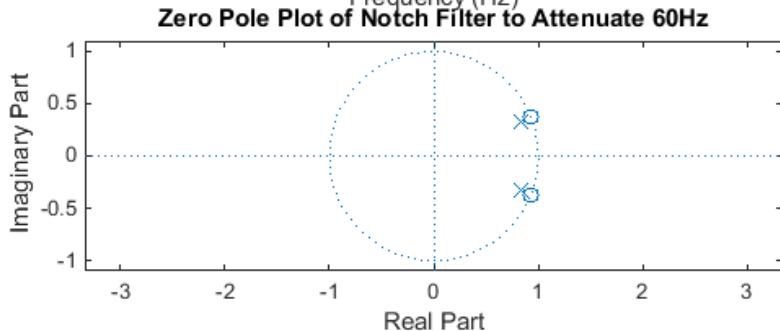
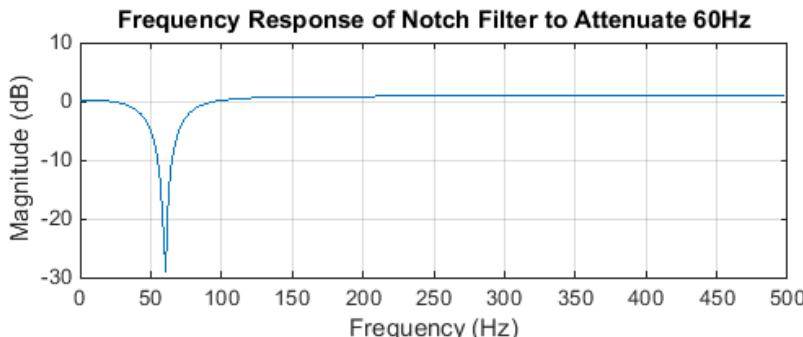
Comb



From Lab2: Remove certain frequency via Notch filter

Removing 60 Hz

$$\theta = \frac{f}{fs} \cdot 2\pi = 60/1000 * 2\pi$$



Parametric Resonators

As another example, consider the design of a simple second-order “resonator” filter whose frequency response is dominated by a single narrow pole peak at some frequency ω_0 .

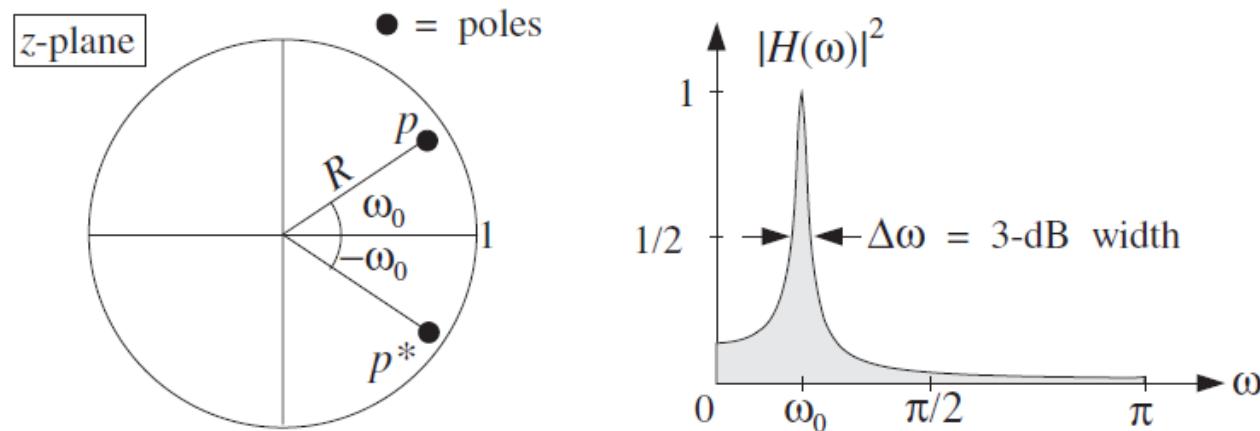


Fig. 6.4.2 Pole/zero pattern and frequency response of resonator filter.

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$$H(z) = \frac{G}{(1 - Re^{j\omega_0}Z^{-1})(1 - Re^{-j\omega_0}Z^{-1})} = \frac{G}{1 + a_1Z^{-1} + a_2Z^{-2}} \quad (6.4.3)$$

where a_1 and a_2 are related to R and ω_0 by

$$[a_1 = -2R \cos \omega_0, \quad a_2 = R^2]$$

The gain G may be fixed so as to normalize the filter to unity at ω_0 , that is, $|H(e^{j\omega})| = 1$. The frequency response of the filter is obtained by the substitution $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{G}{(1 - Re^{j\omega_0}e^{-j\omega})(1 - Re^{-j\omega_0}e^{-j\omega})} = \frac{G}{1 + a_1e^{-j\omega} + a_2e^{-2j\omega}}$$

The normalization requirement $|H(\omega_0)| = 1$ gives the condition:

$$|H(\omega_0)| = \frac{G}{|(1 - Re^{j\omega_0}e^{-j\omega_0})(1 - Re^{-j\omega_0}e^{-j\omega_0})|} = 1$$

which can be solved for G :

$$G = (1 - R)\sqrt{1 - 2R \cos(2\omega_0) + R^2}$$

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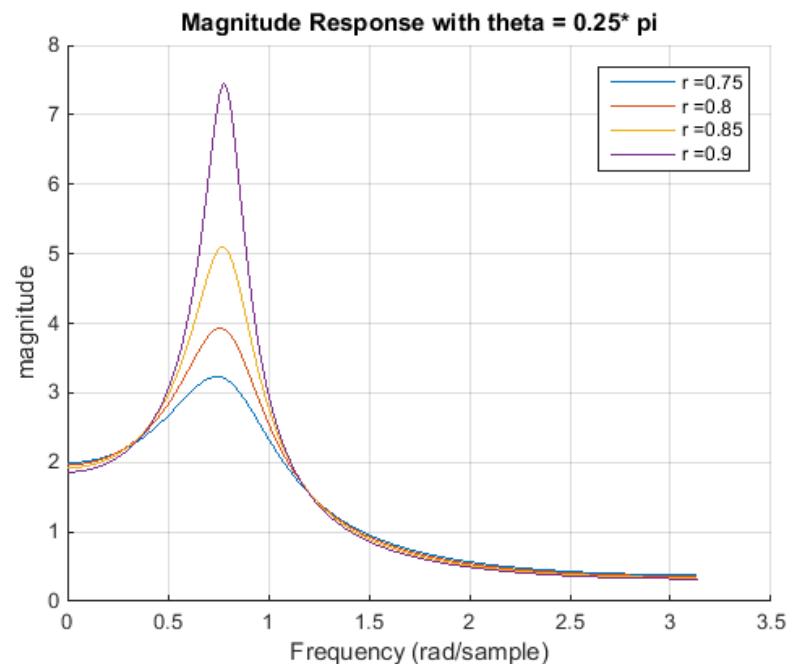
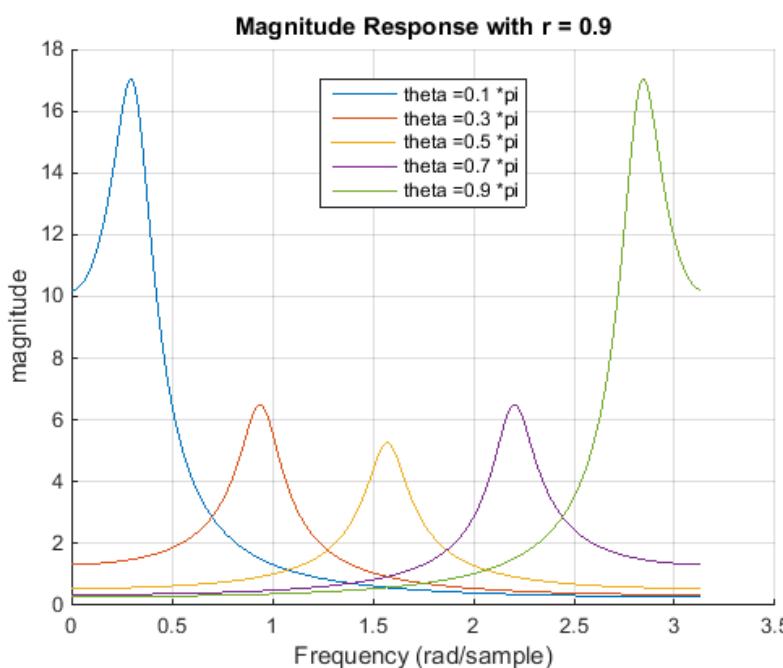


Introduction to Lab#2

FILTERS AND RESONATORS

From Lab2: Resonators

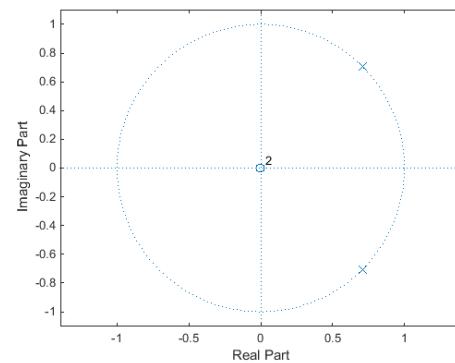
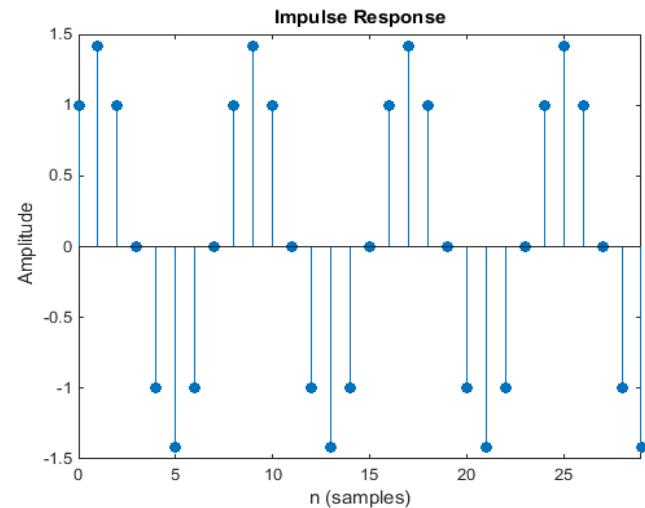
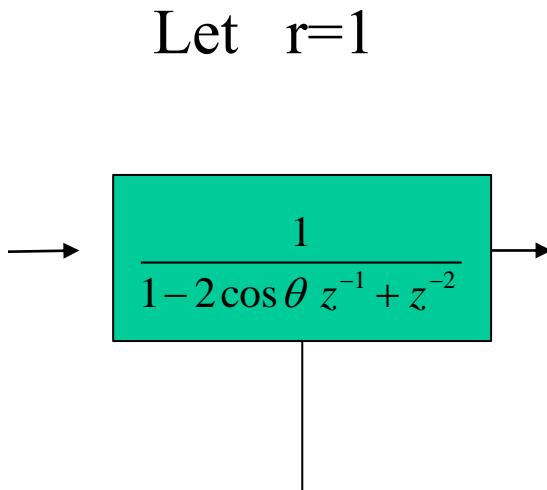
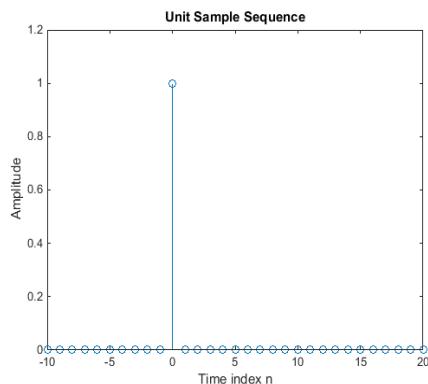
$$H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$$



The Gain $G = |H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2r \cos(2\theta) + r^2}}$

Sinusoid Generator

$$H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$$



$$\text{Frequency} = f = \frac{\theta}{2\pi} \cdot fs$$

THE END